3 Perfection and Eutaxy

Introduction

The aim of this chapter is to give a characterization of extreme lattices, those lattices on which the Hermite invariant attains a local maximum. To this end, we introduce in Section 3.2 the two notions of perfection and eutaxy, and prove in Section 3.4 that they characterize extreme lattices; in particular, we obtain an algorithm which allows us to decide whether a given lattice is extreme, whose justification relies on the technical material in Sections 3.1 and 3.3. Then we give in Sections 3.5 and 3.6 a detailed analysis of the properties of perfection and eutaxy.

In Section 3.7 we show that the "laminating" process of Conway and Sloane produces perfect – but not necessarily extreme – lattices. Section 3.8 is devoted to another kind of extremal property involving both a lattice and its dual.

As usual E denotes an n-dimensional Euclidean vector space.

3.1 Symmetric Endomorphisms

Definition 3.1.1. *Let $u \in \mathrm{End}(E)$. The* transpose *of u is the endomorphism ${}^t u$ such that $u(x) \cdot y = x \cdot {}^t u(y)$ for all x, $y \in E$. We say that u is* symmetric *(resp.* skew-symmetric *or* alternating*) if ${}^t u = u$ (resp. ${}^t u = -u$). We denote by $\mathrm{End}^s(E)$ (resp. $\mathrm{End}^a(E)$) the real vector space of symmetric (resp. skew-symmetric) endomorphisms of E, and by Sym_n the space of $n \times n$ symmetric matrices.*

One easily verifies that the transpose of a given $u \in \mathrm{End}(E)$ exists and is unique, and that $u \mapsto {}^t u$ is an *involution* of $\mathrm{End}(E)$, i.e., that it satisfies the rule ${}^t(v \circ u) = {}^t u \, {}^t v$.

Proposition 3.1.2. *Let $\mathcal{B}$ be a basis for E, let $\mathcal{B}^*$ be its dual basis, let $\mathcal{B}_0$ be an orthogonal basis for E, and let $u, v \in \mathrm{End}(E)$. The following three conditions are equivalent:*

1. *Each of the matrices of u and of v with respect to the bases $\mathcal{B}$ and $\mathcal{B}^*$ is the transpose of the other.*

2. *Each of the matrices of u and of v in the base $\mathcal{B}_0$ is the transpose of the other.*
3. $v = {}^t u$.

Proof. The matrix $(m_{i,j})$ of an element $u \in \text{End}(E)$ with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$ is characterized by the relations $u(e_j) = \sum_k m_{k,j} \, e'_k$, which can also be written in the form $m_{i,j} = e'^{\,*}_i \cdot u(e_j)$. Taking $\mathcal{B}' = \mathcal{B}^*$, we find for u and ${}^t u$ matrices $(m_{i,j})$ and $(m'_{i,j})$ such that $m_{i,j} = e_i \cdot u(e_j)$ and $m'_{i,j} = e_i \cdot {}^t u(e_j) = u(e_i) \cdot e_j$. This first shows the equivalence of (1) and (2), and then that of (2) and (3) because $\mathcal{B}^*_0 = \mathcal{B}_0$. □

Let $u \in \text{End}^s(E)$. The definition of the transpose of u shows that the orthogonal of any subspace of E stable under u is also stable under ${}^t u$. Moreover, we know that u has real eigenvalues (extend for instance u to a Hermitian endomorphism of the complex space $\mathbb{C} \bigotimes E$). Combining these two remarks, we easily prove that the matrix of u is a diagonal matrix in some orthonormal basis for E. [Matrix interpretation: for any real symmetric matrix $A \in \mathcal{M}_n(\mathbb{R})$, there exists an orthogonal matrix $P \in \mathcal{M}_n(\mathbb{R})$ (i.e., we have $P^{-1} = {}^t P$) such that ${}^t P A P$ is a diagonal matrix.]

Definition 3.1.3. *We say that $u \in \text{End}^s(E)$ is* positive *if its eigenvalues are non-negative,* definite *if they are nonzero, and thus* positive definite *if they are strictly positive. We denote by $\text{End}^{s+}(E)$ (resp. $\text{End}^{s++}(E)$) the set of positive (resp. positive definite) symmetric endomorphisms.*

We recover the usual definitions of the theory of real quadratic forms by applying the definitions above to the form $x \mapsto x \cdot u(x)$. The notion of a positive, or definite, or positive definite matrix is defined in the same way.

Example 3.1.4. *For any $u \in \text{End}(E)$, ${}^t u u$ is a (symmetric) positive endomorphism.*

Indeed, if x is any eigenvector for ${}^t u u$ with corresponding eigenvalue λ, we have $\lambda \, (x \cdot x) = {}^t u u(x) \cdot x = u(x) \cdot u(x)$, whence $\lambda = \frac{\|u(x)\|^2}{\|x\|^2} \geq 0$.

