CHAPTER I
Derivatives

§1. FIRST DERIVATIVE

As was said in the Introduction, in this chapter and the next we shall study the infinitesimal properties of functions which are defined on a subset of the real field **R** and take their values in a *Hausdorff topological vector space* E over the field **R**; for brevity we shall say that such a function is a *vector function of a real variable.* The most important case is that where E $=$ **R** (real-valued functions of a real variable). When E $=$ **R**^{n}, consideration of a vector function with values in E reduces to the simultaneous consideration of *n* finite real functions.

Many of the definitions and properties stated in chapter I extend to functions which are defined on a subset of the field **C** of complex numbers and take their values in a topological vector space over **C** (vector functions of a complex variable). Some of these definitions and properties extend even to functions which are defined on a subset of an arbitrary commutative *topological field* K and take their values in a topological vector space over K.

We shall indicate these generalizations in passing (see in particular I, p. 10, *Remark 2*), emphasising above all the case of functions of a complex variable, which are by far the most important, together with functions of a real variable, and will be studied in greater depth in a later Book.

1. DERIVATIVE OF A VECTOR FUNCTION

DEFINITION 1. *Let* **f** *be a vector function defined on an interval* I $\subset$ **R** *which does not reduce to a single point. We say that* **f** *is differentiable at a point* $x_0 \in$ I *if*

$$\lim_{x \to x_0, x \in I, x \neq x_0} \frac{\mathbf{f}(x) - \mathbf{f}(x_0)}{x - x_0}$$ *exists* (*in the vector space where* **f** *takes its values*)*; the value of this limit is called the first derivative* (or simply the *derivative*) *of* **f** *at the point* x_0, *and it is denoted by* $\mathbf{f}'(x_0)$ *or* $\mathrm{D}\mathbf{f}(x_0)$.

If **f** is differentiable at the point x_0, so is the *restriction* of **f** to any interval J $\subset$ I which does not reduce to a single point and such that $x_0 \in$ J; and the derivative of this restriction is equal to $\mathbf{f}'(x_0)$. Conversely, let J be an interval contained in I and containing a neighbourhood of x_0 relative to I; if the restriction of **f** to J admits a derivative at the point x_0, then so does **f**.

We summarise these properties by saying that the concept of derivative is a *local* concept.

Remarks. *1) In Kinematics, if the point $\mathbf{f}(t)$ is the position of a moving point in the space $\mathbf{R}^3$ at time t, then $\dfrac{\mathbf{f}(t) - \mathbf{f}(t_0)}{t - t_0}$ is termed the *average velocity* between the instants t_0 and t, and its limit $\mathbf{f}'(t_0)$ is the *instantaneous velocity* (or simply *velocity*) at the time t_0 (when this limit exists).*

2) If a function $\mathbf{f}$, defined on I, is differentiable at a point $x_0 \in$ I, it is necessarily *continuous relative to* I at this point.

DEFINITION 2. *Let $\mathbf{f}$ be a vector function defined on an interval $\mathrm{I} \subset \mathbf{R}$, and let x_0 be a point of I such that the interval $\mathrm{I} \cap [x_0, +\infty[$ (resp. $\mathrm{I} \cap] - \infty, x_0]$) does not reduce to a single point. We say that $\mathbf{f}$ is differentiable on the right (resp. on the left) at the point x_0 if the restriction of $\mathbf{f}$ to the interval $\mathrm{I} \cap [x_0, +\infty[$ (resp. $\mathrm{I} \cap] - \infty, x_0]$) is differentiable at the point x_0; the value of the derivative of this restriction at the point x_0 is called the right (resp. left) derivative of $\mathbf{f}$ at the point x_0 and is denoted by $\mathbf{f}'_d(x_0)$ (resp. $\mathbf{f}'_g(x_0)$).*

Let $\mathbf{f}$ be a vector function defined on I, and x_0 an *interior* point of I such that $\mathbf{f}$ is continuous at this point; it follows from defs. 1 and 2 that for $\mathbf{f}$ to be differentiable at x_0 it is necessary and sufficient that $\mathbf{f}$ admit both a right and a left derivative at this point, and that these derivatives be *equal;* and then

$$\mathbf{f}'(x_0) = \mathbf{f}'_d(x_0) = \mathbf{f}'_g(x_0).$$

Examples. 1) A *constant* function has zero derivative at every point.

2) An affine linear function $x \mapsto \mathbf{a}x + \mathbf{b}$ has derivative equal to $\mathbf{a}$ at every point.

3) The real function $1/x$ (defined for $x \neq 0$) is differentiable at each point $x_0 \neq 0$, for we have $\left(\dfrac{1}{x} - \dfrac{1}{x_0}\right) \Big/ (x - x_0) = -\dfrac{1}{xx_0}$, and, since $1/x$ is continuous at x_0, the limit of the preceding expression is $-1/x_0^2$.

4) The scalar function $|x|$, defined on $\mathbf{R}$, has right derivative $+1$ and left derivative -1 at $x = 0$; it is not differentiable at this point.

5) The real function equal to 0 for $x = 0$, and to $x \sin 1/x$ for $x \neq 0$, is defined and continuous on $\mathbf{R}$, but has neither right nor left derivative at the point $x \neq 0$. One can give examples of functions which are continuous on an interval and fail to have a derivative at *every* point of the interval (I, p. 35, exerc. 2 and 3).

DEFINITION 3. *We say that a vector function $\mathbf{f}$ defined on an interval $\mathrm{I} \subset \mathbf{R}$ is differentiable (resp. right differentiable, left differentiable) on I if it is differentiable (resp. right differentiable, left differentiable) at each point of I; the function $x \mapsto \mathbf{f}'(x)$ (resp. $x \mapsto \mathbf{f}'_d(x)$, $x \mapsto \mathbf{f}'_g(x)$) defined on I, is called the derived function, or (by abuse of language) the derivative (resp. right derivative, left derivative) of $\mathbf{f}$, and is denoted by $\mathbf{f}'$ or $\mathrm{D}\mathbf{f}$ or $d\mathbf{f}/dx$ (resp. $\mathbf{f}'_d, \mathbf{f}'_g$).*

Remark. A function may be differentiable on an interval without its derivative being continuous at every point of the interval (*cf.* I, p. 36, exerc. 5); *this is shown by the

example of the function equal to 0 for $x = 0$ and to $x^2 \sin 1/x$ for $x \neq 0$; it has a derivative everywhere, but this derivative is discontinuous at the point $x = 0$.$_*$

2. LINEARITY OF DIFFERENTIATION

PROPOSITION 1. *The set of vector functions defined on an interval* $I \subset \mathbf{R}$, *taking values in a given topological vector space* E, *and differentiable at the point* x_0, *is a vector space over* $\mathbf{R}$, *and the map* $\mathbf{f} \mapsto D\mathbf{f}(x_0)$ *is a linear mapping of this space into* E.

In other words, if $\mathbf{f}$ and $\mathbf{g}$ are defined on I and differentiable at the point x_0, then $\mathbf{f} + \mathbf{g}$ and $\mathbf{f}a$ (a an arbitrary scalar) are differentiable at x_0 and their derivatives there are $\mathbf{f}'(x_0) + \mathbf{g}'(x_0)$ and $\mathbf{f}'(x_0)a$ respectively. This follows immediately from the continuity of $\mathbf{x} + \mathbf{y}$ and of $\mathbf{x}a$ on $E \times E$ and E respectively.

COROLLARY. *The set of vector functions defined on an interval* I, *taking values in a given topological vector space* E, *and differentiable on* I, *is a vector space over* $\mathbf{R}$, *and the map* $\mathbf{f} \mapsto D\mathbf{f}$ *is a linear mapping of this space into the vector space of mappings from* I *into* E.

Remark. If one endows the vector space of mappings from I into E and its subspace of differentiable mappings (*cf. Gen. Top.*, X, p. 277) with the topology of simple convergence (or the topology of uniform convergence), the linear mapping $\mathbf{f} \mapsto D\mathbf{f}$ *is not continuous* (in general) *for example, the sequence of functions $\mathbf{f}_n(x) = \sin n^2 x/n$ converges uniformly to 0 on $\mathbf{R}$, but the sequence of derivatives $\mathbf{f}'_n(x) = n \cos n^2 x$ does not converge even simply to 0.$_*$

PROPOSITION 2. *Let* E *and* F *be two topological vector spaces over* $\mathbf{R}$, *and* $\mathbf{u}$ *a continuous linear map from* E *into* F. *If* $\mathbf{f}$ *is a vector function defined on an interval* $I \subset \mathbf{R}$, *taking values in* E, *and differentiable at the point* $x_0 \in I$, *then the composite function* $\mathbf{u} \circ \mathbf{f}$ *has a derivative equal to* $\mathbf{u}(\mathbf{f}'(x_0))$ *at* x_0.

Indeed, since $\dfrac{\mathbf{u}(\mathbf{f}(x)) - \mathbf{u}(\mathbf{f}(x_0))}{x - x_0} = \mathbf{u}\left(\dfrac{\mathbf{f}(x) - \mathbf{f}(x_0)}{x - x_0}\right)$, this follows from the continuity of $\mathbf{u}$.

COROLLARY. *If* φ *is a continuous linear form on* E, *then the real function* $\varphi \circ \mathbf{f}$ *has a derivative equal to* $\varphi(\mathbf{f}'(x_0))$ *at the point* x_0.

Examples. 1) Let $\mathbf{f} = (f_i)_{1 \leqslant i \leqslant n}$ be a function with values in $\mathbf{R}^n$, defined on an interval $I \subset \mathbf{R}$; each real function f_i is none other than the composite function $\mathrm{pr}_i \circ \mathbf{f}$, so is differentiable at the point x_0 if $\mathbf{f}$ is, and, if so, $\mathbf{f}'(x_0) = (f'_i(x_0))_{1 \leqslant i \leqslant n}$.

*2) In Kinematics, if $\mathbf{f}(t)$ is the position of a moving point M at time t, if $\mathbf{g}(t)$ is the position at the same instant of the projection M′ of M onto a plane P (resp. a line D) with kernel a line (resp. a plane) not parallel to P (resp. D), then $\mathbf{g}$ is the composition of the projection $\mathbf{u}$ of $\mathbf{R}^3$ onto P (resp. D) and of $\mathbf{f}$; since $\mathbf{u}$ is a (continuous) linear mapping

one sees that the projection of the velocity of a moving point onto a plane (resp. a line) is equal to the velocity of the projection of the moving point onto the plane (resp. line).$_*$

3) Let f be a complex-valued function defined on an interval $I \subset \mathbf{R}$, and let a be an arbitrary complex number; prop. 2 shows that if f is differentiable at a point x_0 then so is af, and the derivative of this function at x_0 is equal to $af'(x_0)$.

3. DERIVATIVE OF A PRODUCT

Let us now consider p topological vector spaces E_i ($1 \leqslant i \leqslant p$) over $\mathbf{R}$, and a continuous multilinear [1] map (which we shall denote by

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) \mapsto [\mathbf{x}_1.\mathbf{x}_2 \ldots \mathbf{x}_p])$$

of $E_1 \times E_2 \times \cdots \times E_p$ into a topological vector space F over $\mathbf{R}$.

PROPOSITION 3 . *For each index i ($1 \leqslant i \leqslant p$) let $\mathbf{f}_i$ be a function defined on an interval $I \subset \mathbf{R}$, taking values in E_i, and differentiable at the point $x_0 \in I$. Then the function*

$$x \mapsto [\mathbf{f}_1(x).\mathbf{f}_2(x) \ldots \mathbf{f}_p(x)]$$

defined on I *with values in* F *has a derivative equal to*

$$\sum_{i=1}^{p} \left[\mathbf{f}_1(x_0) \ldots \mathbf{f}_{i-1}(x_0).\mathbf{f}_i'(x_0).\mathbf{f}_{i+1}(x_0) \ldots \mathbf{f}_p(x_0) \right] \qquad (1)$$

at x_0.

Let us put $\mathbf{h}(x) = [\mathbf{f}_1(x).\mathbf{f}_2(x) \ldots \mathbf{f}_p(x)]$; then, by the identity

$$\left[\mathbf{b}_1.\mathbf{b}_2 \ldots \mathbf{b}_p \right] - \left[\mathbf{a}_1.\mathbf{a}_2 \ldots \mathbf{a}_p \right] = \sum_{i=1}^{p} \left[\mathbf{b}_1 \ldots \mathbf{b}_{i-1}.(\mathbf{b}_i - \mathbf{a}_i).\mathbf{a}_{i+1} \ldots \mathbf{a}_p \right],$$

we can write

$$\mathbf{h}(x) - \mathbf{h}(x_0) = \sum_{i=1}^{p} \left[\mathbf{f}_1(x) \ldots \mathbf{f}_{i-1}(x).(\mathbf{f}_i(x) - \mathbf{f}_i(x_0)).\mathbf{f}_{i+1}(x_0) \ldots \mathbf{f}_p(x_0) \right].$$

On multiplying both sides by $\dfrac{1}{x - x_0}$ and letting x approach x_0 in I, we obtain the expression (1), since both the map

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) \mapsto [\mathbf{x}_1.\mathbf{x}_2 \ldots \mathbf{x}_p]$$

and addition in F are continuous.

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[1] Recall (*Alg.*, II, p. 265) that a map $\mathbf{f}$ of $E_1 \times E_2 \times \cdots \times E_p$ into F is said to be *multilinear* if each partial mapping

$$\mathbf{x}_i \mapsto \mathbf{f}(\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{x}_i, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_p)$$

from E_i into F ($1 \leqslant i \leqslant p$) is a *linear* map, the $\mathbf{a}_j$ for indices $j \neq i$ being arbitrary in E_j. We note that if the E_i are *finite* dimensional over $\mathbf{R}$ then every multilinear map of $E_1 \times E_2 \times \cdots \times E_p$ into F is necessarily *continuous;* this need not be so if some of these spaces are topological vector spaces of infinite dimension.

When some of the functions $\mathbf{f}_i$ are *constant,* the terms in the expression (1) containing their derivatives $\mathbf{f}'_i(x_0)$ are zero.

Let us consider in detail the particular case $p = 2$, the most important in applications: if $(\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}.\mathbf{y}]$ is a *continuous bilinear* map of $E \times F$ into G, $(E, F, G$ being topological vector spaces over $\mathbf{R})$, and $\mathbf{f}$ and $\mathbf{g}$ are two vector functions, differentiable at x_0, with values in E and F respectively, then the vector function $x \mapsto [\mathbf{f}(x).\mathbf{g}(x)]$ (which we denote by $[\mathbf{f}.\mathbf{g}]$) has a derivative equal to $[\mathbf{f}'(x_0).\mathbf{g}(x_0)] + [\mathbf{f}(x_0).\mathbf{g}'(x_0)]$ at x_0. In particular, if $\mathbf{a}$ is a constant vector, then $[\mathbf{a}.\mathbf{f}]$ (resp. $[\mathbf{f}.\mathbf{a}]$) has a derivative equal to $[\mathbf{a}.\mathbf{f}'(x_0)]$ (resp. $[\mathbf{f}'(x_0).\mathbf{a}]$) at x_0.

If $\mathbf{f}$ and $\mathbf{g}$ are both differentiable on I then so is $[\mathbf{f}.\mathbf{g}]$, and we have

$$[\mathbf{f}.\mathbf{g}]' = [\mathbf{f}'.\mathbf{g}] + [\mathbf{f}.\mathbf{g}']. \tag{2}$$

Examples.   1) Let f be a real function, $\mathbf{g}$ a vector function, both differentiable at a point x_0; the function $\mathbf{g}f$ has a derivative equal to $\mathbf{g}'(x_0)f(x_0) + \mathbf{g}(x_0)f'(x_0)$ at x_0. In particular, if $\mathbf{a}$ is constant, then $\mathbf{a}f$ has derivative $\mathbf{a}f'(x_0)$. This last remark, in conjunction with example 1 of I, p. 5, proves that if $\mathbf{f} = (f_i)_{1 \leqslant i \leqslant n}$ is a vector function with values in $\mathbf{R}^n$, then for $\mathbf{f}$ to be differentiable at the point x_0 it is necessary and sufficient that each of the real functions f_i $(1 \leqslant i \leqslant n)$ be differentiable there: for, if $(\mathbf{e}_i)_{1 \leqslant i \leqslant n}$ is the canonical basis of $\mathbf{R}^n$, we can write $\mathbf{f} = \sum\limits_{i=1}^{n} \mathbf{e}_i f_i$.

2) The real function x^n arises from the multilinear function

$$(x_1, x_2, \ldots, x_n) \mapsto x_1 x_2 \ldots x_n$$

defined on $\mathbf{R}^n$, by substituting x for each of the x_i; so prop. 3 shows that x^n is differentiable on $\mathbf{R}$ and has derivative nx^{n-1}. As a result the polynomial function $\mathbf{a}_0 x^n + \mathbf{a}_1 x^{n-1} + \cdots + \mathbf{a}_{n-1}x + \mathbf{a}_n$ (the $\mathbf{a}_i$ being constant vectors) has derivative

$$n\mathbf{a}_0 x^{n-1} + (n-1)\mathbf{a}_1 x^{n-2} + \cdots + \mathbf{a}_{n-1};$$

when the $\mathbf{a}_i$ are real numbers this function coincides with the derivative of a polynomial function as defined in Algebra (A, IV).

3) The euclidean *scalar product* $(\mathbf{x} \mid \mathbf{y})$ (*Gen. Top.*, VI, p. 40) is a bilinear map (necessarily continuous) of $\mathbf{R}^n \times \mathbf{R}^n$ into $\mathbf{R}$. If $\mathbf{f}$ and $\mathbf{g}$ are two vector functions with values in $\mathbf{R}^n$, and differentiable at the point x_0, then the real function $x \mapsto (\mathbf{f}(x) \mid \mathbf{g}(x))$ has a derivative equal to $(\mathbf{f}'(x_0) \mid \mathbf{g}(x_0)) + (\mathbf{f}(x_0) \mid \mathbf{g}'(x_0))$ at the point x_0. There is an analogous result for the hermitian scalar product on $\mathbf{C}^n$, this space being considered as a vector space *over* $\mathbf{R}$.

Let us consider in particular the case where the euclidean norm $\|\mathbf{f}(x)\|$ is *constant,* so that $(\mathbf{f}(x) \mid \mathbf{f}(x)) = \|\mathbf{f}(x)\|^2$ is also constant; on writing that the derivative of $(\mathbf{f}(x) \mid \mathbf{f}(x))$ vanishes at x_0 we obtain $(\mathbf{f}(x_0) \mid \mathbf{f}'(x_0)) = 0$; in other words, $\mathbf{f}'(x_0)$ is *orthogonal* to $\mathbf{f}(x_0)$.

