Introduction

These notes deal with complex analysis, harmonic analysis and geometric measure theory. My main motivation is to explain recent progress on the Painlevé Problem and to describe their connections with the study of the L^2-boundedness of the Cauchy singular integral operator on Ahlfors-regular sets and the quantitative theory of rectifiability.

Let $E \subset \mathbb{C}$ be a compact set. We say that E is **removable** for bounded analytic functions if, for any open set $U \supset E$, any bounded analytic function $f : U \setminus E \to \mathbb{C}$ has an analytic extension to the whole U. The **Painlevé problem** can be stated as follows:

Find a geometric/metric characterization of such removable sets.

In 1947, L. Ahlfors [**1**] introduced the notion of **analytic capacity** of a compact set E:

$$\gamma(E) = \sup\{|f'(\infty)|, f : \mathbb{C} \setminus E \to \mathbb{C} \text{ is analytic bounded with } ||f||_\infty \leq 1\}$$

and proved that E is removable if and only if $\gamma(E) = 0$. But, as wrote Ahlfors himself (in this quotation, $M(G)$ is the analytic capacity of the boundary of G where G is a complex domain of finite connectivity),

"Of course our theorem is only a rather trivial restatement of Painlevé's Problem in what one might call finite terms. But it shows that a "solution" of Painlevé's Problem will be found if we can construct an explicitly defined quantity, depending on G, which tends to zero simultaneously with $M(G)$. Just what is meant by an explicit definition is of course open to discussion, but most mathematicians would probably agree that the ultimate goal is a definition in purely geometric terms. The solution would then be the same general character as one which refers to measure or capacity."

By Riemann's principle for removable singularities, a singleton is removable. On the other hand, by arguments of complex analysis, (non degenerate) continua or compact sets with non zero area are not removable. This suggests that the metric size of the set should play an important role. This observation can be stated more precisely in terms of **Hausdorff dimension** (denoted by $\dim_H$) and 1-dimensional **Hausdorff measure** (denoted by H^1):

(i) If $H^1(E) = 0$, then E is removable.
(ii) If $\dim_H E > 1$, then E is not removable.

Unfortunately, examples of A. Vitushkin, J. Garnett and L. Ivanov [**42**] [**43**] show that the condition $H^1(E) = 0$ is not necessary for the removability of the compact set E. Sets they considered are purely unrectifiable in the sense of geometric measure theory. This leads to the **Vitushkin Conjecture**:

The compact set $E \subset \mathbb{C}$ is removable for bounded analytic functions if and only if $Fav(E) = 0$.

Here, $\text{Fav}(E)$ is the Favard length of E and is defined by $\text{Fav}(E) = \int_0^\pi |P_\theta(E)| d\theta$ where P_θ is the projection on the line of $\mathbb{C}$ that makes an angle θ with the real axis and $|P_\theta(E)|$ is the Lebesgue measure of the projection of E on this line. If $H^1(E) < \infty$, the condition $\text{Fav}(E) = 0$ is equivalent to $H^1(E \cap \Gamma) = 0$ for any rectifiable curve Γ of $\mathbb{C}$ (that is E is purely unrectifiable in H. Federer's terminology).
The work of P. Mattila [**63**], P. Jones and T. Murai [**51**] showed that the Vitushkin conjecture is not true for general sets, but left open the case of sets with finite length (that is such that $H^1(E) < \infty$). In 1977, A. P. Calderón [**11**] proved the L^2 boundedness of the **Cauchy operator** on Lipschitz graphs with small constant. This famous result implies a solution to the **Denjoy Conjecture** (and therefore one sense of the Vitushkin conjecture for sets of finite length):

Let $E \subset \mathbb{C}$ be a subset of a rectifiable curve Γ. Then, E is removable if and only if $H^1(E) = 0$.

At that time, it was clear that the removability of a compact set E is closely related to the behavior of the Cauchy operator on E. This motivates the following question:

For which Ahlfors 1-regular sets E is the Cauchy operator bounded on $L^2(E)$ (with respect to the restriction of H^1 to E) ?