Lemma 3.1.5. *Let u be a symmetric endomorphism and let m be a positive integer.*

1. *The centralizer of u in $\text{End}(E)$ is the set of the endomorphisms which stabilize all the eigenspaces of u.*
2. *If u is positive or if m is odd, u and u^m have the same centralizer.*

Proof. Let F be an eigenspace of E with corresponding eigenvalue λ, and let $v \in \text{End}(E)$. If u and v commute, we have $u(v(x)) = v(u(x)) = v(\lambda x) = \lambda v(x)$ for all $x \in F$, whence the inclusion $v(F) \subset F$.

Conversely, let $v \in \text{End}(E)$ which stabilizes the eigenspaces $E_1, \dots, E_k$ of u, and for all i, let v_i be the restriction of v to E_i. Since E is the direct sum of the E_i and since the restrictions u_i of u to E_i are homothetic

transformations, we have $uv(x) = u_i v_i(x) = v_i u_i(x) = vu(x)$ for all $x \in E_i$, whence (1).

Denote by λ_i the eigenvalue attached to the eigenspace E_i of u. For $x \in E_i$, we have $u^m(x) = \lambda_i^m x$, which shows that λ_i^m is for all i an eigenvalue of u^m whose corresponding eigenspace E_i' contains E_i. But the hypotheses we have made on u and m show that the map $i \mapsto \lambda_i^m$ is injective. This implies that the sum $\sum_i E_i$ is a direct one, and shows the equality $\sum \dim E_i' \geq \sum \dim E_i = n$, whence $\sum \dim E_i' = n$ and $\dim E_i' = \dim E_i$ for all i (because of the lower bound $\dim E_i' \geq \dim E_i$). Finally, the subspaces E_i' and E_i coincide for all i, and (2) is now an immediate consequence of (1). $\qquad\square$

Theorem 3.1.6. *A positive symmetric endomorphism possesses a unique positive square root.*

Proof. Let $\mathcal{B}$ be an orthonormal basis for E in which the matrix of u is a diagonal matrix D. The endomorphism v defined in the basis $\mathcal{B}$ by the diagonal matrix whose diagonal terms are the square roots of the diagonal terms of D is a positive square root of u. By Lemma 3.1.5 (2), v and $u = v^2$ have the same centralizer, and Lemma 3.1.5 (1) then shows that v is characterized by the equations $v(x) = \sqrt{\lambda}\, x$ for every eigenvalue λ of u and every vector x in the corresponding eigenspace. This proves that v is unique. $\qquad\square$

In the sequel, the unique positive square root of a positive symmetric endomorphism u will be called the *square root of u*, and the same convention applies to real positive symmetric matrices. In both cases, we shall use the usual notation for a square root. The equality $\sqrt{uv} = \sqrt{u}\,\sqrt{v}$ holds whenever u and v commute, and we also have $\sqrt{u^{-1}} = (\sqrt{u})^{-1}$ whenever u is invertible.

We now come to a theorem which will play a prominent role in the sequel: the decomposition of an endomorphism into symmetric and orthogonal components.

Theorem 3.1.7. *Any $u \in \mathrm{GL}(E)$ possesses unique decompositions into each of the forms $u = v\,w$ and $u = w'\,v'$ with positive symmetric v, v' and orthogonal w, w'.*

Proof. From $u = v\,w$ (resp. $u = w'\,v'$), we deduce the relation $u\,{}^t u = v^2$ (resp. ${}^t u\,u = v'^2$). The uniqueness of v and of v' then follows from Theorem 3.1.6, and that of w and of w' from the equalities $w = v^{-1}u$ and $w' = u\,v'^{-1}$.

To prove the existence of these decompositions, we apply Example 3.1.4 and Theorem 3.1.6: let v (resp. v') be the square root of $u\,{}^t u$ (resp. of ${}^t u\,u$), and let $w = v^{-1}u$ (resp. $w' = u\,v'^{-1}$). We then easily verify that ${}^t w\,w = {}^t w'\,w' = \mathrm{Id}$, and thus that w and w' are orthogonal. $\qquad\square$

We now study how the endomorphisms constructed in Theorems 3.1.6 and 3.1.7 vary with u.

Proposition 3.1.8. *1. The map $u \mapsto \sqrt{u}$ from $\mathrm{End}^{s++}(E)$ onto itself is a diffeomorphism.*

2. The maps from $\mathrm{GL}(E)$ to $\mathrm{End}^s(E)$ (resp. to $\mathrm{O}(E)$) which associate with an automorphism of E its left or right symmetric (resp. orthogonal) components are differentiable.

Proof. The way the symmetric orthogonal components have been constructed in the proof of Theorem 3.1.7 shows that the assertions in (2) are immediate consequences of assertion (1), which we now prove.