4) If E is a *topological algebra* over $\mathbf{R}$ (*cf.* Introduction), the product $\mathbf{xy}$ of two elements of E is a continuous bilinear function of $(\mathbf{x}, \mathbf{y})$; if $\mathbf{f}$ and $\mathbf{g}$ have their values in E and are differentiable at the point x_0, then the function $x \mapsto \mathbf{f}(x)\mathbf{g}(x)$ has a derivative equal to $\mathbf{f}'(x_0)\mathbf{g}(x_0) + \mathbf{f}(x_0)\mathbf{g}'(x_0)$ at x_0. In particular, if $U(x) = (\alpha_{ij}(x))$ and $V(x) = (\beta_{ij}(x))$ are two *square matrices* of order n, differentiable at x_0, their product UV has a derivative equal to $U'(x_0)V(x_0) + U(x_0)V'(x_0)$ at x_0 (where $U'(x) = (\alpha'_{ij}(x))$ and $V'(x) = (\beta'_{ij}(x))$).

5) The *determinant* $\det(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ of n vectors $\mathbf{x}_i = (x_{ij})_{1 \leqslant j \leqslant n}$ from the space $\mathbf{R}^n$ (*Alg.*, III, p. 522) being a (continuous) multilinear function of the $\mathbf{x}_i$, one sees that if the

n^2 real functions f_{ij} are differentiable at x_0, then their determinant $g(x) = \det(f_{ij}(x))$ has a derivative equal to

$$\sum_{i=1}^{n} \left[\mathbf{f}_1(x_0), \dots, \mathbf{f}_{i-1}(x_0), \mathbf{f}_i'(x_0), \mathbf{f}_{i+1}(x_0), \dots, \mathbf{f}_n(x_0) \right]$$

at x_0, where $\mathbf{f}_i(x) = (f_{ij}(x))_{1 \leqslant j \leqslant n}$; in other words, one obtains the derivative of a determinant of order n by taking the sum of the n determinants formed by replacing, for each i, the terms of the ith column by their derivatives.

Remark. If $U(x)$ is a square matrix which is differentiable and invertible at the point x_0, then the derivative of its determinant $\Delta(x) = \det(U(x))$ can be expressed through the derivative of $U(x)$ by the formula

$$\Delta'(x_0) = \Delta(x_0).\mathrm{Tr}(U'(x_0)U^{-1}(x_0)). \tag{3}$$

Indeed, let us put $U(x_0 + h) = U(x_0) + hV$; then, by definition, V tends to $U'(x_0)$ when h tends to 0. One can write

$$\Delta(x_0 + h) = \Delta(x_0).\det(I + hVU^{-1}(x_0)).$$

Now $\det(I + hX) = 1 + h\mathrm{Tr}(X) + \sum_{k=2}^{n} \lambda_k h^k$, the λ_k ($k \geqslant 2$) being polynomials in the elements of the matrix X; since the elements of $VU^{-1}(x_0)$ have a limit when h tends to 0, we indeed obtain the formula (3).

4. DERIVATIVE OF THE INVERSE OF A FUNCTION

PROPOSITION 4. *Let* E *be a complete normed algebra with a unit element over* **R** *and let* **f** *be a function defined on an interval* $I \subset \mathbf{R}$, *taking values in* E, *and differentiable at the point* $x_0 \in I$. *If* $\mathbf{y}_0 = \mathbf{f}(x_0)$ *is invertible* [2] *in* E, *then the function* $x \mapsto (\mathbf{f}(x))^{-1}$ *is defined on a neighbourhood of* x_0 *(relative to* I), *and has a derivative equal to* $-\left(\mathbf{f}(x_0)\right)^{-1} \mathbf{f}'(x_0) \left(\mathbf{f}(x_0)\right)^{-1}$ *at* x_0.

Indeed, the set of invertible elements in E is an open set on which the function $\mathbf{y} \mapsto \mathbf{y}^{-1}$ is continuous (*Gen. Top.*, IX, p. 178); since $\mathbf{f}$ is continuous (relative to I) at x_0, $\left(\mathbf{f}(x)\right)^{-1}$ is defined on a neighbourhood of x_0, and we have

$$\left(\mathbf{f}(x)\right)^{-1} - \left(\mathbf{f}(x_0)\right)^{-1} = \left(\mathbf{f}(x)\right)^{-1} \left(\mathbf{f}(x_0) - \mathbf{f}(x)\right) \left(\mathbf{f}(x_0)\right)^{-1}.$$

The proposition thus follows from the continuity of $\mathbf{y}^{-1}$ on a neighbourhood of $\mathbf{y}_0$ and the continuity of $\mathbf{xy}$ on $E \times E$.

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[2] Recall from (*Alg.*, I, p. 15) that an element $\mathbf{z} \in E$ is said to be *invertible* if there exists an element of E, denoted by $\mathbf{z}^{-1}$, such that $\mathbf{zz}^{-1} = \mathbf{z}^{-1}\mathbf{z} = \mathbf{e}$ (**e** being the unit element of E).

Examples. 1) The most important particular case is that where E is one of the fields **R** or **C** : if f is a function with real or complex values, differentiable at the point x_0, and such that $f(x_0) \neq 0$, then $1/f$ has derivative equal to $-f'(x_0)/(f(x_0))^2$ at x_0.

2) If $U = (\alpha_{ij}(x))$ is a square matrix of order n, differentiable at x_0 and invertible at this point, then U^{-1} has derivative equal to $-U^{-1}U'U^{-1}$ at x_0.

5. DERIVATIVE OF A COMPOSITE FUNCTION

PROPOSITION 5. *Let f be a real function defined on an interval* $\mathrm{I} \subset \mathbf{R}$, *and* **g** *a vector function defined on an interval of* **R** *containing* $f(\mathrm{I})$. *If f is differentiable at the point x_0 and* **g** *is differentiable at the point $f(x_0)$ then the composite function* **g** $\circ$ f *has a derivative equal to* $\mathbf{g}'(f(x_0))f'(x_0)$ *at x_0.*

Let us put $\mathbf{h} = \mathbf{g} \circ f$; for $x \neq x_0$ we can write

$$\frac{\mathbf{h}(x) - \mathbf{h}(x_0)}{x - x_0} = \mathbf{u}(x) \frac{f(x) - f(x_0)}{x - x_0}$$

where we set $\mathbf{u}(x) = \dfrac{\mathbf{g}(f(x)) - \mathbf{g}(f(x_0))}{f(x) - f(x_0)}$ if $f(x) \neq f(x_0)$, and $\mathbf{u}(x) = \mathbf{g}'(f(x_0))$ otherwise. Now $f(x)$ has limit $f(x_0)$ when x tends to x_0, so $\mathbf{u}(x)$ has limit $\mathbf{g}'(f(x_0))$, from which the proposition follows in view of the continuity of the function $\mathbf{y}x$ on E × **R**.

6. DERIVATIVE OF AN INVERSE FUNCTION

PROPOSITION 6. *Let f be a homeomorphism of an interval* $\mathrm{I} \subset \mathbf{R}$ *onto an interval* $\mathrm{J} = f(\mathrm{I}) \subset \mathbf{R}$, *and let g be the inverse homeomorphism*[3]. *If f is differentiable at the point $x_0 \in \mathrm{I}$, and if $f'(x_0) \neq 0$, then g has a derivative equal to $1/f'(x_0)$ at* $y_0 = f(x_0)$.

For each $y \in \mathrm{J}$ we have $g(y) \in \mathrm{I}$ and $u = f(g(y))$; we thus can write $\dfrac{g(y) - g(y_0)}{y - y_0} = \dfrac{g(y) - x_0}{f(g(y)) - f(x_0)}$, for $y \neq y_0$. When y tends to y_0 while remaining in J and $\neq y_0$, then $g(y)$ tends to x_0 remaining in I and $\neq x_0$, and the right-hand side in the preceding formula thus has limit $1/f'(x_0)$, since by hypothesis $f'(x_0) \neq 0$.

COROLLARY. *If f is differentiable on* I *and if $f'(x) \neq 0$ on* I, *then g is differentiable on* J *and its derivative at each point $y \in \mathrm{J}$ is $1/f'(g(y))$.*

For example, for each integer $n > 0$, the function $x^{1/n}$ is a homeomorphism of $\mathbf{R}_+$ onto itself, is the inverse of x^n, and has derivative $\frac{1}{n}x^{\frac{1}{n}-1}$ at each $x > 0$.

One deduces easily, from prop. 5, that for every rational number $r = p/q > 0$ the function $x^r = \left(x^{1/q}\right)^p$ has derivative rx^{r-1} at every $x > 0$.

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[3] For f to be a homeomorphism of I onto a subset of **R** we know that it is necessary and sufficient that f be continuous and strictly monotone on I (*Gen. Top.*, IV, p. 338, th. 5).

Remarks. 1) All the preceding propositions, stated for functions differentiable at a point x_0, immediately yield propositions for functions which are right (resp. left) differentiable at x_0, when, instead of the functions themselves, one considers their restrictions to the intersection of their intervals of definition with the interval $[x_0, +\infty[$ (resp. $]-\infty, x_0]$); we leave it to the reader to state them.

2) The preceding definitions and propositions (except for those concerning right and left derivatives) extend easily to the case where one replaces **R** by an arbitrary *commutative non-discrete topological field* K, and the topological vector spaces (resp. topological algebras) over **R** by topological vector spaces (resp. topological algebras) over K. In def. 1 and props. 1, 2 and 3 it is enough to replace I by a *neighbourhood* of x_0 in K; in prop. 4 one must assume further that the map $\mathbf{y} \mapsto \mathbf{y}^{-1}$ is defined and continuous on a neighbourhood of $\mathbf{f}(x_0)$ in E. Prop. 5 generalizes in the following manner: let K′ be a non-discrete subfield of the topological field K, let E be a topological vector space *over* K; let f be a function defined on a neighbourhood $V \subset K'$ of $x_0 \in K'$, with values in K (considered as a topological vector space over K′), differentiable at x_0, and let $\mathbf{g}$ be a function defined on a neighbourhood of $f(x_0) \in K$, with values in E, and differentiable at the point $f(x_0)$; then the map $\mathbf{g} \circ f$ is differentiable at x_0 and has derivative $\mathbf{g}'(f(x_0))f'(x_0)$ there (E being then considered as a topological vector space *over* K′).

With the same notation, let **f** be a function defined on a neighbourhood V of $a \in K$, with values in E, and differentiable at the point a; if $a \in K'$, then the *restriction* of **f** to $V \cap K'$ is differentiable at a, and has derivative $\mathbf{f}'(a)$ there. These considerations apply above all, in practice, to the case where $K = \mathbf{C}$ and $K' = \mathbf{R}$.

Finally, prop. 6 extends to the case where one replaces I by a neighbourhood of $x_0 \in K$, and f by a homeomorphism of I onto a neighbourhood $J = f(I)$ of $y_0 = f(x_0)$ in K.

7. DERIVATIVES OF REAL-VALUED FUNCTIONS

The preceding definitions and propositions may be augmented in several respects when we deal with *real-valued* functions of a real variable.

In the first place, if f is such a function, defined on an interval $I \subset \mathbf{R}$, and continuous relative to I at a point $x_0 \in I$, it can happen that when x tends to x_0 while remaining in I and $\neq x_0$, that $\dfrac{f(x) - f(x_0)}{x - x_0}$ has a limit equal to $+\infty$ or to $-\infty$; one then says that f is differentiable at x_0 and has derivative $+\infty$ (resp. $-\infty$) there; if the function f has a derivative $f'(x)$ (finite or infinite) at every point x of I, then the function f' (with values in $\overline{\mathbf{R}}$) is again called the derived function (or simply the derivative) of f. One generalizes the definitions of right and left derivative similarly.

Example. At the point $x = 0$ the function $x^{1/3}$ (the inverse function of x^3, a homeomorphism of **R** onto itself) has a derivative, equal to $+\infty$; at $x = 0$ the function $|x|^{1/3}$ has right derivative $+\infty$ and left derivative $-\infty$.

The formulae for the derivative of a sum, of a product of differentiable real functions, and for the inverse of a differentiable function (props. 1, 3 and 4), as well as for the derivative of a (real-valued) composition of functions (prop. 5) remain valid when the derivatives that occur are infinite, so long as all the expressions that occur in these formulae make sense (*Gen. Top.*, IV, p. 345–346). In fact, if in prop. 6 one supposes that f is strictly increasing (resp. strictly decreasing) and continuous on I, and if $f'(x_0) = 0$, then the inverse function g has a derivative equal to $+\infty$

(resp. $-\infty$) at the point $y_0 = f(x_0)$; if $f'(x_0) = +\infty$ (resp. $-\infty$), then g has derivative 0. There are similar results for right and left derivatives, which we leave to the care of the reader.

Let C be the *graph* or *representing curve* of a finite real function f, the subset of the plane $\mathbf{R}^2$ formed by the points $(x, f(x))$ where x runs through the set where f is defined. If the function f has a finite right derivative at a point $x_0 \in$ I, then the half-line with origin at the point $M_{x_0} = (x_0, f(x_0))$ of C, and direction numbers $(1, f'_d(x_0))$ is called the *right half-tangent* to the curve C at the point M_{x_0}; when $f'_d(x_0) = +\infty$ (resp. $f'_d(x_0) = -\infty$) we use the same terminology for the half-line with origin M_{x_0} and direction numbers $(0, 1)$ (resp. $(0, -1)$). In the same way one defines the *left half-tangent* at M_{x_0} when $f'_g(x_0)$ exists. With these definitions one can verify quickly that the angle which the right (resp. left) half-tangent makes with the abscissa is the *limit* of the angle made by this axis with the half-line originating at M_{x_0} and passing through the point $M_x = (x, f(x))$ of C, as x tends to x_0 while remaining $> x_0$ (resp. $< x_0$).

One can also say that the right (resp. left) half-tangent is the *limit* of the half-line originating at M_{x_0} passing through M_x, on endowing the set of half-lines having the same origin with the quotient space topology $C^*/\mathbf{R}_+^*$ (*Gen. Top.*, VIII, p. 107).

If the two half-tangents exist at a point M_{x_0} of C, they are in opposite directions only when f has a *derivative* (finite or not) at the point x_0 (assumed interior to I); they are identical only when $f'_d(x_0)$ and $f'_g(x_0)$ are infinite and of opposite sign. In these two cases we say that the line containing these two half-tangents is the *tangent* to C at the point M_{x_0}.

When the tangent at M_{x_0} exists it is the *limit* of the line passing through M_{x_0} and M_x as x tends to x_0 remaining $\neq x_0$, the topology on the set of lines which pass through a given fixed point being that of the quotient space $C^*/\mathbf{R}^*$ (*Gen. Top.*, VIII, p. 114).

The concepts of tangent and half-tangent to a graph are particular cases of general concepts which will be defined in the part of this Series devoted to differentiable varieties.

DEFINITION 4. *We say that a real function f, defined on a subset* A *of a topological space* E, *has a relative maximum* (resp. *strict relative maximum, relative minimum, strict relative minimum*) *at a point $x_0 \in$ A, relative to* A, *if there is a neighbourhood* V *of x_0 in* E *such that at every point $x \in$ V∩A distinct from x_0 one has $f(x) \leqslant f(x_0)$* (resp. $f(x) < f(x_0)$, $f(x) \geqslant f(x_0)$, $f(x) > f(x_0)$).

It is clear that if f attains its least upper bound (resp. greatest lower bound) over A at a point of A, then it has a relative maximum (resp. relative minimum) relative to A at this point; the converse is of course incorrect.

Note that if B $\subset$ A, and if f admits (for example) a relative maximum at a point $x_0 \in$ B *relative to* B, then f does not necessarily have a relative maximum *relative to* A at this point.

PROPOSITION 7. *Let f be a finite real function, defined on an interval* I $\subset \mathbf{R}$. *If f admits a relative maximum* (resp. *relative minimum*) *at a point x_0 interior to* I,

and has both right and left derivatives at this point, then one has $f'_d(x_0) \leqslant 0$ *and* $f'_g(x_0) \geqslant 0$ (resp. $f'_d(x_0) \geqslant 0$ *and* $f'_g(x_0) \leqslant 0$); *in particular, if* f *is differentiable at the point* x_0, *then* $f'(x_0) = 0$.

The proposition follows trivially from the definitions.

We can say further that if at a point x_0 interior to I the function f is differentiable and admits a relative maximum or minimum, then the tangent to its graph is *parallel to the abscissa.* The converse is incorrect, as is shown by the example of the function x^3 which has zero derivative at the point $x = 0$, but has neither relative maximum nor minimum at this point.

§ 2. THE MEAN VALUE THEOREM

The hypotheses and conclusions demonstrated in § 1 are *local* in character: they concern the properties of the functions under consideration only on an *arbitrarily small* neighbourhood of a fixed point. In contrast, the questions which we treat in this section involve the properties of a function on *all of an interval.*

1. ROLLE'S THEOREM

PROPOSITION 1 ("Rolle's theorem"). *Let* f *be a real function which is finite and continuous on a closed interval* I $= [a,\ b]$ *(where* $a < b$), *has a derivative* (finite or not) *at every point of* $]a,\ b[$, *and is such that* $f(a) = f(b)$. *Then there exists a point* c *of* $]a,\ b[$ *such that* $f'(c) = 0$.

The proposition is evident if f is constant: if not, f takes, for example, values $> f(a)$, and so attains its least upper bound at a point c *interior* to I (*Gen. Top.*, IV, p. 359, th. 1). Since f has a relative maximum at this point we have $f'(c) = 0$ (I, p. 20, prop. 7).

COROLLARY. *Let* f *be a real function which is finite and continuous on* $[a,\ b]$ *(where* $a < b$), *and has a derivative* (finite or not) *at every point. Then there exists a point* c *of* $]a,\ b[$ *such that* $f(b) - f(a) = f'(c)(b - a)$.

We need only apply prop. 1 to the function $f(x) - \dfrac{f(b) - f(a)}{b - a}(x - a)$.

This corollary signifies that there is a point $M_c = (c, f(c))$ on the graph C of f such that $a < c < b$ and such that the tangent to C at this point is *parallel* to the line joining the points $M_a = (a, f(a))$ and $M_b = (b, f(b))$.

2. THE MEAN VALUE THEOREM FOR REAL-VALUED FUNCTIONS

The following important result is a consequence of the corollary to prop. 1: if one has $m \leqslant f'(x) \leqslant M$ on $]a,\ b[$, then also $m \leqslant \dfrac{f(b) - f(a)}{b - a} \leqslant M$. In other words, a *bound for the derivative of* f' on the whole interval with endpoints a, b implies the *same bound* for $\dfrac{f(b) - f(a)}{b - a}$ (the ratio of the "increment" of the function to the "increment" of the variable on the interval). We shall make this fundamental result more precise, and generalize it, in the sequel.

PROPOSITION 2. *Let* f *be a real function which is finite and continuous on the closed bounded interval* $I = [a,\ b]$ *(where* $a < b$*) and has a right derivative* (finite or not) *at all the points of the relative complement in* $[a,\ b)$ *of a* countable *subset* A *of this interval. If* $f_d'(x) \geqslant 0$ *at every point of* $[a,\ b[$ *not belonging to* A, *then one has* $f(b) \geqslant f(a)$*; if, further,* $f_d'(x) > 0$ *for at least one point of* $[a,\ b[$*, then* $f(b) > f(a)$.