A set E in $\mathbb{C}$ is Ahlfors 1-regular if there exists $C > 0$ such that

$$C^{-1}R \leq H^1(E \cap B(x,R)) \leq CR$$

whenever $x \in E$ and $R \in (0, \text{diam}E)$. The example of Lipschitz graphs show that rectifiability properties of the set should play a role. Recall that a set $E \subset \mathbb{C}$ is 1-**rectifiable** if there exist Lipschitz curves Γ_j such that $H^1(E \setminus \cup_j \Gamma_j) = 0$. For this, P. Jones [**50**] (for 1-dimensional sets), G. David and S. Semmes [**28**] [**29**] (in higher dimensions) have developed a quantitative theory of rectifiability.

In 1995, M. Melnikov [**71**] rediscovered the **Menger curvature** and used it to study the semi-additivity of the analytic capacity. The Menger curvature $c(x, y, z)$ of three non collinear points x, y and z of $\mathbb{C}$ is the inverse of the radius of the circumference passing through x, y and z. If the points x, y and z are collinear, we set $c(x, y, z) = 0$. If μ is a positive Radon measure on $\mathbb{C}$, the Menger curvature $c^2(\mu)$ of μ is

$$c^2(\mu) = \int\int\int c(x,y,z)^2 d\mu(x) d\mu(y) d\mu(z).$$

If we assume that $c^2(\mu) < +\infty$, our intuition says that most (with respect to μ) triples are nearly collinear, in other words μ is "flat". In fact, G. David (unpublished) and J.C. Léger [**56**] proved that, if E is a compact set of $\mathbb{C}$ which satisfies $H^1(E) < +\infty$ and $c^2(H^1_{\lceil E}) < +\infty$, then E is 1-rectifiable.

Using the Menger curvature, M. Melnikov and J. Verdera [**72**] gave a simple and geometric proof of the L^2 boundedness of the Cauchy operator on Lipschitz graphs, and with P. Mattila [**69**], they proved that the Cauchy operator is bounded on a

Ahlfors-regular set E if and only if E is contained in a regular curve (that is E is uniformly rectifiable in the sense of G. David and S. Semmes). From this and a previous work of M. Christ [**18**], they proved the Vitushkin conjecture for Ahlfors regular sets. The general case was solved by G. David [**26**].

Very recently, X. Tolsa gave a characterization of removable sets in terms of Menger curvature :

A compact set E of $\mathbb{C}$ is not removable for bounded analytic functions if and only if E supports a positive Radon measure with linear growth and finite Menger curvature.

Recall that a measure μ in $\mathbb{C}$ has linear growth if there exists $C > 0$ such that $\mu(B) \leq C\text{diam}B$ whenever B is a ball in $\mathbb{C}$.

In this book, I would like to tell you this very beautiful story, and I will follow the following plan. In Chapter 1, basic notions of geometric measure theory (like Hausdorff measures, Hausdorff dimension, rectifiable and purely unrectifiable sets) are defined. In particular, we will give several characterizations of rectifiable sets. We will conclude with the proof of a covering lemma by Ahlfors-regular sets. Chapter 2 is devoted to the geometric traveling salesman theorem of P. Jones and the theory of uniformly rectifiable sets of G. David and S. Semmes. In Chapter 3, we will define the Menger curvature and describe some of its properties. In particular, we will show that the Menger curvature is a useful tool to study the geometry of sets and measures in the complex plane. In this part, the reader will find the proofs of some unpublished results of P. Jones. In Chapter 4 is given an overview of the theory of Calderón-Zygmund operators. We also include Melnikov-Verdera's proof of the L^2 boundedness of the Cauchy operator on Lipschitz graphs. The last part of this Chapter will be devoted to the proof of Mattila-Melnikov-Verdera's characterization of Ahlfors-regular sets on which the Cauchy operator is bounded. In Chapter 5, we will define the analytic capacity and we will prove some of its basic properties. The Denjoy and Vitushkin conjectures are proved in Chapter 6. In the last Chapter, we will describe X. Tolsa's characterization of removable sets and we will discuss some open problems.

This book is almost self contained. Only a basic knowledge of real analysis, complex analysis and measure theory is required. Most of the proofs are given. When a proof is omitted or sketched, a reference is indicated where the reader can find a complete proof.