Let $\varepsilon > 0$ be arbitrary, and denote by $(a_n)_{n \geqslant 1}$ a sequence obtained by listing the countable set A. Let J be the set of points $y \in I$ such that one has

$$f(x) - f(a) \geqslant -\varepsilon(x - a) - \varepsilon \sum_{a_n < x} \frac{1}{2^n} \tag{1}$$

for all x with $a \leqslant x \leqslant y$, the sum in the second term of the right-hand side being taken over all indices n for which $a_n < x$. We shall show that if $f_d'(x) \geqslant 0$ at every point of $[a,\ b[$ distinct from the a_n, then J = I.

It is clear that J is not empty, since $a \in$ J; moreover the definition of this set shows that if $y \in$ J one has $x \in$ J for $a \leqslant x \leqslant y$, so J is an *interval* with left-hand endpoint a (*Gen. Top.*, IV, p. 336, prop. 1); let c be its right-hand endpoint. One has $c \in$ J; this is clear if $c = a$; if not, for every $x < c$ we have the inequality (1), and *a fortiori*

$$f(x) - f(a) \geqslant -\varepsilon(c - a) - \varepsilon \sum_{a_n < c} \frac{1}{2^n}$$

from which it follows, on letting x tend to c in this inequality (since f is continuous), that c satisfies (1).

This being so, we shall see that we must have $c = b$. Indeed, if one had $c < b$, then certainly one would have $c \notin$ A; now $f_d'(c)$ exists, and since $f_d'(c) \geqslant 0$ by hypothesis, there exists a y such that $c < y \leqslant b$ and such that for $c \leqslant x \leqslant y$ one has

$$f(x) - f(c) \geqslant -\varepsilon(x - c)$$

from which, taking account of (1), where x is replaced by c,

$$f(x) - f(a) \geqslant -\varepsilon(x - a) - \varepsilon \sum_{a_n < c} \frac{1}{2^n} \geqslant -\varepsilon(x - a) - \varepsilon \sum_{a_n < x} \frac{1}{2^n}$$

which signifies that $y \in \mathrm{J}$, contradicting the definition of c. Thus we have $c = a_k$ for some index k; since f is continuous at the point a_k there is a y such that $c < y \leqslant b$ and such that for $c < x \leqslant y$ one has

$$f(x) - f(c) \geqslant -\frac{\varepsilon}{2^k}$$

from which, taking account of (1), where x is replaced by c,

$$f(x) - f(a) \geqslant -\varepsilon(c - a) - \varepsilon \sum_{a_n < x} \frac{1}{2^n} \geqslant -\varepsilon(x - a) - \varepsilon \sum_{a_n < x} \frac{1}{2^n}$$

which again leads to a contradiction; we thus have $c = b$, and in consequence

$$f(b) - f(a) \geqslant -\varepsilon(b - a) - \varepsilon \sum_{a_n < b} \frac{1}{2^n} \geqslant -\varepsilon(b - a) - \varepsilon. \tag{2}$$

Since $\varepsilon > 0$ is arbitrary we deduce from (2) that $f(b) \geqslant f(a)$, which demonstrates the first part of the proposition.

We remark now that this result applied to an interval $[x, \ y]$ where $a \leqslant x < y \leqslant b$ proves that f is *increasing* on I; if one had $f(b) = f(a)$ one could deduce that f is *constant* on I, and then that $f_d'(x) = 0$ at every point of $[a, \ b[$; the second part follows from this.

COROLLARY. *Let f be a finite continuous real function on $[a, \ b]$ (where $a < b$) and have a right derivative at all points of the complement in $[a, \ b[$ of a countable subset* A *of this interval. For f to be increasing on* I *it is necessary and sufficient that $f_d'(x) \geqslant 0$ at every point of $[a, \ b[$ that does not belong to* A; *for f to be strictly increasing it is necessary and sufficient that that the preceding condition holds, and further that the set of points x where $f_d'(x) > 0$ be dense in $[a, \ b]$.*

Remarks. 1) Prop. 2 remains true when one replaces the interval $[a, \ b[$ by $]a, \ b]$ and the words "right derivative" by "left derivative".

2) The hypothesis of *continuity* on f on the closed interval I (and not just *right continuity* [4] at every point of $[a, \ b[$) is essential for the validity of prop. 2 (*cf.* I, p. 36, exerc. 8).

3) The conclusion of prop. 2 is not guaranteed if one merely supposes that the set A of "exceptional" points is nowhere dense in I, but not countable (*cf.* I, p. 37, exerc. 3).

Prop. 2 entails the following fundamental theorem (which appears to be more general):

THEOREM 1 (mean value theorem). *Let f and g be two finite continuous real-valued functions defined on a closed bounded interval* $\mathrm{I} = [a, \ b]$ *and having a*

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[4] A function defined on an interval $\mathrm{I} \subset \mathbf{R}$ is said to be *right continuous* at a point $x_0 \in \mathrm{I}$ if its restriction to the interval $\mathrm{I} \cap [x_0, +\infty[$ is continuous at the point x_0 relative to this interval; it comes to the same to say that the right limit of this function exists at this point and is equal to the value of the function at this point.

right derivative (finite or not) *at all points of the relative complement in* $[a, b[$ *of a countable subset of this interval. Suppose further that* $f'_d(x)$ *and* $g'_r(x)$ *are not simultaneously infinite except at the points of a countable subset of* I *and that there are finite numbers* m, M *such that*

$$mg'_r(x) \leqslant f'_d(x) \leqslant Mg'_r(x) \tag{3}$$

except at the points of a countable subset of I *(replacing* $Mg'_r(x)$ *(resp.* $mg'_r(x)$*) by 0 if* M $= 0$ *(resp.* $m = 0$*) and* $g'_r(x) = \pm\infty$*). Under these conditions one has*

$$m(g(b) - g(a)) < f(b) - f(a) < M(g(b) - f(a)) \tag{4}$$

except when one has $f(x) = Mg(x) + k$, *or* $f(x) = mg(x) + k$ *(k constant) for all* $x \in$ I.

It suffices to apply prop. 2 to the functions $Mg - f$ and $f - mg$, which, under our hypotheses, have a positive right derivative except at the points of a countable subset of I.

Remark.   Th. 1 fails if one allows f'_d and g'_r to be simultaneously infinite on an uncountable subset of I (*cf.* I, p. 37, exerc. 3).

COROLLARY. *Let* f *be a finite continuous function on* $[a, b]$ *(where* $a < b$*) and have a right derivative* (finite or not) *at all points of the relative complement* B *in* $[a, b[$ *of a countable subset of this interval. If* m *and* M *are the greatest lower and least upper bounds of* f'_d *on* B *then one has*

$$m(b - a) < f(b) - f(a) < M(b - a) \tag{5}$$

if f *is not an affine linear function; if* f *is affine linear one has*

$$m = M = \frac{f(b) - f(a)}{b - a}.$$

The inequalities (5) are consequences of (4) when m and M are finite; the case when one or the other of these numbers is infinite is trivial.

Remark.   The inequalities (5) prove that a continuous function cannot have right derivative equal to $+\infty$ at all points of an interval (*cf.* I, p. 38, exerc. 6).

3. THE MEAN VALUE THEOREM FOR VECTOR FUNCTIONS

THEOREM 2.   *Let* **f** *be a vector function defined and continuous on a closed bounded interval* I $= [a, b]$ *of* **R** *(where* $a < b$*) and taking values in a normed space* E *over* **R**; *let* g *be a continuous increasing real function on* I. *Suppose that* **f** *and* g *have right derivatives at all points of the relative complement in* $[a, b[$ *of a countable*

subset A *of this interval* (allowing $g_r'(x)$ to be infinite at some of the points $x \notin$ A), *and suppose that at each of these points we have*

$$\left\| \mathbf{f}_d'(x) \right\| \leqslant g_r'(x). \tag{6}$$

Under these hypotheses one has

$$\|\mathbf{f}(b) - \mathbf{f}(a)\| \leqslant g(b) - g(a). \tag{7}$$

The proof proceeds similarly to that of prop. 2. Let $\varepsilon > 0$ be arbitrary, and (a_n) the sequence obtained by enumerating A in some order. Let J be the set of points $y \in$ I such that, for all x such that $a \leqslant x \leqslant y$ one has

$$\|\mathbf{f}(x) - \mathbf{f}(a)\| \leqslant g(x) - g(a) + \varepsilon(x - a) + \varepsilon \sum_{a_n < x} \frac{1}{2^n}; \tag{8}$$

we shall show that J $=$ I. One sees immediately, as in prop. 2, that J is an interval with left-hand endpoint a; if c is its right-hand endpoint then $c \in$ J; indeed, for all $x < c$ one has (8), and *a fortiori*

$$\|\mathbf{f}(x) - \mathbf{f}(a)\| \leqslant g(c) - g(a) + \varepsilon(c - a) + \varepsilon \sum_{a_n < c} \frac{1}{2^n}$$

from which, letting x tend to c in this inequality, it follows from the continuity of $\mathbf{f}$ that c satisfies (8).

Let us show that we must have $c = b$. So suppose that $c < b$ and that moreover $c \notin$ A : then $\mathbf{f}_d'(c)$ and $g_r'(c)$ exist and satisfy (6); suppose in the first place that $g_r'(c)$ (which is necessarily $\geqslant 0$) is finite; then one can always write $\mathbf{f}_d'(c) = \mathbf{u} g_r'(c)$, with $\|\mathbf{u}\| \leqslant 1$; since the function $\mathbf{f}(x) - \mathbf{u} g(x)$ has zero right derivative at the point c there must exist a y such that $c < y \leqslant b$ and such that for $c \leqslant x \leqslant y$ one has

$$\|\mathbf{f}(x) - \mathbf{f}(c) - \mathbf{u}(g(x) - g(c))\| \leqslant \varepsilon(x - c)$$

from which

$$\|\mathbf{f}(x) - \mathbf{f}(c)\| \leqslant g(x) - g(c) + \varepsilon(x - c)$$

and, taking account of (8), in which x is replaced by c,

$$\|\mathbf{f}(x) - \mathbf{f}(a)\| \leqslant g(x) - g(a) + \varepsilon(x - a) + \varepsilon \sum_{a_n < c} \frac{1}{2^n}$$

$$\leqslant g(x) - g(a) + \varepsilon(x - a) + \varepsilon \sum_{a_n < x} \frac{1}{2^n}.$$

Thus one has $y \in$ J, which is a contradiction. Suppose next that $c \notin$ A and that $g_r'(c) = +\infty$; then there is a y such that $c < y \leqslant b$ and such that for $c \leqslant x \leqslant y$ one has on the one hand

$$\|\mathbf{f}(x) - \mathbf{f}(c)\| \leqslant \left(\left\|\mathbf{f}_d'(c)\right\| + 1\right)(x - c)$$

while on the other hand

$$g(x) - g(c) \geqslant \left(\left\|\mathbf{f}_d'(c)\right\| + 1\right)(x - c)$$

from which

$$\|\mathbf{f}(x) - \mathbf{f}(c)\| \leqslant g(x) - g(c)$$

and one concludes as above. Finally, if one has $c = a_k$, then there is a y such that $c < y \leqslant b$, and such that for $c < x \leqslant y$ one has

$$\|\mathbf{f}(x) - \mathbf{f}(c)\| \leqslant \frac{\varepsilon}{2^k}$$

from which, taking account of (8), with x replaced by c,

$$\|\mathbf{f}(x) - \mathbf{f}(a)\| \leqslant g(c) - g(a) + \varepsilon(c - a) + \varepsilon \sum_{a_n < x} \frac{1}{2^n}$$

$$\leqslant g(x) - g(a) + \varepsilon(x - a) + \varepsilon \sum_{a_n < x} \frac{1}{2^n}$$

which again entails a contradiction. The proof finishes as that of prop. 2.

<div align="right">Q.E.D.</div>

Remarks. 1) Here again, in the statement of th. 2 one can replace the interval $[a, b[$ by $]a, b]$ and "right derivative" by "left derivative".
   2) We shall show later how to identify the case of equality in (7), and also how to generalize th. 2 to the case where E is an arbitrary locally convex space, with the help of another method of proof which allows one to deduce th. 2 from th. 1.

COROLLARY. *For a continuous vector function on an interval* $\mathrm{I} \subset \mathbf{R}$, *with values in a normed space* E *over* $\mathbf{R}$, *to be constant on* I *it suffices that it have zero right derivative at all points of the complement* (relative to I) *of a countable subset of* I.

Remark.   The proofs of ths. 1 and 2 rely in an essential manner on the special topological properties of the field $\mathbf{R}$; one can give examples of valued fields K for which there are nonconstant linear maps of K to itself with zero derivative at every point (*cf.* I, p. 37, exerc. 2).

PROPOSITION 3. *Let* $\mathbf{f}$ *be a vector function with values in a normed space* E *over* $\mathbf{R}$, *defined and continuous on an interval* $\mathrm{I} \subset \mathbf{R}$, *and right differentiable on the complement* B (relative to I) *of a countable subset of* I; *then for all points* $x_0 \in \mathrm{B}, x \in \mathrm{I}, y \in \mathrm{I}$, *one has* (supposing that $x < y$, for example)

$$\left\|\mathbf{f}(y) - \mathbf{f}(x) - \mathbf{f}_d'(x_0)(y - x)\right\| \leqslant (y - x) \sup_{z \in \mathrm{B},\, x < z < y} \left\|\mathbf{f}_d'(z) - \mathbf{f}_d'(x_0)\right\|. \qquad (9)$$

Indeed it suffices to apply th. 2, replacing $\mathbf{f}$ by the function

$$\mathbf{f}(z) - \mathbf{f}_d'(x_0)z,$$

and g by the linear function whose derivative is $\displaystyle \sup_{z \in \mathrm{B},\, x < z < y} \left\|\mathbf{f}_d'(z) - \mathbf{f}_d'(x_0)\right\|$.

Theorem 2 extends to vector functions of a *complex* variable:

PROPOSITION 4. *Let* **f** *be a continuous differentiable function of a complex variable defined on a convex open subset* A *of the field* **C**, *with values in a normed space* E *over the field* **C**. *If one has* $\left\|\mathbf{f}'(z)\right\| \leqslant m$ *for all* $z \in$ A, *then one has* $\|\mathbf{f}(b) - \mathbf{f}(a)\| \leqslant m\,|b - a|$ *for every pair of points* a, b *of* A.

We put $\mathbf{g}(t) = \dfrac{1}{b - a}\, \mathbf{f}(a + t(b - a))$ for $0 \leqslant t \leqslant 1$; since $\mathbf{g}'(t) = \mathbf{f}'(a + t(b - a))$, applying th. 2 to the function **g** yields the proposition immediately.

COROLLARY. *For a vector function* **f** *of a complex variable, defined and continuous on an open set* A $\subset$ **C**, *and with values in a normed space over* **C**, *to be constant, it suffices that it have zero derivative at every point of* A.

Indeed, let a be an arbitrary point of A; the set B of points z of A where $\mathbf{f}(z) = \mathbf{f}(a)$ is *closed* because **f** is continuous; it is also *open,* as is shown by applying prop. 4 (with $m = 0$) to a convex open neighbourhood, contained in A, of an arbitrary point of B; so is identical to A.

PROPOSITION 5. *Let* **f** *be a vector function of a complex variable, defined, continuous and differentiable on a convex open set* A $\subset$ **C**, *taking values in a normed space over the field* **C**; *then, no matter what the points* x_0, x *and* y *in* A, *one has*

$$\left\|\mathbf{f}(y) - \mathbf{f}(x) - \mathbf{f}'_d(x_0)(y - x)\right\| \leqslant |y - x| \sup_{z \in A} \left\|\mathbf{f}'(z) - \mathbf{f}'(x_0)\right\|. \qquad (10)$$

It suffices to apply th. 2 to the function

$$\mathbf{g}(t) = \mathbf{f}(x + t(y - x)) - \mathbf{f}'(x_0)(y - x)t$$

on the interval $[0,\ 1]$.

4. CONTINUITY OF DERIVATIVES

PROPOSITION 6. *Let* I *be an open interval in* **R**, *let* x_0 *be one of the endpoints of* I, *and* **f** *a vector function defined and continuous on* I, *with values in a complete normed space* E *over* **R**; *suppose that* **f** *has a right derivative at the points of the complement* B *in* I *of a countable subset of* I. *Then for* $\mathbf{f}'_d(x)$ *to have a limit as* x *tends to* x_0 *while remaining in* B *and* $\neq x_0$ *it is necessary and sufficient that* $\dfrac{\mathbf{f}(y) - \mathbf{f}(x)}{y - x}$ *have a limit* **c** *as* (x, y) *tends to* (x_0, x_0) *subject to* $x \in$ I, $y \in$ I, $x \neq x_0$, $y \neq x_0$ *and* $x \neq y$. *Under these conditions* **f** *extends by continuity to the point* x_0, *the right*

derivative $\mathbf{f}_d'(x)$ *tends to* $\mathbf{c}$ *as* x *tends to* x_0 (while remaining in B) *and the function* $\mathbf{f}$ *extended* (defined on $I \cup \{x_0\}$) *has derivative at* x_0 *equal to* $\mathbf{c}$.

Suppose for example that x_0 is the right-hand endpoint of I. Let us first show that if $\mathbf{f}_d'(x)$ tends to $\mathbf{c}$ as x tends to x_0 while remaining in B and $\neq x_0$, then $\dfrac{\mathbf{f}(y) - \mathbf{f}(x)}{y - x}$ tends to $\mathbf{c}$; this follows immediately from th. 2 applied to the function $\mathbf{f}(z) - \mathbf{c}z$, which yields

$$\|\mathbf{f}(y) - \mathbf{f}(x) - \mathbf{c}(y - x)\| \leqslant (y - x) \sup_{z \in B, \, x < z < y} \left\|\mathbf{f}_d'(z) - \mathbf{c}\right\|$$

for $x < y < x_0$. Conversely, if $\dfrac{\mathbf{f}(y) - \mathbf{f}(x)}{y - x}$ tends to $\mathbf{c}$, then for every $\varepsilon > 0$ there exists an $h > 0$ such that the conditions $|x - x_0| < h$, $|y - x_0| < h$ ($x \neq x_0, y \neq x_0$) imply

$$\|\mathbf{f}(y) - \mathbf{f}(x) - \mathbf{c}(y - x)\| \leqslant \varepsilon |y - x| . \tag{11}$$

But for all $x \in B$ and $\neq x_0$ such that $|x - x_0| < h$ there exists a $k > 0$ (depending on x) such that the relation $x < y < x + k$ entails

$$\left\|\mathbf{f}(y) - \mathbf{f}(x) - \mathbf{f}_d'(x)(y - x)\right\| \leqslant \varepsilon |y - x| \tag{12}$$

from which, considering (11):

$$\left\|\mathbf{f}_d'(x) - \mathbf{c}\right\| \leqslant 2\varepsilon$$

for $|x - x_0| < h$, $x \in B$ and $x \neq x_0$, which proves that $\mathbf{f}_d'(x)$ tends to $\mathbf{c}$. Moreover, from the relation (11) one has immediately that

$$\|\mathbf{f}(y) - \mathbf{f}(x)\| \leqslant (\|\mathbf{c}\| + \varepsilon) |y - x| ,$$

which proves (by Cauchy's criterion) that $\mathbf{f}$ has a limit $\mathbf{d}$ at the point x_0 as x tends to this point while remaining in I and $\neq x_0$; now, letting x approach x_0 in (11), for $y \in I$, $y \neq x_0$ and $|y - x_0| \leqslant h$, we have

$$\left\|\frac{\mathbf{f}(y) - \mathbf{d}}{y - x_0} - \mathbf{c}\right\| \leqslant \varepsilon$$

which proves that $\mathbf{c}$ is the derivative at the point x_0 of the function $\mathbf{f}$ extended by continuity to $I \cup \{x_0\}$.

Remark. A similar argument, based on th. 1, shows that if f is a real function such that $f_d'(x)$ tends to $+\infty$ at the point x_0 then the ratio

$$(f(y) - f(x))/(y - x)$$

also tends to $+\infty$, and conversely; if moreover f has a finite limit at the point x_0 (which is not a consequence of the present hypothesis), then the function f extended by continuity to x_0 has a derivative equal to $+\infty$ at this point.

§3. DERIVATIVES OF HIGHER ORDER

1. DERIVATIVES OF ORDER n

Let $\mathbf{f}$ be a vector function of a real variable, defined, continuous and differentiable on an interval I. If the derivative $\mathbf{f}'$ exists on a neighbourhood (with respect to I) of a point $x_0 \in$ I, and is differentiable at the point x_0, then its derivative is called the *second derivative* of $\mathbf{f}$ at the point x_0, and is denoted by $\mathbf{f}''(x_0)$ or $D^2\mathbf{f}(x_0)$. If this second derivative exists at every point of I (which implies that $\mathbf{f}'$ exists and is continuous on I), then $x \mapsto \mathbf{f}''(x)$ is a vector function which one denotes by $\mathbf{f}''$ or $D^2\mathbf{f}$. We define, in the same way, recursively, the nth *derivative* (or *derivative of order n*) of $\mathbf{f}$, and denote it by $\mathbf{f}^{(n)}$ or $D^n\mathbf{f}$; by definition, its value at the point $x_0 \in$ I is the derivative of the function $\mathbf{f}^{(n-1)}$ at the point x_0 : this definition presupposes the existence of *all* the derivatives $\mathbf{f}^{(k)}$ of order $k \leqslant n - 1$ on a *neighbourhood* of x_0 relative to I, and the differentiability of $\mathbf{f}^{(n-1)}$ at the point x_0.

We will say that $\mathbf{f}$ is *n times differentiable* at the point x_0 (resp. in an interval) if it admits an nth derivative at this point (resp. in this interval). One says that $\mathbf{f}$ is *indefinitely differentiable* on I if for each integer $n > 0$ it admits a derivative of order n on I.

By induction on m one sees that

$$D^m(D^n\mathbf{f}) = D^{m+n}\mathbf{f}. \tag{1}$$

More precisely, when one of the two terms in (1) is defined, then so is the other, and is equal to it.

PROPOSITION 1. *The set of vector functions defined on an interval* I $\subset \mathbf{R}$, *taking values in a given topological vector space* E, *and having an nth derivative on* I, *is a vector space over* $\mathbf{R}$, *and* $\mathbf{f} \mapsto D^n\mathbf{f}$ *is a linear mapping of this space into the vector space of linear mappings from* I *into* E.

One proves the formulae

$$D^n(\mathbf{f} + \mathbf{g}) = D^n\mathbf{f} + D^n\mathbf{g} \tag{2}$$

$$D^n(\mathbf{f}a) = D^n\mathbf{f}.a \tag{3}$$

by induction on n when $\mathbf{f}$ and $\mathbf{g}$ have an nth derivative on I (a being constant).

PROPOSITION 2 $\left(\text{"Leibniz' formula"}\right)$.   *Let* E, F, G *be three topological vector spaces over* $\mathbf{R}$, *and* $(\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}.\mathbf{y}]$ *a continuous bilinear mapping of* E $\times$ F *into* G. *If* $\mathbf{f}$ *(resp.* $\mathbf{g}$*) is defined on an interval* I $\subset \mathbf{R}$, *takes its values in* E *(resp.* F*) and has an nth derivative on* I, *then* $[\mathbf{f}.\mathbf{g}]$ *has an nth derivative on* I, *given by the formula*

$$D^n[\mathbf{f}.\mathbf{g}] = [\mathbf{f}^{(n)}.\mathbf{g}] + \binom{n}{1}[\mathbf{f}^{(n-1)}.\mathbf{g}'] + \cdots + \binom{n}{p}[\mathbf{f}^{(n-p)}.\mathbf{g}^{(p)}] + \cdots + [\mathbf{f}.\mathbf{g}^{(n)}]. \tag{4}$$

Formula (4) is proved by induction on n (using the relation $\binom{n}{p} = \binom{n-1}{p} + \binom{n-1}{p-1}$ for the binomial coefficients).

In the same way one can verify the following formula (where the hypotheses are the same as in prop. 2):

$$[\mathbf{f}^{(n)}.\mathbf{g}] + (-1)^{n-1}[\mathbf{f}.\mathbf{g}^{(n)}] = D\big([\mathbf{f}^{(n-1)}.\mathbf{g}] - [\mathbf{f}^{(n-2)}.\mathbf{g}'] + \cdots$$
$$+ (-1)^{n-1}[\mathbf{f}.\mathbf{g}^{(n-1)}]\big). \tag{5}$$

The preceding propositions have been stated for functions that are n times differentiable on an interval; we leave it to the reader to formulate the analogous propositions for functions that are n times differentiable at a point.

2. TAYLOR'S FORMULA

Let $\mathbf{f}$ be a vector function defined on an interval $I \subset \mathbf{R}$, with values in a *normed* space E over $\mathbf{R}$; to say that $\mathbf{f}$ has a derivative at a point $a \in I$ signifies that

$$\lim_{x \to a,\, x \in I,\, x \neq a} \frac{\mathbf{f}(x) - \mathbf{f}(a) - \mathbf{f}'(a)(x - a)}{x - a} = 0; \tag{6}$$

or, otherwise, that $\mathbf{f}$ is "approximately equal" to the *linear* function $\mathbf{f}(a) + \mathbf{f}'(a)(x-a)$ on a neighbourhood of a (*cf.* chap. V, where this concept is developed in a general manner). We shall see that the existence of the nth order derivative of $\mathbf{f}$ at the point a entails in the same way that $\mathbf{f}$ is "approximately equal" to a *polynomial of degree n in x*, with coefficients in E (*Gen. Top.*, X, p. 315) on a neighbourhood of a. To be precise:

THEOREM 1. *If the function $\mathbf{f}$ has an nth derivative at the point a then*

$$\lim_{x \to a,\, x \in I,\, x \neq a} \frac{\mathbf{f}(x) - \mathbf{f}(a) - \mathbf{f}'(a)\frac{(x-a)}{1!} - \cdots - \mathbf{f}^{(n)}(a)\frac{(x-a)^n}{n!}}{(x - a)^n} = 0. \tag{7}$$

We proceed by induction on n. The theorem holds for $n = 1$. For arbitrary n one can, by the induction hypothesis, apply it to the derivative $\mathbf{f}'$ of $\mathbf{f}$: for any $\varepsilon > 0$ there is an $h > 0$ such that, if one puts

$$\mathbf{g}(x) = \mathbf{f}(x) - \mathbf{f}(a) - \mathbf{f}'(a)\frac{(x - a)}{1!} - \mathbf{f}''(a)\frac{(x - a)^2}{2!} - \cdots - \mathbf{f}^{(n)}(a)\frac{(x - a)^n}{n!}$$

one has, for $|y - a| \leqslant h$ and $y \in I$,

$$\big\|\mathbf{g}'(y)\big\| = \Big\|\mathbf{f}'(y) - \mathbf{f}'(a) - \mathbf{f}''(a)\frac{(y - a)}{1!} - \cdots - \mathbf{f}^{(n)}(a)\frac{(y - a)^{n-1}}{(n - 1)!}\Big\|$$
$$\leqslant \varepsilon \, |y - a|^{n-1}.$$

We apply the mean value theorem (I, p. 15, th. 2) on the interval with endpoints a, x (with $|x - a| \leqslant h$) to the vector function $\mathbf{g}$ and to the real increasing function

equal to $\varepsilon\,|y-a|^n\,/n$ if $x > a$, and to $-\varepsilon\,|y-a|^n\,/n$ if $x < a$; it follows that $\|\mathbf{g}(x)\| \leqslant \varepsilon\,|x-a|^n\,/n$, which proves the theorem.

We thus can write

$$
\begin{aligned}
\mathbf{f}(x) = \mathbf{f}(a) &+ \mathbf{f}'(a)\frac{(x-a)}{1!} + \mathbf{f}''(a)\frac{(x-a)^2}{2!} + \cdots \\
&+ \mathbf{f}^{(n)}(a)\frac{(x-a)^n}{n!} + \mathbf{u}(x)\frac{(x-a)^n}{n!}
\end{aligned}
\tag{8}
$$

where $\mathbf{u}(x)$ approaches 0 as x approaches a while remaining in I; this formula is called *Taylor's formula of order n* at the point a, and the right-hand side of (8) is called the *Taylor expansion of order n* of the function $\mathbf{f}$ at the point a. The last term $\mathbf{r}_n(x) = \mathbf{u}(x)(x-a)^n/n!$ is called the *remainder* in the Taylor formula of order n.

When $\mathbf{f}$ has a *derivative of order $n+1$* on I, one can estimate $\|\mathbf{r}_n(x)\|$ in terms of this $n+1^{th}$ derivative, on all of I, and not just on an unspecified neighbourhood of a :

PROPOSITION 3.   *If* $\left\|\mathbf{f}^{(n+1)}(x)\right\| \leqslant \mathrm{M}$ *on* I, *then we have*

$$
\|\mathbf{r}_n(x)\| \leqslant \mathrm{M}\,\frac{|x-a|^{n+1}}{(n+1)!}
\tag{9}
$$

on I.

Indeed, the formula holds for $n = 0$, by I, p. 15, th. 2. Let us prove it by induction on n : by the induction hypothesis applied to $\mathbf{f}'$, one has

$$
\left\|\mathbf{r}_n'(y)\right\| \leqslant \mathrm{M}\frac{|y-a|^n}{n!}
$$

from which the formula (9) follows by the mean value theorem (I, p. 23, th. 2).

COROLLARY.   *If f is a finite real function with a derivative of order $n+1$ on* I, *and if $m \leqslant f^{(n+1)}(x) \leqslant \mathrm{M}$ on* I, *then for all $x \geqslant a$ in* I *one has*

$$
m\,\frac{(x-a)^{n+1}}{(n+1)!} \leqslant r_n(x) \leqslant \mathrm{M}\,\frac{(x-a)^{n+1}}{(n+1)!}
\tag{10}
$$

and the second term cannot be equal to the first (resp. *to the third*) *unless $f^{(n+1)}$ is constant and equal to m* (resp. M) *on the interval* $[a, x]$.

The proof proceeds in the same way, but applying th. 1 of I, p. 14.

Remarks. 1) We have already noticed in the proof of th. 1 that if $\mathbf{f}$ has a derivative of order n on I, and if

$$
\mathbf{f}(x) = \mathbf{a}_0 + \mathbf{a}_1(x-a) + \mathbf{a}_2(x-a)^2 + \cdots + \mathbf{a}_n(x-a)^n + \mathbf{r}_n(x)
\tag{11}
$$

is its Taylor expansion of order n at the point a, then the Taylor expansion of order $n-1$ for $\mathbf{f}'$ at the point a is

$$
\mathbf{f}'(x) = \mathbf{a}_1 + 2\mathbf{a}_2(x-a) + \cdots + n\mathbf{a}_n(x-a)^{n-1} + \mathbf{r}_n'(x).
\tag{12}
$$

We say that it is obtained from the expansion (11) of $\mathbf{f}$ by *differentiating term-by-term*.

2) With the same hypotheses, the coefficients $\mathbf{a}_i$ in (11) are determined recursively by the relations

$$\mathbf{a}_0 = \mathbf{f}(a)$$

$$\mathbf{a}_1 = \lim_{x \to a} \frac{\mathbf{f}(x) - \mathbf{f}(a)}{x - a}$$

$$\mathbf{a}_2 = \lim_{x \to a} \frac{\mathbf{f}(x) - \mathbf{f}(a) - \mathbf{a}_1(x - a)}{(x - a)^2}$$

$$\cdots$$

$$\mathbf{a}_n = \lim_{x \to a} \frac{\mathbf{f}(x) - \mathbf{f}(a) - \mathbf{a}_1(x - a) - \cdots - \mathbf{a}_{n-1}(x - a)^{n-1}}{(x - a)^n}.$$

In the case $a = 0$ one concludes, in particular, that if $\mathbf{f}(x^p)$ (p an integer > 0) has a derivative of order pn on a neighbourhood of 0 then the Taylor expansion of order pn of this function is simply

$$\mathbf{f}(x^p) = \mathbf{a}_0 + \mathbf{a}_1 x^p + \mathbf{a}_2 x^{2p} + \cdots + \mathbf{a}_n x^{np} + \mathbf{r}_n(x^p) \tag{13}$$

where $\mathbf{r}_n(x^p)$ is the remainder in the expansion (*cf.* V, p. 222).

3) The definition of the derivative of order n and the preceding results generalize immediately to functions of a complex variable; we shall not pursue this topic further here; it will be treated in detail in a later Book in this Series.

§ 4. CONVEX FUNCTIONS OF A REAL VARIABLE

Let H be a subset of $\mathbf{R}$, f a finite real function defined on H, and let G be the *graph* or representative set of the function f in $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$, the set of points $M_x = (x, f(x))$, where x runs through H. It is convenient to say that a point (a, b) of $\mathbf{R}^2$ such that $a \in H$ lies *above* (resp. *strictly above, below, strictly below*) G if one has $b \geqslant f(a)$ (resp. $b > f(a), b \leqslant f(a), b < f(a)$). If $A = (a, a')$ and $B = (b, b')$ are two points of $\mathbf{R}^2$ we denote by AB the closed segment with endpoints A and B; if $a < b$ then AB is the graph of the linear function $a' + \dfrac{b' - a'}{b - a}(x - a)$ defined on $[a, b]$; we denote the slope $\dfrac{b' - a'}{b - a}$ of this segment by $p(AB)$, and will make use of the following lemma, whose verification is immediate:

Lemma.   Let $A = (a, a')$, $B = (b, b')$, $C = (c, c')$ *be three points in* $\mathbf{R}^2$ *such that* $a < b < c$. *The following statements are equivalent:*
a) B *is below* AC;
b) C *lies above the line passing through* A *and* B;
c) A *is above the line passing through* B *and* C;
d) $p(AB) \leqslant p(AC)$;
e) $p(AC) \leqslant p(BC)$.

The lemma still holds when one replaces "above" (resp. "below") by "strictly above" (resp. "strictly below") and the sign $\leqslant$ by $<$ (fig. 1).
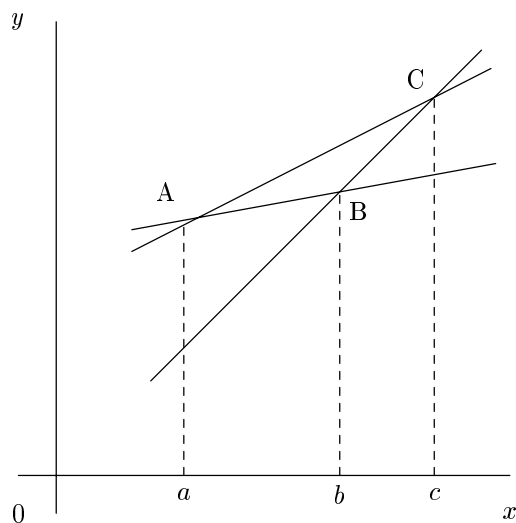

Fig. 1

1. DEFINITION OF A CONVEX FUNCTION

DEFINITION 1. *We say that a finite numerical function f, defined on an interval $I \subset \mathbf{R}$, is convex on I if, no matter what the points x, x' of I, $(x < x')$, every point M_z of the graph G of f such that $x \leqslant z \leqslant x'$ lies below the segment $M_x M_{x'}$ (or, what comes to the same, if every point of this segment lies above G) (fig. 2).*
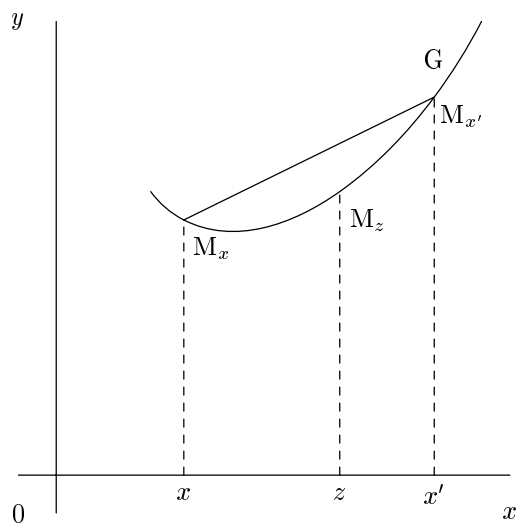

Fig. 2

Taking account of the parametric representation of a segment (*Gen. Top.*, VI, p. 35), the condition for f to be convex on I is that one has the inequality

$$f(\lambda x + (1-\lambda)x') \leqslant \lambda f(x) + (1-\lambda)f(x') \qquad (1)$$

for each pair of points (x, x') of I and every $\lambda \in [0, 1]$.

Definition 1 is equivalent to the following: *the set of points in* $\mathbf{R}^2$ *lying above the graph* G *of* f *is convex.* Indeed, this condition is clearly sufficient for f to be convex on I; it is also necessary, for if f is convex on I, and if (x, y), (x', y') are two points lying above G, then one has $y \geqslant f(x)$, $y' \geqslant f(x')$, from which, for $0 \leqslant \lambda \leqslant 1$,

$$\lambda y + (1-\lambda)y' \geqslant \lambda f(x) + (1-\lambda)f(x') \geqslant f(\lambda x + (1-\lambda)x')$$

by (1), which shows that every point of the segment with endpoints (x, y) and (x', y') lies above G.