There are good surveys about the subject of this book [**25**] [**67**] [**106**]. I hope that these notes are a complement to these papers and a modest continuation of J. Garnett's Lecture Notes [**42**].

These notes are based on lectures given at the Ecole Normale Supérieure de Lyon and on a graduate course given at Yale University. I would like to thank J. P. Otal and P. Jones for their kind invitation.
I am very grateful to G. David, J. Garnett, N. Kang, P. Mattila and J. Verdera for their suggestions and encouragements. Part of this work was done while the author

has the benefit of a “delegation” at the CNRS. The author is partially supported by the European Commission (European TMR Network “Harmonic Analysis”).

Notations and conventions

If x, $y \in \mathbb{R}^n$, the Euclidean distance between x and y is denoted by $|x-y|$. If $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, $d(x,A) = \inf\{|x-a|; a \in A\}$, and if $B \subset \mathbb{R}^n$, $d(A,B) = \inf\{|a-b|; a \in A, b \in B\}$.
The open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ is denoted by $B(x,r)$. In the special case $n = 2$, that is in $\mathbb{C}$, we will also use the notation $D(x,r)$. For instance, the unit disc in $\mathbb{C}$ is $D(0,1) = \{z \in \mathbb{C}, |z| < 1\}$.
If B is a ball in $\mathbb{R}^n$, we often denote by R_B the radius of B. If $k \in \mathbb{R}^{+*}$, kB is the ball with the same center as B, but whose radius R_{kB} is $k.R_B$.

If E et F are two sets in $\mathbb{R}^n$, then $E + F = \{x+y, x \in E, y \in F\}$ and, for any $x \in \mathbb{R}^n$, $x + F = \{x+y, y \in F\}$.

A measure μ on $\mathbb{R}^n$ for us will be a non-negative, monotonic, subadditive set function which vanishes for empty sets. We always assume that $\mu(\mathbb{R}^n) \neq 0$. A set $A \subset \mathbb{R}^n$ is μ measurable if $\mu(E) = \mu(E \cap A) + \mu(E \setminus A)$ for all $E \subset \mathbb{R}^n$. The measure μ is a Borel measure if all Borel sets are μ measurable. The measure μ is a Radon measure if μ is a Borel measure and satisfies
(i) $\mu(K) < +\infty$ whenever K is a compact set in $\mathbb{R}^n$;
(ii) $\mu(O) = \sup\{\mu(K), K \subset O \text{ compact}\}$ whenever O is an open set in $\mathbb{R}^n$;
(iii) $\mu(A) = \inf\{\mu(O); A \subset O, O \text{ is open }\}$.
If μ is a measure on $\mathbb{R}^n$ and if $E \subset \mathbb{R}^n$, then $\mu_{\restriction E}$ will denote the restriction of μ to E.
The support of a measure μ in $\mathbb{R}^n$ (denoted by $\mathrm{Supp}\mu$) is the smallest closed set K such that $\mu(\mathbb{R}^n \setminus K) = 0$.

The Lebesgue measure in $\mathbb{R}^n$ will be denoted by $\mathcal{L}^n$.

A dyadic cube Q in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form $Q = \Pi_{j=1}^n [k_j 2^{-k}, (k_j+1)2^{-k}]$ where $k \in \mathbb{Z}$ and $k_j \in \mathbb{Z}$. We denote by Δ the set of all dyadic cubes in $\mathbb{R}^n$ and by Δ_k the subset of Δ of k-th generation, that is of side length 2^{-k}. In the special case $n = 1$ (respectively $n = 2$), an element Q of Δ will be called "dyadic interval" (respectively "dyadic square").
Let Q be a dyadic cube in $\mathbb{R}^n$ whose side length is $l(Q)$. Then, if $k \in \mathbb{N}$, kQ denotes the cube with sides parallel to the axis, whose center is the center of Q, but whose side length is $kl(Q)$.

If $E \subset \mathbb{R}^n$, the characteristic function of E is denoted by χ_E.

A constant without a subscript (like C) may vary throughout all the book.

If $A(X)$ and $B(X)$ are two quantities depending on the same variable(s) X, we will say that A and B are comparable if there exists $C \geq 1$ not depending on X such that $C^{-1}A(X) \leq B(X) \leq CA(X)$ for every X.