Remark. On sees in the same way that the set of points lying *strictly above* G is convex. Conversely, if this set is convex one has

$$\lambda y + (1-\lambda)y' > f(\lambda x + (1-\lambda)x')$$

for $0 \leqslant \lambda \leqslant 1$ and $y > f(x)$, $y' > f(x')$; on letting y tend to $f(x)$ and y' approach $f(x')$ in this formula it follows that f is convex.

Examples. 1) Every (real) affine linear function $ax + b$ is convex on $\mathbf{R}$.
2) The function x^2 is convex on $\mathbf{R}$, since one has

$$\lambda x^2 + (1-\lambda)x'^2 - \left(\lambda x + (1-\lambda)x'\right)^2 = \lambda(1-\lambda)(x - x')^2 \geqslant 0$$

for $0 \leqslant \lambda \leqslant 1$.

3) The function $|x|$ is convex on $\mathbf{R}$, since

$$\left|\lambda x + (1-\lambda)x'\right| \leqslant \lambda |x| + (1-\lambda)|x'|$$

for $0 \leqslant \lambda \leqslant 1$.

It is clear that if f is convex on I, then its restriction to any interval $J \subset I$ is convex on J.

Let f be a convex function on I, and x, x' two points of I such that $x < x'$; if $z \in I$ is *exterior* to $[x, x']$ then M_z lies *above* the line D joining M_x and $M_{x'}$; this is an immediate consequence of the lemma.

One deduces from this that if z is a point such that $x < z < x'$ and such that M_z lies *on* the segment $M_x M_{x'}$, then, for *every other point* z' such that $x < z' < x'$ the point $M_{z'}$ also lies *on* the segment $M_x M_{x'}$, for it follows from the above that $M_{z'}$ is at the same time both above and below this segment; in other words, f is then equal to an *affine linear* function on $[x, x']$.

DEFINITION 2. *We say that a finite real function* f *defined on an interval* $I \subset \mathbf{R}$ *is strictly convex on* I *if, for any points* x, x' *of* I $(x < x')$, *every point* M_z *of the graph* G *of* f *such that* $x < z < x'$ *lies strictly below the segment* $M_x M_{x'}$ *(or, what comes to the same, if every point of the segment, apart from the endpoints, lies strictly above G).*

In other words, we must have the inequality

$$f(\lambda x + (1 - \lambda)x') < \lambda f(x) + (1 - \lambda)f(x') \tag{2}$$

for every pair of distinct points (x, x') of I and every λ such that $0 < \lambda < 1$.

The remarks that precede def. 2 show that for a convex function f to be strictly convex on I it is necessary and sufficient that there be no interval contained in I (not reducing to a single point) such that the restriction of f to this interval is *affine linear*.

Of the examples above, the first and third are not strictly convex; on the other hand, x^2 is strictly convex on **R**; a similar calculation shows that $1/x$ is strictly convex on $]0, +\infty[$.

PROPOSITION 1. *Let f be a finite real function, convex* (resp. *strictly convex*) *on an interval* $I \subset \mathbf{R}$. *For every family* $(x_i)_{1 \leqslant i \leqslant p}$ *of* $p \geqslant 2$ *distinct points of* I, *and every family* $(\lambda_i)_{1 \leqslant i \leqslant p}$ *of* p *real numbers such that* $0 < \lambda_i < 1$ *and* $\sum_{i=1}^{p} \lambda_i = 1$, *we have*

$$f\left(\sum_{i=1}^{p} \lambda_i x_i\right) \leqslant \sum_{i=1}^{p} \lambda_i f(x_i) \tag{3}$$

(resp.

$$f\left(\sum_{i=1}^{p} \lambda_i x_i\right) < \sum_{i=1}^{p} \lambda_i f(x_i)). \tag{4}$$

Since the proposition (for convex functions) reduces to the inequality (1) for $p = 2$ we argue by induction for $p > 2$. The number $\mu = \sum_{i=1}^{p-1} \lambda_i$ is > 0; it is immediate that if a and b are the smallest and largest of the x_i then $a \leqslant \sum_{i=1}^{p-1} \lambda_i x_i \Big/ \sum_{i=1}^{p-1} \lambda_i \leqslant b$; in other words, the point $x = \frac{1}{\mu} \sum_{i=1}^{p-1} \lambda_i x_i$ belongs to I, and the induction hypothesis implies that $\mu f(x) \leqslant \sum_{i=1}^{p-1} \lambda_i f(x_i)$; moreover we have, from (1), that

$$f\left(\sum_{i=1}^{p} \lambda_i x_i\right) = f(\mu x + (1 - \mu)x_p) \leqslant \mu f(x) + (1 - \mu)f(x_p) \leqslant \sum_{i=1}^{p} \lambda_i f(x_i).$$

One argues in the same way for strictly convex functions, starting from the inequality (2).

We say that a finite real function f is *concave* (resp. *strictly concave*) on I if $-f$ is convex (resp. strictly convex) on I. It comes to the same to say that for every pair (x, x') of distinct points of I and every λ such that $0 < \lambda < 1$ one has

$$f(\lambda x + (1 - \lambda)x') \geqslant \lambda f(x) + (1 - \lambda)f(x')$$

$$(\text{resp. } f(\lambda x + (1 - \lambda)x') > \lambda f(x) + (1 - \lambda)f(x')).$$

2. FAMILIES OF CONVEX FUNCTIONS

PROPOSITION 2. *Let f_i $(1 \leqslant i \leqslant p)$ be p convex functions on an interval $\mathrm{I} \subset \mathbf{R}$, and c_i $(1 \leqslant i \leqslant p)$ be p arbitrary positive numbers; then the function $f = \sum_{i=1}^{p} c_i f_i$ is convex on I. Further, if for at least one index j the function f_j is strictly convex on I, and $c_j > 0$, then f is strictly convex on I.*

This follows immediately by applying the inequality (1) (resp. (2)) to each of the f_i, multiplying the inequality for f_i by c_i, and then adding term-by-term.

PROPOSITION 3. *Let (f_α) be a family of convex functions on an interval $\mathrm{I} \subset \mathbf{R}$; if the upper envelope g of this family is finite at every point of I then g is convex on I.*

Indeed, the set of points $(x, y) \in \mathbf{R}^2$ lying above the graph of g is the intersection of the convex sets formed by the points lying above the graph of each of the functions f_α; so it is convex.

PROPOSITION 4. *Let H be a set of convex functions on an interval $\mathrm{I} \subset \mathbf{R}$; if $\mathfrak{F}$ is a filter on H which converges pointwise on I to a finite real function f_0, then this function is convex on I.*

To see this it suffices to pass to the limit along $\mathfrak{F}$ in the inequality (1).

3. CONTINUITY AND DIFFERENTIABILITY OF CONVEX FUNCTIONS

PROPOSITION 5. *For a real finite function f to be convex (resp. strictly convex) on an interval I it is necessary and sufficient that for all $a \in \mathrm{I}$ the gradient*

$$p(\mathrm{M}_a \mathrm{M}_x) = \frac{f(x) - f(a)}{x - a}$$

be an increasing (resp. strictly increasing) function of x on $\mathrm{I} \cap \complement\{a\}$.

This proposition is an immediate consequence of definitions 1 and 2 and of the lemma of I, p. 23.

PROPOSITION 6. *Let f be a finite real function convex on an interval $\mathrm{I} \subset \mathbf{R}$. Then at every* interior *point a of I the function f is continuous, has finite right and left derivatives, and $f'_g(a) \leqslant f'_d(a)$.*

Indeed, for $x \in \mathrm{I}$ and $x > a$ the function $x \mapsto \dfrac{f(x) - f(a)}{x - a}$ is increasing (prop. 5) and bounded below, since if $y < a$ and $y \in \mathrm{I}$ we have

$$\frac{f(y) - f(a)}{y - a} \leqslant \frac{f(x) - f(a)}{x - a} \tag{5}$$

by prop. 5; this function therefore has a finite right limit at the point a; in other words, $f'_d(a)$ exists and is finite; further, letting x approach a ($x > a$) in (5), it follows that

$$\frac{f(y) - f(a)}{y - a} \leqslant f'_d(a) \tag{6}$$

for all $y < a$ belonging to I. In the same way one shows that $f'_g(a)$ exists and that

$$f'_d(a) \leqslant \frac{f(x) - f(a)}{x - a} \tag{7}$$

for $x \in$ I and $x > a$. On letting x approach a ($x > a$) in this last inequality we obtain $f'_g(a) \leqslant f'_d(a)$. The existence of the left and right derivatives at the point a clearly ensures the continuity of f at this point.

COROLLARY 1. *Let f be a convex* (resp. *strictly convex*) *function on* I; *if a and b are two interior points of* I *such that $a < b$ one has* (fig. 3)
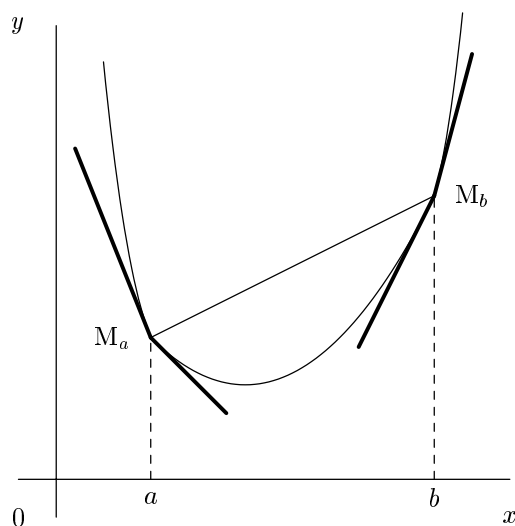
Fig. 3

$$f'_d(a) \leqslant \frac{f(b) - f(a)}{b - a} \leqslant f'_g(b) \tag{8}$$

(resp.

$$f'_d(a) < \frac{f(b) - f(a)}{b - a} < f'_g(b) \,). \tag{9}$$

The double inequality (8) results from (6) and (7) by a simple change of notation. On the other hand, if f is strictly convex and c is such that $a < c < b$ one has, from (8) and prop. 5,

$$f'_d(a) \leqslant \frac{f(c) - f(a)}{c - a} < \frac{f(b) - f(a)}{b - a} < \frac{f(b) - f(c)}{b - c} \leqslant f'_g(b)$$

from which (9).

COROLLARY 2. *If f is convex* (resp. *strictly convex*) *on* I *then* f'_d *and* f'_g *are increasing* (resp. *strictly increasing*) *on the interior of* I*; the set of points in* I *at which f is not differentiable is countable, and f'_d and f'_g are continuous at every point where f is differentiable.*

The first part follows immediately from (8) (resp. (9)) and the inequality

$$f'_g(a) \leqslant f'_d(a).$$

On the other hand, let E be the set of interior points x of I where f is not differentiable (that is $f'_g(x) < f'_d(x)$). For each $x \in$ E let J_x be the open interval $]f'_g(x), f'_d(x)[$; it follows from (8) that if x and y are two points of E such that $x < y$, then $u < v$ for all $u \in J_x$ and all $v \in J_y$; in other words, as x runs through E the open nonempty intervals J_x are pairwise disjoint; the set of such intervals is thus countable, and hence so is E. Finally, f'_d (resp. f'_g) being increasing, it has a right limit and a left limit at every interior point x of I; prop. 6 of I, p. 18 now shows that the right limit of f'_d (resp. f'_g) at the point x is equal to $f'_d(x)$, and its left limit is $f'_g(x)$; from which we have the last part of the corollary.

Let f be a convex function on I, a an interior point of I, and D a line passing through the point M_a, with equation $y - f(a) = \alpha(x - a)$. It follows from the inequalities (8) that if $f'_g(a) \leqslant \alpha \leqslant f'_d(a)$ then every point of the graph G lies *above* D, and, if f is strictly convex, M_a is the only point common to D and G; one says that D is a *support line* to G at the point M_a. Conversely, if G lies above D, one has $f(x) - f(a) \geqslant \alpha(x - a)$ for every $x \in$ I, from which $\dfrac{f(x) - f(a)}{x - a} \geqslant \alpha$ for $x > a$, and $\dfrac{f(x) - f(a)}{x - a} \leqslant \alpha$ for $x < a$; on letting x tend to a in these inequalities it follows that $f'_g(a) \leqslant \alpha \leqslant f'_d(a)$.

In particular, if f is differentiable at the point a there is *only one* supporting line to G at the point M_a, the *tangent* to G at M_a.

Remark. If f is a strictly convex function on an open interval I then f'_d is strictly increasing on I, so there are three possible cases, according to prop. 2 of I, p. 13:
    1° f is strictly decreasing on I;
    2° f is strictly increasing on I;
    3° there is an $a \in$ I such that f is strictly decreasing for $x \leqslant a$, and is strictly increasing for $x \geqslant a$.
    When f is convex on I, but not strictly convex, f can be constant on an interval contained in I; let $J =]a, b[$ be the largest open interval on which f is constant (that is

to say, the interior of the interval where $f'_d(x) = 0$); then f is strictly decreasing on the interval formed by the points $x \in I$ such that $x \leqslant a$ (if it exists), strictly increasing on the interval formed by the points $x \in I$ such that $x \geqslant b$ (if it exists).

In all cases one sees that f possesses a *right limit* at the left-hand endpoint of I (in $\overline{\mathbf{R}}$), and a *left limit* at the right-hand endpoint; these limits may be finite or infinite (*cf.* I, p. 46, exerc. 5, 6 and 7). By abuse of language one sometimes says that the continuous function (with values in $\overline{\mathbf{R}}$), equal to f on the interior of I, and extended by continuity to the endpoints of I, is *convex on* $\overline{\mathrm{I}}$.

4. CRITERIA FOR CONVEXITY

PROPOSITION 7.   *Let f be a finite real function defined on an interval* $\mathrm{I} \subset \mathbf{R}$. *For f to be convex on* I *it is necessary and sufficient that for every pair of numbers a, b of* I *such that $a < b$, and for every real number μ, the function $f(x) + \mu x$ attains its supremum on* $[a, b]$ *at one of the points a, b.*

The condition is *necessary;* indeed, since μx is convex on $\mathbf{R}$, the function $f(x) + \mu x$ is convex on I; one can therefore restrict oneself to the case $\mu = 0$. Then, for

$$x = \lambda a + (1 - \lambda)b \quad (0 \leqslant \lambda \leqslant 1),$$

one has

$$f(x) \leqslant \lambda f(a) + (1 - \lambda)f(b) \leqslant \mathrm{Max}(f(a), f(b)).$$

The condition is *sufficient.* Let us take $\mu = -\dfrac{f(b) - f(a)}{b - a}$ and let $g(x) = f(x) + \mu x$; one has $g(a) = g(b)$ and therefore $g(x) \leqslant g(a)$ for all $x \in [a, b]$, and one can check immediately that this inequality is equivalent to the inequality (1) where one replaces z by a and x' by b.

PROPOSITION 8.   *For a finite real function f to be convex (resp. strictly convex) on an open interval* $\mathrm{I} \subset \mathbf{R}$ *it is necessary and sufficient that it be continuous on* I, *have a derivative at every point of the complement* B *relative to* I *of a countable subset of this interval, and that the derivative be increasing (resp. strictly increasing) on* B.

The condition is necessary, from prop. 6 and its corollary 2 (I, p. 27); let us show that it is sufficient. Suppose, therefore, that f' is increasing on B, and that f is not convex; there then exist (I, p. 27, prop. 5) three points a, b, c of I, such that $a < c < b$, and $\dfrac{f(c) - f(a)}{c - a} > \dfrac{f(b) - f(c)}{b - c}$; but from the mean value theorem (I, p. 14, th. 1) one has

$$\frac{f(c) - f(a)}{c - a} \leqslant \sup_{x \in \mathrm{B},\ a < x < c} f'(x) \qquad \text{and} \qquad \frac{f(b) - f(c)}{b - c} \geqslant \inf_{x \in \mathrm{B},\ c < x < b} f'(x).$$

One thus has $\sup\limits_{x\in\text{B},\ a<x<c} f'(x) > \inf\limits_{x\in\text{B},\ c<x<b} f'(x)$, contrary to the hypothesis that
f' is increasing on B.

If now we assume that f' is strictly increasing on B, then f is convex and cannot be equal to an affine linear function on any open interval contained in I, for then f' would be constant on this interval, contrary to the hypothesis.

COROLLARY. *Let f be a finite real function, continuous and twice differentiable on an interval* $\text{I} \subset \mathbf{R}$; *for f to be convex on* I *it is necessary and sufficient that $f''(x) \geqslant 0$ for all $x \in$ I; for f to be strictly convex on* I *it is necessary and sufficient that the previous condition be satisfied and further that the set of points $x \in$ I where $f''(x) > 0$ be dense in* I.

This follows immediately from the preceding proposition, and from the corollary at I, p. 14.

Example. *On the interval $]0, +\infty[$ the function x^r (r any real number) has a second derivative equal to $r(r-1)x^{r-2}$; thus it is strictly convex if $r > 1$ or $r < 0$, and strictly concave if $0 < r < 1$.*

In order to be able to formulate another criterion for convexity we make the following definition: given the graph G of a finite real function defined on an interval $\text{I} \subset \mathbf{R}$ and an interior point a of I, we shall say that a line D passing through $\text{M}_a = (a, f(a))$ is *locally above* (resp. *locally below*) G if there exists a neighbourhood $\text{V} \subset \text{I}$ of a such that every point of D contained in $\text{V} \times \mathbf{R}$ is above (resp. below) G; we shall say that D is *locally on* G at the point M_a if there is a neighbourhood $\text{V} \subset \text{I}$ of a such that the intersection of D and $\text{V} \times \mathbf{R}$ is identical to that of G and $\text{V} \times \mathbf{R}$ (in other words, if D is simultaneously locally above and locally below G).

PROPOSITION 9. *Let f be a real finite function which is upper semi-continuous on an open interval* $\text{I} \subset \mathbf{R}$. *For f to be convex on* I *it is necessary and sufficient that for every point M_x of the graph G of f every line locally above G at this point should be locally on* G *(at the point M_x).*

The condition is *necessary:* indeed, if f is convex on I then at every point M_a of the graph G of f there exists a *support line* Δ to G; now Δ is below G, so *a fortiori* locally below G (I, p. 29); if a line D is locally above G at the point M_a it is locally above Δ, so must coincide with Δ, and consequently is locally on G at the point M_a.

The condition is *sufficient.* Indeed, suppose it is satisfied, and suppose that f is not convex on I; then there are two points a, b of I ($a < b$) such that there are points M_x of G strictly above the segment M_aM_b (fig. 4). In other words, the function
$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$ takes values > 0 on $[a, b]$; since g is finite and upper semi-continuous on this compact interval its least upper bound k on $[a, b]$ is finite and > 0, and
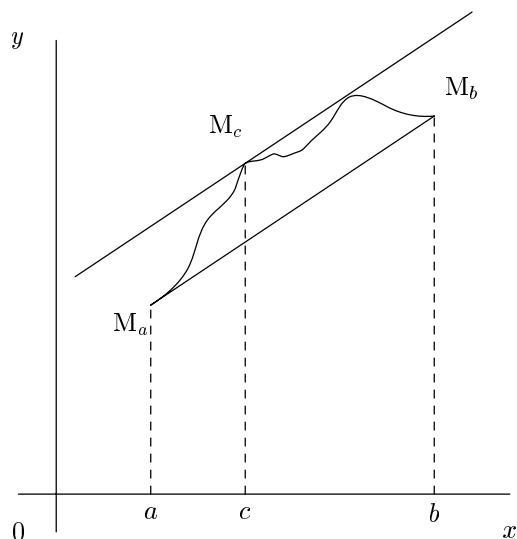
Fig. 4

the set $\overset{-1}{g}(k)$ is closed and not empty (*Gen. Top.*, IV, p. 361, th. 3 and prop. 1). Let c be the greatest lower bound of $\overset{-1}{g}(k)$; we have $a < c < b$, and at the point M_c the line D with equation $y = f(c) + \dfrac{f(b) - f(a)}{b - a}(x - c)$ lies locally above G; but it cannot be locally on G at this point, since, for $a < x < c$, one has $g(x) < k$, which signifies that M_x is strictly below D. This has led us to a contradiction, which establishes the proposition.

COROLLARY 1. *For a real finite function* f *defined on an open interval* $I \subset \mathbf{R}$ *and upper semi-continuous on* I *to be convex on* I *it is necessary and sufficient that for all* $x \in I$ *there should exist an* $\varepsilon > 0$ *such that the relation* $|h| \leqslant \varepsilon$ *entails*

$$f(x) \leqslant \frac{1}{2}\left(f(x + h) + f(x - h)\right).$$

We have only to show that the condition is *sufficient*. Indeed, if at a point M_a of the graph G of f a line D is locally above G, then it is locally on G at this point; for, in the opposite case, for example, a point M_{a+h} would be strictly below D, while a point M_{a-h} would be below D; the mid-point of the segment $M_{a-h}M_{a+h}$ would thus be strictly above D (fig. 5), and, in virtue of the hypothesis, M_a would *a fortiori* be strictly below D, which is absurd.
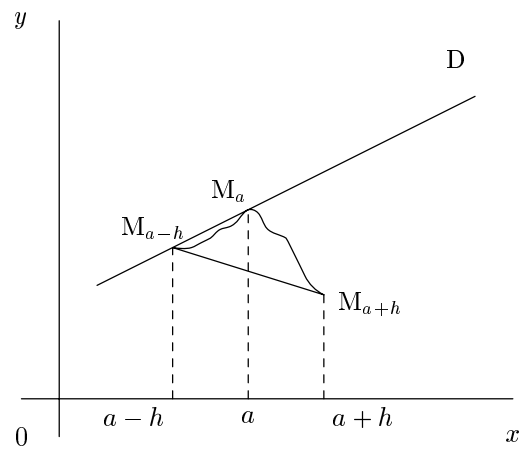
Fig. 5

COROLLARY 2. *Let f be a finite real function defined on an open interval $I \subset \mathbf{R}$. If for every point $x \in I$ there is an open interval $J_x \subset I$ containing x and such that the restriction of f to J_x is convex on J_x, then f is convex on I.*

It is clear that f satisfies the criterion of prop. 9.

EXERCISES

§1.

1) Let f be a vector function of a real variable, defined on an interval $I \subset \mathbf{R}$ and differentiable at a point x_0 interior to I. Show that the quotient

$$\frac{f(x_0 + h) - f(x_0 - k)}{h + k}$$

tends to $f'(x_0)$ as h and k tend to 0 *through values* > 0. Converse.

Show that the function f equal to $x^2 \sin 1/x$ for $x \neq 0$, and to 0 for $x = 0$, is everywhere differentiable, but that $(f(y) - f(z))/(y - z)$ does not approach $f'(0)$ as y and z tend to 0, while remaining distinct and > 0.

2) On the interval $I = [0, 1]$ we define a sequence of continuous real functions (f_n) inductively as follows: We take $f_0(x) = x$; for each integer $n \geqslant 1$ the function f_n is affine linear on each of the 3^n intervals $\left[\dfrac{k}{3^n}, \dfrac{k+1}{3^n} \right]$ for $0 \leqslant k \leqslant 3^n - 1$; further, we take

$$f_{n+1}\left(\frac{k}{3^n} \right) = f_n\left(\frac{k}{3^n} \right)$$

$$f_{n+1}\left(\frac{k}{3^n} + \frac{1}{3^{n+1}} \right) = f_n\left(\frac{k}{3^n} + \frac{2}{3^{n+1}} \right), \quad f_{n+1}\left(\frac{k}{3^n} + \frac{2}{3^{n+1}} \right) = f_n\left(\frac{k}{3^n} + \frac{1}{3^{n+1}} \right).$$

Show that the sequence (f_n) converges uniformly on I to a continuous function which has no derivative (finite or infinite) at any point of the interval $]0, 1[$ (use exerc. 1).

3) Let $\mathscr{C}(I)$ be the complete space of continuous finite real functions defined on the compact interval $I = [a, b]$ of $\mathbf{R}$, and endow $\mathscr{C}(I)$ with the topology of uniform convergence (*Gen. Top.*, X, p. 277). Let A be the subset of $\mathscr{C}(I)$ formed by the functions x such that for *at least one* point $t \in [a, b[$ (depending on the function x) the function x has a *finite* right derivative. Show that A is a *meagre* set in $\mathscr{C}(I)$ (*Gen. Top.*, IX, p. 192), and hence its complement, that is, the set of continuous functions on I not having a finite right derivative at *any point* of $[a, b[$ is a Baire subspace of $\mathscr{C}(I)$ (*Gen. Top.*, IX, p. 192). (Let A_n be the set of functions $x \in \mathscr{C}(I)$ such that for at least one value of t satisfying $a \leqslant t \leqslant b - 1/n$ (and depending on x) one has $|x(t') - x(t)| \leqslant n |t' - t|$ for all t' such that $t \leqslant t' \leqslant t + 1/n$. Show that each A_n is a closed *nowhere dense* set in $\mathscr{C}(I)$: remark that in $\mathscr{C}(I)$ each ball contains a function having bounded right derivative on $[a, b[$; on the other hand, for every $\varepsilon > 0$ and every integer $m > 0$ there exists on I a continuous function having at every point of $[a, b[$ a finite right derivative such that, for all $t \in [a, b[$ one has $|y(t)| \leqslant \varepsilon$ and $\left| y_r'(t) \right| \geqslant m$.)

4) Let E be a topological vector space over $\mathbf{R}$ and $\mathbf{f}$ a continuous vector function defined on an open interval $I \subset \mathbf{R}$, and having a right derivative and a left derivative at every point of I.

a) Let U be a nonempty open set in E, and A the subset of I formed by the points x such that $\mathbf{f}'_d(x) \in U$. Given a number $\alpha > 0$ let B be the subset of I formed by the points x such that there exists at least one $y \in I$ satisfying the conditions $x - \alpha \leqslant y < x$ and $(\mathbf{f}(x) - \mathbf{f}(y))/(x - y) \in U$; show that the set B is open and that $A \cap \complement B$ is countable (remark that this last set is formed by the left-hand endpoints of intervals contiguous to $\complement B$). Deduce that the set of points $x \in A$ such that $\mathbf{f}'_g(x) \notin \overline{U}$ is countable.

b) Suppose that E is a *normed* space; the image $\mathbf{f}(I)$ is then a metric space having a countable base, and the same is true for the closed vector subspace F of E generated by $\mathbf{f}(I)$, a subspace which contains $\mathbf{f}'_d(I)$ and $\mathbf{f}'_g(I)$. Deduce from a) that the set of points $x \in I$ such that $\mathbf{f}'_d(x) \neq \mathbf{f}'_g(x)$ is *countable*. (If (U_m) is a countable base for the topology of F note that for two distinct points a, b of F there exist two disjoint sets U_p, U_q such that $a \in U_p$ and $b \in U_q$.)

c) Take for E the product $\mathbf{R}^I$ (the space of mappings from I into $\mathbf{R}$, endowed with the topology of simple convergence), and for each $x \in I$ denote by $\mathbf{g}(x)$ the map $t \mapsto |x - t|$ of I into $\mathbf{R}$. Show that $\mathbf{g}$ is continuous and that, for every $x \in I$, one has $\mathbf{g}'_r(x) \neq \mathbf{g}'_l(x)$.

5) Let $\mathbf{f}$ be a continuous vector function defined on an open interval $I \subset \mathbf{R}$ with values in a normed space E over $\mathbf{R}$, and admitting a right derivative at every point of I.

a) Show that the set of points $x \in I$ such that $\mathbf{f}'_d$ is bounded on a neighbourhood of x is an open set dense in I (use th. 2 of *Gen. Top.*, IX, p. 194).

b) Show that the set of points of I where $\mathbf{f}'_d$ is continuous is the complement of a *meagre* subset of I (*cf. Gen. Top.*, IX, p. 255, exerc. 21).

6) Let (r_n) be the sequence formed by the rational numbers in $[0, 1]$, arranged in a certain order. Show that the function $f(x) = \sum_{n=0}^{\infty} 2^{-n}(x - r_n)^{1/3}$ is continuous and differentiable at every point of $\mathbf{R}$, and has an infinite derivative at every point r_n. (To see that f is differentiable at a point x distinct from the r_n, distinguish two cases, according to whether the series with general term $2^{-n}(x - r_n)^{-2/3}$ has sum $+\infty$ or converges; in the second case, note for all $x \neq 0$ and all $y \neq x$, one has

$$0 \leqslant (y^{1/3} - x^{1/3})/(y - x) \leqslant 4/3x^{2/3}).$$

7) Let f be a real function defined on an interval $I \subset \mathbf{R}$, admitting a right derivative $f'_d(x_0) = 0$ at a point of I, and let $\mathbf{g}$ be a vector function defined on a neighbourhood of $y_0 = f(x_0)$, having a right derivative and a left derivative (not necessarily equal) at this point. Show that $\mathbf{g} \circ f$ has a right derivative equal to 0 at the point x_0.

8) Let f be a mapping from $\mathbf{R}$ to itself such that the set C of points of $\mathbf{R}$ where f is continuous is dense in $\mathbf{R}$, and such that the complement A of C is also dense. Show that the set D of points of C where f is right differentiable is meagre. (For each integer n, let E_n be the set of points $a \in \mathbf{R}$ such that there exist two points x, y such that $0 < x - a < 1/n$, $0 < y - a < 1/n$ and

$$\left| \frac{f(x) - f(a)}{x - a} - \frac{f(y) - f(a)}{y - a} \right| > 1.$$

Show that the interior of E_n is dense in $\mathbf{R}$. For this, note that for every open nonempty interval I in $\mathbf{R}$ there is a point $b \in I \cap A$; show that for $a < b$ and $b - a$ sufficiently small one has $a \in E_n$.)

9) Let $\mathcal{B}(\mathbf{N})$ be the space of bounded sequences $\mathbf{x} = (x_n)_{n \in \mathbf{N}}$ of real numbers, endowed with the norm $\|\mathbf{x}\| = \sup |x_n|$; give an example of a continuous map $t \mapsto \mathbf{f}(t) = (f_n(t))_{n \in \mathbf{N}}$ of $\mathbf{R}$ into $\mathcal{B}(\mathbf{N})$ such that each of the functions f_n is differentiable for $t = 0$, but $\mathbf{f}$ is not differentiable at this point.

§ 2.

1) Let f be a real function defined and left continuous on an open interval $I = \,]a, b[$ in $\mathbf{R}$; suppose that at all the points of the complement B with respect to I of a countable subset of I the function f is *increasing to the right,* that is, at every point $x \in B$ there exists a y such that $x < y \leqslant b$ and such that for all z such that $x \leqslant z < y$ one has $f(x) \leqslant f(z)$. Show that f is increasing on I (argue as in prop. 2).

2) In the field $\mathbf{Q}_p$ of p-adic numbers (*Gen. Top.*, III, p. 322, exerc. 23) every p-adic integer $x \in \mathbf{Z}_p$ has one and only one expansion in the form $x = a_0 + a_1 p + \cdots + a_n p^n + \cdots$, where the a_j are rational integers such that $0 \leqslant a_j \leqslant p - 1$ for each j. For each $z \in \mathbf{Z}_p$ put

$$f(x) = a_0 + a_1 p^2 + \cdots + a_n p^{2n} + \cdots;$$

show that, on $\mathbf{Z}_p$, f is a continuous function which is not constant on a neighbourhood of any point yet has a *zero derivative* at every point.

3) *a*) Let K be the triadic Cantor set (*Gen. Top.*, IV, p. 338), let $I_{n,p}$ be the 2^n contiguous intervals of K with length $1/3^{n+1}$ ($1 \leqslant p \leqslant 2^n$), and $K_{n,p}$ the 2^{n+1} closed intervals of length $1/3^{n+1}$ whose union is the complement of the union of the $I_{m,p}$ for $m \leqslant n$. Let α be a number such that $1 < \alpha < 3/2$; for each n we denote by f_n the continuous increasing function on $[0, 1]$ which is equal to 0 for $x = 0$, constant on each of the intervals $I_{m,p}$ for $m \leqslant n$, is affine linear on each of the intervals $K_{n,p}$ ($1 \leqslant p \leqslant 2^{n+1}$) and such that $f'_d(x) = \alpha^{n+1}$ on each of the interiors of these last intervals. Show that the series with general term f_n is uniformly convergent on $[0, 1]$, that its sum is a function f which admits a right derivative (finite or not) everywhere in $[0, 1[$, and that one has $f'_r(x) = +\infty$ at every point of K distinct from the left-hand endpoints of the contiguous intervals $I_{n,p}$.

b) Let g be a continuous increasing map of $[0, 1]$ onto itself, constant on each of the intervals $I_{n,p}$ (*Gen. Top.*, IV, p. 403, exerc. 9). If $h = f + g$, show that h admits a right derivative equal to $f'_d(x)$ at every point x of $[0, 1[$.

4) Let f be a finite real function, continuous on a compact interval $[a, b]$ in $\mathbf{R}$, and having a right derivative at every point of the open interval $]a, b[$. Let m and M be the greatest lower bound and least upper bound (finite or not) of f'_d over $]a, b[$.

a) Show that when x and y run through $]a, b[$ keeping $x \neq y$, the set of values of $(f(x) - f(y))/(x - y)$ contains $]m, M[$ and is contained in $[m, M]$. (Reduce to proving that if f'_d takes two values of opposite sign at the two points c, d of $]a, b[$ (with $c < d$), then there exist two distinct points of the interval $]c, d[$ where f takes the same value).

b) If, further, f has a left derivative at every point of $]a, b[$ then the infima (resp. suprema) of f'_d and f'_g over $]a, b[$ are equal.

c) Deduce that if f is differentiable on $]a,\ b[$ then the image under f' of every interval contained in $]a,\ b[$ is itself an interval, and consequently *connected* (use *a*)).

5) Let **f** be the vector mapping of $I = [0,\ 1]$ into $\mathbf{R}^3$ defined as follows: for $0 \leqslant t \leqslant \frac{1}{4}$, $\mathbf{f}(t) = (-4t, 0, 0)$; for $\frac{1}{4} \leqslant t \leqslant \frac{1}{2}$ let $\mathbf{f}(t) = (-1, 4t - 1, 0)$; for $\frac{1}{2} \leqslant t \leqslant \frac{3}{4}$ let $\mathbf{f}(t) = (-1, 1, 4t - 2)$; finally, for $\frac{3}{4} \leqslant t \leqslant 1$ take $\mathbf{f}(t) = (4t - 4, 1, 1)$. Show that the convex set generated by the set $\mathbf{f}'_d(I)$ is not identical to the closure of the set of values of $\dfrac{\mathbf{f}(y) - \mathbf{f}(x)}{y - x}$ as (x, y) runs through the set of pairs of distinct points of I (*cf.* exerc. 4 *a*)).

6) On the interval $I = [-1, +1]$ consider the vector function **f**, with values in $\mathbf{R}^2$, defined as follows: $\mathbf{f}(t) = (0, 0)$ for $-1 \leqslant t \leqslant 0$;

$$\mathbf{f}(t) = \left(t^2 \sin \frac{1}{t}, \ t^2 \cos \frac{1}{t} \right)$$

for $0 \leqslant t \leqslant 1$. Show that **f** is differentiable on $]-1, +1[$ but that the image of this interval under $\mathbf{f}'$ is not a connected set in $\mathbf{R}^2$ (*cf.* exerc. 4 *c*)).

7) Let **f** be a continuous vector function defined on an open interval $I \subset \mathbf{R}$, with values in a normed space E over $\mathbf{R}$, and admitting a right derivative at every point of I. Show that the set of points of I where **f** admits a derivative is the complement of a meagre subset of I (use exerc. 5 *b*) of I, p. 36, and prop. 6 of I, p. 18).

8) Consider, on the interval $[0, 1]$, a family $(I_{n,p})$ of pairwise disjoint open intervals, defined inductively as follows: the integer n takes all values $\geqslant 0$; for each value of n the integer p takes the values $1, 2, \ldots, 2^n$; one has $I_{0,1} =]\frac{1}{3}, \frac{2}{3}[$; if J_n is the union of the intervals $I_{m,p}$ corresponding to the numbers $m \leqslant n$, the complement of J_n is the union of 2^{n+1} pairwise disjoint closed intervals $K_{n,p}$ $(1 \leqslant p \leqslant 2^{n+1})$. If $K_{n,p}$ is an interval $[a,\ b]$ one then takes for $I_{n+1,p}$ the open interval with endpoints $b - \dfrac{b - a}{3}\left(1 + \dfrac{1}{2^n}\right)$ and $b - \dfrac{b - a}{3 \cdot 2^n}$. Let E be the perfect set which is the complement with respect to $[0, 1]$ of the union of the $I_{n,p}$. Define on $[0, 1]$ a continuous real function f which admits a right derivative at every point of $[0, 1[$, but fails to have a left derivative at the uncountable subset of E of points distinct from the endpoints of intervals contiguous to E (*cf.* exerc. 7). (Take $f(x) = 0$ on E, define f suitably on each of the intervals $I_{n,p}$ in such a way that for every $x \in E$ there are points $y < x$ not belonging to E, arbitrarily close to x, and such that $\dfrac{f(y) - f(x)}{y - x} = -1$.)

9) Let f and g be two finite real functions, continuous on $[a,\ b]$, both having a finite derivative on $]a,\ b[$; show that there exists a c such that $a < c < b$ and that

$$\begin{vmatrix} f(b) - f(a) & g(b) - g(a) \\ f'(c) & g'(c) \end{vmatrix} = 0.$$

¶ 10) Let f and g be two finite real functions, strictly positive, continuous and differentiable on an open interval I. Show that if f' and g' are strictly positive and f'/g' is strictly increasing on I, then either f/g is strictly increasing on I, or else there exists a number

$c \in$ I such that f/g is strictly decreasing for $x \leqslant c$ and strictly increasing for $x \geqslant c$ (note that if one has $f'(x)/g'(x) < f(x)/g(x)$ then also

$$f'(y)/g'(y) < f(y)/g(y)$$

for all $y < x$).

11) Let f be a complex function, continuous on an open interval I, vanishing nowhere, and admitting a right derivative at every point of I. For $|f|$ to be increasing on I it is necessary and sufficient that $\mathcal{R}(f'_d/f) \geqslant 0$ on I.

¶ 12) Let f be a differentiable real function on an open interval I, g its derivative on I, and $[a, b]$ a compact interval contained in I; suppose that g is differentiable on the open interval $]a, b[$ but not necessarily right (resp. left) continuous at the point a (resp. b); show that there exists c such that $a < c < b$ and that

$$g(b) - g(a) = (b - a)g'(c)$$

(use exerc. 4 c) of I, p. 36).

13) One terms the *symmetric derivative* of a vector function $\mathbf{f}$ at a point x_0 interior to the interval where $\mathbf{f}$ is defined, the limit (when it exists) of $\dfrac{\mathbf{f}(x_0 + h) - \mathbf{f}(x_0 - h)}{2h}$ as h tends to 0 remaining > 0.

a) Generalize to the symmetric derivative the rules of calculus established in § 1 for the derivative.

b) Show that theorems 1 and 2 of § 2 remain valid when one replaces the words "right derivative" by "symmetric derivative".

14) Let $\mathbf{f}$ be a vector function defined and continuous on a compact interval I $= [a, b]$ in $\mathbf{R}$, with values in a normed space over $\mathbf{R}$. Suppose that $\mathbf{f}$ admits a right derivative at all points of the complement with respect to $[a, b[$ of a countable subset A of this interval. Show that there exists a point $x \in]a, b[\cap \complement A$ such that

$$\|\mathbf{f}(b) - \mathbf{f}(a)\| \leqslant \|\mathbf{f}'_d(x)\| (b - a).$$

(Argue by contradiction, decomposing $[a, b[$ into three intervals $[a, t[,\ [t, t + h[$ and $[t + h, b[$ with $t \notin$ A; if $k = \|\mathbf{f}(b) - \mathbf{f}(a)\|/(b - a)$, note that for h sufficiently small one has $\|\mathbf{f}(t + h) - \mathbf{f}(t)\| < k.h$, and use th. 2 of I, p. 15 for the other intervals.)

§ 3.

1) With the same hypotheses as in prop. 2 of I, p. 20 prove the formula

$$[\mathbf{f}^{(n)}.\mathbf{g}] = \sum_{p=0}^{n} (-1)^p \binom{n}{p} D^{n-p}[\mathbf{f}.\mathbf{g}^{(p)}].$$

2) With the notation of prop. 2 of I, p. 28 suppose that the relation $[\mathbf{a}.\mathbf{y}] = 0$ for all $\mathbf{y} \in$ F implies that $\mathbf{a} = 0$ in E. Under these conditions, if $\mathbf{g}_i$ ($0 \leqslant i \leqslant n$) are $n + 1$ vector

functions with values in E, defined on an interval I of $\mathbf{R}$ and such that for *every* vector function $\mathbf{f}$ with values in F and n times differentiable on I, one has identically

$$[\mathbf{g}_0.\mathbf{f}] + [\mathbf{g}_1.\mathbf{f}'] + \cdots + [\mathbf{g}_n.\mathbf{f}^{(n)}] = 0$$

then the functions $\mathbf{g}_i$ are identically zero.

3) With the notation of exerc. 2 and the same hypothesis on $[\mathbf{x}.\mathbf{y}]$ suppose that each of the functions $\mathbf{g}_k$ is n times differentiable on I; for each function $\mathbf{f}$ which is n times differentiable on I, with values in F, put

$$[\mathbf{g}_0.\mathbf{f}] - [\mathbf{g}_1.\mathbf{f}]' + [\mathbf{g}_2.\mathbf{f}]'' + \cdots + (-1)^n [\mathbf{g}_n.\mathbf{f}]^{(n)} = [\mathbf{h}_0.\mathbf{f}] + [\mathbf{h}_1.\mathbf{f}'] + \cdots + [\mathbf{h}_n.\mathbf{f}^{(n)}],$$

which defines the functions $\mathbf{h}_i$ $(0 \leqslant i \leqslant n)$ without ambiguity (exerc. 2); show that one has

$$[\mathbf{h}_0.\mathbf{f}] - [\mathbf{h}.\mathbf{f}]' + [\mathbf{h}_2.\mathbf{f}]'' + \cdots + (-1)^n [\mathbf{h}_n.\mathbf{f}]^{(n)} = [\mathbf{g}_0.\mathbf{f}] + [\mathbf{g}_1.\mathbf{f}'] + \cdots + [\mathbf{g}_n.\mathbf{f}^{(n)}]$$

identically.

4) Let $\mathbf{f}$ be a vector function which is n times differentiable on an interval I $\subset \mathbf{R}$. Show that for $1/x \in$ I one has identically

$$\frac{1}{x^{n+1}}\mathbf{f}^{(n)}\left(\frac{1}{x}\right) = (-1)^n D^n\left(x^{n-1}\mathbf{f}\left(\frac{1}{x}\right)\right)$$

(argue inductively on n).

5) Let u and v be two real functions which are n times differentiable on an interval I $\subset \mathbf{R}$. If one puts $D^n(u/v) = (-1)^n w_n / v^{n+1}$ at every point where $v(x) \neq 0$, show that

$$w_n = \begin{vmatrix} u & v & 0 & 0 & \ldots & 0 \\ u' & v' & v & 0 & \ldots & 0 \\ u'' & v'' & 2v' & v & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ u^{(n-1)} & v^{(n-1)} & \binom{n-1}{1}v^{(n-2)} & \binom{n-1}{2}v^{(n-3)} & \ldots & v \\ u^{(n)} & v^{(n)} & \binom{n}{1}v^{(n-1)} & \binom{n}{2}v^{(n-2)} & \ldots & \binom{n}{n-1}v' \end{vmatrix}$$

(put $w = u/v$ and differentiate n times the relation $u = wv$).

6) Let $\mathbf{f}$ be a vector function defined on an open interval I $\subset \mathbf{R}$, taking values in a normed space E.

Put $\Delta\mathbf{f}(x; h_1) = \mathbf{f}(x + h_1) - \mathbf{f}(x)$, and then, inductively, define

$$\Delta^p\mathbf{f}(x; h_1, h_2, \ldots, h_{p-1}, h_p) = \Delta^{p-1}\mathbf{f}(x + h_p; h_1, \ldots, h_{p-1}) - \Delta^{p-1}\mathbf{f}(x; h_1, \ldots, h_{p-1});$$

these functions are defined for each $x \in$ I when the h_i are small enough.

a) If the function **f** is *n* times differentiable *at the point x* (and so $n-1$ times differentiable on a neighbourhood of *x*), one has

$$\lim_{\substack{(h_1, \ldots, h_n) \to (0, \ldots, 0) \\ h_1 h_2 \ldots h_n \neq 0}} \frac{\Delta^n \mathbf{f}(x; h_1, \ldots, h_n)}{h_1 h_2 \ldots h_n} = \mathbf{f}^{(n)}(x)$$

(argue by induction on *n*, using the mean value theorem).

b) If **f** is *n* times differentiable *on the interval* I, one has

$$\left\| \Delta^n \mathbf{f}(x; h_1, \ldots, h_n) - \mathbf{f}^{(n)}(x_0) h_1 h_2 \ldots h_n \right\|$$
$$\leqslant |h_1 h_2 \ldots h_n| \ \sup \left\| \mathbf{f}^{(n)}(x + t_1 h_1 + \cdots + t_n h_n) - \mathbf{f}^{(n)}(x_0) \right\|$$

the supremum being taken over the set of (t_i) such that $0 \leqslant t_i \leqslant 1$ for $1 \leqslant i \leqslant n$ (same method).

c) If *f* is a real function which is *n* times differentiable on I, one has

$$\Delta^n f(x; h_1, h_2, \ldots, h_n) = h_1 h_2 \ldots h_n \, f^{(n)}(x + \theta_1 h_1 + \cdots + \theta_n h_n)$$

the numbers θ_i belonging to $[0, 1]$ (same method, using I, p. 22, corollary).

7) Let *f* be a finite real function *n* times differentiable at the point x_0, and **g** a vector function which is *n* times differentiable at the point $y_0 = f(x_0)$. Let

$$f(x_0 + h) = a_0 + a_1 h + \cdots + a_n h^n + r_n(h)$$
$$\mathbf{g}(y_0 + k) = \mathbf{b}_0 + \mathbf{b}_1 k + \cdots + \mathbf{b}_n k^n + \mathbf{s}_n(k)$$

be the Taylor expansions of order *n* of *f* and **g** at the points x_0 and y_0 respectively. Show that the sum of the $n+1$ terms of the Taylor expansion of order *n* of the composite function $\mathbf{g} \circ f$ at the point x_0 is equal to the sum of the terms of degree $\leqslant n$ in the polynomial

$$\mathbf{b}_0 + \mathbf{b}_1(a_1 h + \cdots + a_n h^n) + \mathbf{b}_2(a_1 h + \cdots + a_n h^n)^2 + \cdots + \mathbf{b}_n(a_1 h + \cdots + a_n h^n)^n.$$

Deduce the two following formulae:

a)
$$D^n(\mathbf{g}(f(x))) = \sum \frac{n!}{m_1! m_2! \ldots m_q!} \, \mathbf{g}^{(p)}(f(x)) \left(\frac{f'(x)}{1!} \right)^{m_1} \ldots \left(\frac{f^{(q)}(x)}{q!} \right)^{m_q}$$

the sum being taken over all systems of positive integers $(m_i)_{1 \leqslant i \leqslant q}$ such that

$$m_1 + 2m_2 + \cdots + q m_q = n$$

where *p* denotes the sum $m_1 + m_2 + \cdots + m_q$.

b)
$$D^n(\mathbf{g}(f(x))) = \sum_{p=1}^n \frac{1}{p!} \, \mathbf{g}^{(p)}(f(x)) \left(\sum_{q=1}^p \binom{p}{q} (-f(x))^{p-q} \, D^n((f(x))^q) \right).$$

8) Let *f* be a real function defined and *n* times differentiable on an interval I, let $x_1, x_2, \ldots, x_p$ be distinct points of I, and n_i $(1 \leqslant i \leqslant p)$ be *p* integers > 0 such that

$$n_1 + n_2 + \cdots + n_p = n.$$

Suppose that at the point x_i the function f vanishes together with its first $n_i - 1$ derivatives for $1 \leqslant i \leqslant p$: show that there is a point ξ interior to the smallest interval that contains the x_i and such that $f^{(n-1)}(\xi) = 0$.

9) With the same notation as in exerc. 8 suppose that f is n times differentiable on I but otherwise arbitrary. Let g be the polynomial of degree $n - 1$ (with real coefficients) such that at the point x_i $(1 \leqslant i \leqslant p)$ both g and its first $n_i - 1$ derivatives are respectively equal to f and its first $n_i - 1$ derivatives. Show that we have

$$f(x) = g(x) + \frac{(x - x_1)^{n_1}(x - x_2)^{n_2} \ldots (x - x_p)^{n_p}}{n!} \, f^{(n)}(\xi)$$

where ξ is interior to the smallest interval containing the points x_i $(1 \leqslant i \leqslant p)$ and x. (Apply exerc. 8 to the function of t

$$f(t) - g(t) - a \frac{(t - x_1)^{n_1}(t - x_2)^{n_2} \ldots (t - x_p)^{n_p}}{n!}$$

where a is a suitably chosen constant.)

10) Let g be an *odd* real function defined on a neighbourhood of 0, and 5 times differentiable on this neighbourhood. Show that

$$g(x) = \frac{x}{3} \left(g'(x) + 2g'(0) \right) - \frac{x^5}{180} g^{(5)}(\xi) \qquad\qquad (\xi = \theta x, \quad 0 < \theta < 1)$$

(same method as in exerc. 9).

Deduce that if f is a real function defined on $[a, b]$ and 5 times differentiable on this interval, then

$$f(b) - f(a) = \frac{b - a}{6} \left[f'(a) + f'(b) + 4f'\left(\frac{a + b}{2}\right) \right] - \frac{(b - a)^5}{2880} f^{(5)}(\xi)$$

with $a < \xi < b$ ("Simpson's formula").

11) Let $f_1, f_2, \ldots, f_n$ and $g_1, g_2, \ldots, g_n$ be $2n$ real functions which are $n - 1$ times differentiable on an interval I. Let $(x_i)_{1 \leqslant i \leqslant n}$ be a strictly increasing sequence of points in I. Show that the ratio of the two determinants

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_n) \\ \cdots & \cdots & \cdots & \cdots \\ f_n(x_1) & f_n(x_2) & \cdots & f_n(x_n) \end{vmatrix} : \begin{vmatrix} g_1(x_1) & g_1(x_2) & \cdots & g_1(x_n) \\ g_2(x_1) & g_2(x_2) & \cdots & g_2(x_n) \\ \cdots & \cdots & \cdots & \cdots \\ g_n(x_1) & g_n(x_2) & \cdots & g_n(x_n) \end{vmatrix}$$

is equal to the ratio of the two determinants

$$\begin{vmatrix} f_1(\xi_1) & f_1'(\xi_2) & \cdots & f_1^{(n-1)}(\xi_n) \\ f_2(\xi_1) & f_2'(\xi_2) & \cdots & f_2^{(n-1)}(\xi_n) \\ \cdots & \cdots & \cdots & \cdots \\ f_n(\xi_1) & f_n'(\xi_2) & \cdots & f_n^{(n-1)}(\xi_n) \end{vmatrix} : \begin{vmatrix} g_1(\xi_1) & g_1'(\xi_2) & \cdots & g_1^{(n-1)}(\xi_n) \\ g_2(\xi_1) & g_2'(\xi_2) & \cdots & g_2^{(n-1)}(\xi_n) \\ \cdots & \cdots & \cdots & \cdots \\ g_n(\xi_1) & g_n'(\xi_2) & \cdots & g_n^{(n-1)}(\xi_n) \end{vmatrix}$$

where

$$\xi_1 = x_1, \quad \xi_1 < \xi_2 < x_2, \quad \xi_2 < \xi_3 < x_3, \quad \ldots, \quad \xi_{n-1} < \xi_n < x_n$$

(apply exerc. 9 of I, p. 38).

Particular case where $g_1(x) = 1, g_2(x) = x, \ldots, g_n(x) = x^{n-1}$.

¶ 12) *a*) Let **f** be a vector function defined and continuous on the finite interval $I = [-a, +a]$, taking its values in a normed space E over **R** and twice differentiable on I. If one puts $M_0 = \sup_{x \in I} \|\mathbf{f}(x)\|$, $M_2 = \sup_{x \in I} \|\mathbf{f}''(x)\|$, show that for all $x \in I$ one has

$$\|\mathbf{f}'(x)\| \leqslant \frac{M_0}{a} + \frac{x^2 + a^2}{2a} M_2$$

(express each of the differences $\mathbf{f}(a) - \mathbf{f}(x)$, $\mathbf{f}(-a) - \mathbf{f}(x)$).

b) Deduce from *a*) that if **f** is a twice differentiable function on an interval I (bounded or not), and if $M_0 = \sup_{x \in I} \|\mathbf{f}(x)\|$ and $M_2 = \sup_{x \in I} \|\mathbf{f}''(x)\|$ are finite, then so is $M_1 = \sup_{x \in I} \|\mathbf{f}'(x)\|$, and one has:

$$M_1 \leqslant 2\sqrt{M_0 M_2} \qquad \text{if I has length} \geqslant 2\sqrt{\frac{M_0}{M_2}}$$

$$M_1 \leqslant \sqrt{2}\sqrt{M_0 M_2} \qquad \text{if I} = \mathbf{R}.$$

Show that in these two inequalities the numbers 2 and $\sqrt{2}$ respectively cannot be replaced by smaller numbers (consider first the case where one supposes merely that **f** admits a second right derivative, and show that in this case the two terms of the preceding inequalities can become equal, taking for **f** a real function equal "in pieces" to second degree polynomials).

c) Deduce from *b*) that if **f** is *p* times differentiable on **R**, and if $M_p = \sup_{x \in \mathbf{R}} \|\mathbf{f}^{(p)}(x)\|$ and $M_0 = \sup_{x \in \mathbf{R}} \|\mathbf{f}(x)\|$ are finite, then each of the numbers $M_k = \sup_{x \in \mathbf{R}} \|\mathbf{f}^{(k)}(x)\|$ is finite (for $1 \leqslant k \leqslant p - 1$) and

$$M_k \leqslant 2^{k(p-k)/2} \, M_0^{1-k/p} \, M_p^{k/p}.$$

¶ 13) *a*) Let f be a twice differentiable real function on **R**, such that $(f(x))^2 \leqslant a$ and $(f'(x))^2 + (f''(x))^2 \leqslant b$ on **R**; show that

$$(f(x))^2 + (f'(x))^2 \leqslant \max(a, b)$$

on **R** (argue by contradiction, noting that if the function $f^2 + f'^2$ takes a value $c > \max(a, b)$ at a point x_0 then there exist two points x_1, x_2 such that $x_1 < x_0 < x_2$ and that at x_1 and x_2 the function f' takes values small enough that $f^2 + f'^2$ takes values $< c$; then consider a point of $[x_1, x_2]$ where $f^2 + f'^2$ attains its supremum on this interval).

b) Let f be a real function n times differentiable on **R**, and such that $(f(x))^2 \leqslant a$ and $(f^{(n-1)}(x))^2 + (f^{(n)}(x))^2 \leqslant b$ on **R**; show that then

$$(f^{(k-1)}(x))^2 + (f^{(k)}(x))^2 \leqslant \max(a, b)$$

on **R** for $1 \leqslant k \leqslant n$. (Argue by induction on n; note that, by exerc. 12 the supremum c of $(f'(x))^2$ on **R** is finite; show that one necessarily has $c \leqslant \max(a, b)$ by reducing to a contradiction: assuming that $c > \max(a, b)$ choose the constants λ and μ so that

for the function $g = \lambda f + \mu$ one has $|g(x)| \leqslant 1$, $|g'(x)| \leqslant 1$, yet one cannot have $(g(x))^2 + (g'(x))^2 \leqslant 1$ for all x.)

¶ 14) Let $\mathbf{f}$ be a function which is $n - 1$ times differentiable on an interval I containing 0, and let $\mathbf{f}_n$ be the vector function defined for $x \neq 0$ on I by the relation

$$\mathbf{f}(x) = \mathbf{f}(0) + \mathbf{f}'(0)\frac{x}{1!} + \mathbf{f}''(0)\frac{x^2}{2!} + \cdots + \mathbf{f}^{(n-1)}(0)\frac{x^{n-1}}{(n-1)!} + \mathbf{f}_n(x)x^n.$$

a) Show that if $\mathbf{f}$ has an $(n + p)^{th}$ derivative at the point 0 then $\mathbf{f}_n$ has a pth derivative at the point 0 and an $(n + p - 1)^{th}$ derivative at all points of a neighbourhood of 0 distinct from 0; moreover, one has $\mathbf{f}_n^{(k)}(0) = \dfrac{k!}{(n+k)!}\mathbf{f}^{(n+k)}(0)$ for $0 \leqslant k \leqslant p$, and $\mathbf{f}_n^{(p+k)}(x)x^k$ tends to 0 with x, for $1 \leqslant k \leqslant n - 1$ (express the derivatives of $\mathbf{f}_n$ with the help of the Taylor expansions of the successive derivatives of $\mathbf{f}$, and use prop. 6 of I, p. 18).

b) Conversely, let $\mathbf{f}_n$ be a vector function admitting an $(n + p - 1)^{th}$ derivative on a neighbourhood of 0 in I, and such that $\mathbf{f}_n^{(p+k)}(x)x^k$ has a limit for $0 \leqslant k \leqslant n - 1$. Show that the function $\mathbf{f}_n(x)x^n$ has an $(n + p - 1)^{th}$ derivative on I; if, further, $\mathbf{f}_n$ admits a pth derivative at the point 0, then $\mathbf{f}_n(x)x^n$ admits an $(n + p)^{th}$ derivative at the point 0.

c) Suppose that I is symmetric with respect to 0 and that $\mathbf{f}$ is *even* ($\mathbf{f}(-x) = \mathbf{f}(x)$ on I). Show, with the help of a), that if $\mathbf{f}$ is $2n$ times differentiable on I, then there exists a function $\mathbf{g}$ defined and n times differentiable on I, such that $\mathbf{f}(x) = \mathbf{g}(x^2)$ on I.

¶ 15) Let I be an open interval in $\mathbf{R}$, and $\mathbf{f}$ a vector function defined and continuous on I; suppose that there are n vector functions $\mathbf{g}_i$ ($1 \leqslant i \leqslant n$) defined on I, and such that the function of x

$$\frac{1}{h^n}\left(\mathbf{f}(x + h) - \mathbf{f}(x) - \sum_{p=1}^{n} \frac{h^p}{p!}\,\mathbf{g}_p(x)\right)$$

tends *uniformly* to 0 on every compact interval contained in I as h tends to 0.

a) We put $\mathbf{f}_p(x, h) = \Delta^p \mathbf{f}(x; h, h, \ldots, h)$ (I, p. 40, exerc. 6). Show that, for $1 \leqslant p \leqslant n$, $(1/h^p)\mathbf{f}_p(x, h)$ tends *uniformly* to $\mathbf{g}_p(x)$ on every compact subinterval of I as h tends to 0, and that the $\mathbf{g}_p$ are continuous on I (prove this successively for $p = n$, $p = n - 1$, etc.)

b) Deduce from this that $\mathbf{f}$ has a continuous n^{th} derivative and that $\mathbf{f}^{(p)} = \mathbf{g}_p$ for $1 \leqslant p \leqslant n$ (taking account of the relation $\mathbf{f}_{p+1}(x, h) = \mathbf{f}(x + h, h) - \mathbf{f}_p(x, h)$).

¶ 16) Let f be a real function n times differentiable on I $= \ ]-1, +1[$, and such that $|f(x)| \leqslant 1$ on this interval.

a) Show that if $m_k(\lambda)$ denotes the minimum of $\left|f^{(k)}(x)\right|$ on an interval of length λ contained in I then one has

$$m_k(\lambda) \leqslant \frac{2^{k(k+1)/2}\,k^k}{\lambda^k} \qquad\qquad (1 \leqslant k \leqslant n).$$

(Note that if the interval of length λ is decomposed into three intervals of lengths α, β, γ, one has

$$m_k(\lambda) \leqslant \frac{1}{\beta}(m_{k-1}(\alpha) + m_{k-1}(\gamma)).)$$

b) Deduce from a) that there exists a number μ_n depending only on the integer n such that if $|f'(0)| \geqslant \mu_n$, then the derivative $f^{(n)}(x)$ vanishes on at least $n - 1$ distinct points of I (show by induction on k that $f^{(k)}$ vanishes at least $k - 1$ times on I).

17) *a*) Let $\mathbf{f}$ be a vector function having derivatives of all orders on an open interval $I \subset \mathbf{R}$. Suppose that, on I, one has $\left\| \mathbf{f}^{(n)}(x) \right\| \leqslant a\, n! r^n$, where a and r are two numbers > 0 and independent of x and n; show that at each point x_0 the "Taylor series" with general term $(1/n!)\, \mathbf{f}^{(n)}(x_0)(x - x_0)^n$ is convergent, and has sum $\mathbf{f}(x)$ on some neighbourhood of x_0.

b) Conversely, if the Taylor series for $\mathbf{f}$ at a point x_0 converges on a neighbourhood of x_0 there exist two numbers a and r (depending on x_0) such that $\left\| \mathbf{f}^{(n)}(x_0) \right\| \leqslant a.n! r^n$ for every integer $n > 0$.

c) Deduce from *a*) and exerc. 16 *b*) that if, on an open interval $I \subset \mathbf{R}$, a real function f is indefinitely differentiable and if there is an integer p independent of n such that, for all n, the function $f^{(n)}$ does not vanish at more than p distinct points of I, then the Taylor series of f on a neighbourhood of each point $x_0 \in I$ is convergent, and has sum $f(x)$ at every point of a neighbourhood of x_0.

18) Let $(a_n)_{n \geqslant 0}$ be an arbitrary sequence of complex numbers. For each $n \geqslant 0$ put $s_n^{(0)} = a_n$, and, inductively, for $k \geqslant 0$, define

$$s_n^{(k+1)} = s_0^{(k)} + s_1^{(k)} + \cdots + s_{n-1}^{(k)}.$$

a) Prove "Taylor's formula for sequences": for each integer

$$\left| s_{n+h}^{(k)} - s_n^{(k)} - h s_n^{(k-1)} - \binom{h}{2} s_n^{(k-2)} - \cdots - \binom{h}{k-1} s_n^{(1)} \right| \leqslant \binom{h}{k} \sup_{0 \leqslant j \leqslant h-1} \left| a_{n+j} \right|$$

(proceed by induction on k).

b) Suppose that there is a number C such that $|na_n| \leqslant C$ for all n, and that the sequence $(s^{(2)}/n)$ formed by the arithmetic means $(s_0 + \cdots + s_{n-1})/n$ of the partial sums $s_n = a_0 + \cdots + a_{n-1}$ tends to a limit σ. Show that the series with general term a_n is convergent and has sum σ ("Hardy-Littlewood tauberian theorem").   (Write

$$s_n = \frac{1}{h} \left(s_{n+h}^{(2)} - s^{(2)} \right) + \frac{h-1}{2} r_n$$

where $|r_n|$ is bounded above with the aid of the inequality $|na_n| \leqslant C$, and h is chosen suitably as a function of n.)

§ 4.

1) *a*) Let H be a set of convex functions on a compact interval $[a,\, b] \subset \mathbf{R}$; suppose that the sets $H(a)$ and $H(b)$ are bounded above in $\mathbf{R}$ and that there exists a point c such that $a < c < b$ and that $H(c)$ is bounded below in $\mathbf{R}$; show that H is an *equicontinuous* set on $]a,\, b[$ (*Gen. Top.*, X, p. 283).

b) Let H be a set of convex functions on an interval $I \subset \mathbf{R}$, and let $\mathfrak{F}$ be a filter on H which converges pointwise on I to a function f_0; show that $\mathfrak{F}$ converges uniformly to f_0 on every compact interval contained in I.

2) Show that every convex function f on a compact interval $I \subset \mathbf{R}$ is the limit of a decreasing uniformly convergent sequence of convex functions on I which admit a second derivative on I (first consider the function $(x - a)^+$, and approximate f by the sum of an affine linear function and a linear combination $\sum_j c_j (x - a_j)^+$ with coefficients $c_j \geqslant 0$).

3) Let f be a convex function on an interval $I \subset \mathbf{R}$.

a) Show that if f is not constant it cannot attain its least upper bound at an interior point of I.

b) Show that if I is relatively compact in $\mathbf{R}$ then f is bounded below on I.

c) Show that if $I = \mathbf{R}$ and f is not constant, then f is not bounded above on I.

4) For a function f to be convex on a compact interval $[a, b] \subset \mathbf{R}$ it is necessary and sufficient that it be convex on $]a, b[$ and that one has $f(a) \geqslant f(a+)$ and $f(b) \geqslant f(b-)$.

5) Let f be a convex function on an open interval $]a, +\infty[$; if there exists a point $c > a$ such that f is strictly increasing on $]c, +\infty[$ then $\lim\limits_{x \to +\infty} f(x) = +\infty$.

6) Let f be a convex function on an interval $]a, +\infty[$; show that $f(x)/x$ has a limit (finite or equal to $+\infty$) as x tends to $+\infty$; this limit is also that of $f'_d(x)$ and of $f'_g(x)$; it is > 0 if $f(x)$ tends to $+\infty$ as x tends to $+\infty$.

7) Let f be a convex function on the interval $]a, b[$ where $a \geqslant 0$; show that on this interval the function $x \mapsto f(x) - xf'(x)$ (the "ordinate at the origin" of the right semi-tangent at the point x to the graph of f) is decreasing (strictly decreasing if f is strictly convex).

   Deduce that:

a) If f admits a finite right limit at the point a then $(x - a)f'_d(x)$ has a right limit equal to 0 at this point.

b) On $]a, b[$ either $f(x)/x$ is increasing, or $f(x)/x$ is decreasing, or else there exists a $c \in]a, b[$ such that $f(x)/x$ is decreasing on $]a, c[$ and increasing on $]c, b[$.

c) Suppose that $b = +\infty$: show that if

$$\beta = \lim_{x \to +\infty} \left(f(x) - xf'_d(x) \right)$$

is finite, then so is $\alpha = \lim\limits_{x \to +\infty} f(x)/x$, and that the line $y = \alpha x + \beta$ is *asymptotic* [5] to the graph of f, and lies *below* this graph (strictly below if f is strictly convex).

8) Let f be a finite real function, upper semi-continuous on an open interval $I \subset \mathbf{R}$. Then f is convex if and only if $\limsup\limits_{h \to 0, h \neq 0} \dfrac{f(x + h) + f(x - h) - 2f(x)}{h^2} \geqslant 0$ for all $x \in I$. (First show that, for all $\varepsilon > 0$ the function $f(x) + \varepsilon x^2$ is convex, using prop. 9 of I, p. 31.)

¶9) Let f be a finite real function, lower semi-continuous on an interval $I \subset \mathbf{R}$. For f to be convex on I it suffices that, for every pair of points a, b of I such that $a < b$ there exists *one* point z such that $a < z < b$, and that M_z be below the segment $M_a M_b$ (argue by contradiction, noting that the set of points x such that M_x lies strictly above $M_a M_b$ is open).

¶10) Let f be a finite real function defined on an interval $I \subset \mathbf{R}$, such that

$$f\left(\frac{x + y}{2}\right) \leqslant \frac{1}{2}\left(f(x) + f(y) \right)$$

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[5] That is to say, $\lim\limits_{x \to +\infty} \left(f(x) - (\alpha x + \beta) \right) = 0$.

for all x, y in I. Show that if f is bounded above on *one* open interval $]a$, $b[$ contained in I, then f is convex on I (show first that f is bounded above on *every* compact interval contained in I, then that f is continuous at every interior point of I).

¶ 11) Let f be a continuous function on an open interval I $\subset$ **R**, having a finite right derivative at every point of I. If for every $x \in$ I and *every* $y \in$ I such that $y > x$ the point $M_y = (y, f(y))$ lies above the right semi-tangent to the graph of f at the point $M_x = (x, f(x))$, show that f is convex on I (using the mean value theorem show that

$$f_d'(y) \geqslant \frac{f(y) - f(x)}{y - x} \text{ for } x < y).$$

Give an example of a function which is not convex, has a finite right derivative everywhere, and such that for every $x \in$ I there exists a number $h_x > 0$ depending on x such that M_y lies above the right semi-tangent at the point M_x for all y such that $x \leqslant y \leqslant x + h_x$. This last condition is nevertheless sufficient for f to be convex, if one supposes further that f is differentiable on I (use I, p. 12, corollary).

¶ 12) Let f be a continuous real function on an open interval I $\subset$ **R**; suppose that for every pair (a, b) of points of I such that $a < b$ the graph of f lies either entirely above *or* entirely below the segment $M_a M_b$ on the interval $[a$, $b]$. Show that f is convex on all of I or concave on all I (if in $]a$, $b[$ there is a point c such that M_c lies strictly above the segment $M_a M_b$ show that for every $x \in$ I such that $x > a$ the graph of f lies above the segment $M_a M_x$ on the interval $[a, x]$).

13) Let f be a differentiable real function on an open interval I $\subset$ **R**. Suppose that for every pair (a, b) of points of I such that $a < b$ there exists a *unique* point $c \in]a, b[$ such that $f(b) - f(a) = (b - a)f'(c)$; show that f is strictly convex on I or strictly concave on I (show that f' is strictly monotone on I).

14) Let f be a convex real function and strictly monotone on an open interval I $\subset$ **R**; let g be the inverse function of f (defined on the interval $f(I)$). Show that if f is decreasing (resp. increasing) on I, then g is convex (resp. concave) on $f(I)$.

15) Let I be an interval contained in $]0, +\infty[$; show that if $f(1/x)$ is convex on I then so is $xf(x)$, and conversely.

* 16) Let f be a positive convex function on $]0, +\infty[$, and a, b two arbitrary real numbers. Show that the function $x^a f(x^{-b})$ is convex on $]0, +\infty[$ in the following cases:

1° $a = \frac{1}{2}(b + 1)$, $|b| \geqslant 1$;

2° $x^a f(x^{-b})$ is increasing, $a(b - a) \geqslant 0$, $a \geqslant \frac{1}{2}(b + 1)$;

3° $x^a f(x^{-b})$ is decreasing, $a(b - a) \geqslant 0$, $a \leqslant \frac{1}{2}(b + 1)$.

Under the same hypotheses on f show that $e^{x/2} f(e^{-x})$ is convex (use exerc. 2 of I, p. 45).$_*$

17) Let f and g be two positive convex functions on an interval $I = [a, b]$; suppose that there exists a number $c \in$ I such that in each of the intervals $[a, c]$ and $[c, b]$ the functions f and g vary in the same sense. Show that the product fg is convex on I.

18) Let f be a convex function on an interval $I \subset \mathbf{R}$ and g a convex increasing function on an interval containing $f(I)$; show that $g \circ f$ is convex on I.

¶ 19) Let f and g be two finite real functions, f being defined and continuous on an interval I, and g defined and continuous on $\mathbf{R}$. Suppose that for every pair (λ, μ) of real numbers the function $g(f(x) + \lambda x + \mu)$ is convex on I.
a) Show that g is convex and monotone on $\mathbf{R}$.
b) If g is increasing (resp. decreasing) on $\mathbf{R}$, show that f is convex (resp. concave) on I (use prop. 7).

20) Show that the set $\mathfrak{K}$ of convex functions on an interval $I \neq \mathbf{R}$ is reticulated for the order "$f(x) \leqslant g(x)$ for every $x \in I$" (*Set Theory*, III, p. 146). Give an example of two convex functions f, g on I such that their infimum in $\mathfrak{K}$ takes a value different from $\inf(f(x), g(x))$ at certain points. Give an example of an infinite family (f_α) of functions in $\mathfrak{K}$ such that $\inf_\alpha f_\alpha(x)$ is finite at every point $x \in I$ and yet there is no function in $\mathfrak{K}$ less than all the f_α.

21) Let f be a finite real function, upper semi-continuous on an open interval $I \subset \mathbf{R}$. For f to be strictly convex on I it is necessary and sufficient that there be no line locally above the graph G of f at a point of G.

22) Let $f_1, \ldots, f_n$ be continuous convex functions on a compact interval $I \subset \mathbf{R}$; suppose that for all $x \in I$ one has $\sup(f_j(x)) \geqslant 0$. Show that there exist n numbers $\alpha_j \geqslant 0$ such that $\sum_{j=1}^n \alpha_j = 1$ and that $\sum_{j=1}^n \alpha_j f_j(x) \geqslant 0$ on I. (First treat the case $n = 2$, considering a point x_0 where the upper envelope $\sup(f_1, f_2)$ attains its minimum; when x_0 is interior to I determine α_1 so that the left derivative of $\alpha_1 f_1 + (1 - \alpha_1) f_2$ is zero at x_0. Pass to the general case by induction on n; use the induction hypothesis for the restrictions of $f_1, \ldots, f_{n-1}$ to the compact interval where $f_n(x) \leqslant 0$.)

23) Let f be a continuous real function on a compact interval $I \subset \mathbf{R}$; among the functions $g \leqslant f$ which are convex on I there exists one, g_0, larger than all the others. Let $F \subset I$ be the set of $x \in I$ where $g_0(x) = f(x)$; show that F is not empty and that on each of the open intervals contiguous to F the function g_0 is equal to an affine linear function (argue by contradiction).

24) Let $P(x)$ be a polynomial of degree n with real coefficients all of whose roots are real and contained in the interval $[-1, 1]$. Let k be an integer such that $1 \leqslant k \leqslant n$. Show that the rational function
$$f(x) = x + \frac{P^{(k-1)}(x)}{P^{(k)}(x)}$$
is increasing on every interval of $\mathbf{R}$ on which it is defined; if $c_1 < c_2 < \cdots < c_r$ are its poles (contained in $[-1, 1]$), then f is convex for $x < c_1$ and concave for $x > c_r$. Deduce that when a runs through $[-1, 1]$ the length of the largest interval containing the zeros of the kth derivative of $(x - a)P(x)$ attains its largest value when $a = 1$ or $a = -1$.

25) One says that a real function f defined on $[0, +\infty[$ is *superadditive* if one has $f(x + y) \geqslant f(x) + f(y)$ for $x \geqslant 0, y \geqslant 0$, and if $f(0) = 0$.
a) Give examples of discontinuous superadditive functions.

b) Show that every convex function f on $[0, +\infty[$ such that $f(0) = 0$ is superadditive.

c) If f_1 and f_2 are superadditive then so is $\inf(f_1, f_2)$; using this, exhibit examples of nonconvex continuous superadditive functions.

d) If f is continuous and $\geqslant 0$ on an interval $[0, a]$ $(a > 0)$, such that $f(0) = 0$ and $f(x/n) \leqslant f(x)/n$ for each integer $n \geqslant 1$, show that f has a right derivative at the point 0 (argue by contradiction). In particular, every continuous superadditive function which is $\geqslant 0$ admits a right derivative at the point 0.