Let $u \in \mathrm{End}^{s++}(E)$ and let $h \in \mathrm{End}^s(E)$. The identity $(u+h)^2 = u^2 + hu + uh + h^2$ shows that the tangent map to $u \mapsto u^2$ (the differential map) is $h \mapsto hu + uh$. If h lies in its kernel, we have $uh = -hu$, hence $u^2 h = hu^2$, and thus $uh = hu$ by Lemma 3.1.5 (2). We therefore have $uh = 0$, hence also $h = 0$. We now know that the map $u \mapsto u^2$ is one-to-one and that its differential map is invertible at every point of $\mathrm{End}^{s++}(E)$. The reciprocal map then possesses the same properties. □

The uniqueness assertion of Theorem 3.1.7 allows the extension of the decomposition to some subspaces of E. Here is an example that we shall need later to study families of lattices possessing a prescribed automorphism group:

Proposition 3.1.9. *Let G be a subgroup of $\mathrm{O}(E)$ and let u be an endomorphism of E which commutes with G. Then the symmetric and orthogonal components (on each side) of u also commute with G.*

Proof. With the notation of Theorem 3.7.1, let $g \in G$. We have

$$u = gug^{-1} = (gvg^{-1})(gwg^{-1}).$$

Then gwg^{-1} is obviously orthogonal and gvg^{-1} is symmetric, for $g^{-1} = {}^t g$. We thus have $gvg^{-1} = v$ and $gwg^{-1} = w$, and similar equalities hold for the other decomposition. □

To obtain suitable geometrical interpretations of some of the notions we are going to introduce in the next section, we shall need to work sometimes with the dual space of $\mathrm{End}^s(E)$. To this end, we introduce a Euclidean structure on this space of endomorphisms.

Proposition 3.1.10. *The map $u \mapsto \mathrm{Tr}(u^2)$ is a positive definite quadratic form on $\mathrm{End}^s(E)$ with corresponding bilinear form $(u, v) \mapsto \mathrm{Tr}(uv)$, and the resulting identification of $\mathrm{End}^s(E)$ with $\mathrm{End}^s(E)^*$ transforms $\mathrm{Id} \in \mathrm{End}^s(E)$ into $\mathrm{Tr} \in \mathrm{End}^s(E)^*$.*

We denote by $\langle u, v \rangle$ the scalar product $\mathrm{Tr}(uv)$, and call it the Voronoi scalar product. We use the same notation on the space Sym_n of real symmetric matrices.

Proof. The map $u \mapsto \mathrm{Tr}(u^2)$ is clearly a quadratic form with corresponding bilinear form $(u, v) \mapsto \mathrm{Tr}(uv)$. Let $u \in \mathrm{End}^s(E)$. Since its eigenvalues are real, those of u^2 are non-negative. This shows that the form $u \mapsto \mathrm{Tr}(u^2)$ is positive. Moreover, if $\mathrm{Tr}(u^2) = 0$, the eigenvalues of u are all zero, since their sum is zero, and u itself is then zero, since it possesses a diagonal form. Hence $u \mapsto \mathrm{Tr}(u^2)$ is a definite form. Finally, the last assertion is a reformulation of the equalities $\langle \mathrm{Id}, u \rangle = \mathrm{Tr}(\mathrm{Id} \circ u) = \mathrm{Tr}(u)$. $\square$

Among the various symmetric endomorphisms of E, the orthogonal projections onto lines in E will play a crucial rôle. Recall (Proposition 1.3.4) that p_D denotes the orthogonal projection onto a line D; given $x \neq 0$ in E, we set $p_x = p_{\mathbb{R}x}$. We have

$$p_x(y) = \frac{x \cdot y}{N(x)}\, x\,.$$

Proposition 3.1.11. *1. For all $u \in \mathrm{End}^s(E)$, we have*

$$\mathrm{Tr}(u \circ p_x) = \mathrm{Tr}(p_x \circ u) = \frac{u(x) \cdot x}{N(x)}\,.$$

2. The (orthogonal) projections onto the various lines of E span $\mathrm{End}^s(E)$.

Proof. (1) Consider a basis $\mathcal{B} = (f_1, \ldots, f_n)$ for E with $f_1 = x$ and $f_1 \cdot f_i = 0$ for all $i > 1$. We have $p_x(f_i) = 0$ for $i > 1$. The map $u \circ p_x$ is thus zero on $\mathbb{R}x^\perp$, so that the trace of $u \circ p_x$ is the coefficient of $u(f_1)$ on f_1. If $u(f_1) = \sum_{i=1}^n x_i\, f_i$, we have $u(f_1) \cdot f_1 = x_1\,(f_1 \cdot f_1)$, whence $x_1 = \frac{u(x) \cdot x}{N(x)}$. Taking into account the symmetry of the bilinear trace form, this completes the proof of (1).

(2) It suffices to show that an element $u \in \mathrm{End}^s(E)$ which is orthogonal to all the projections is the null map. By (1), we have $\langle u, p_x \rangle = \frac{u(x) \cdot x}{N(x)}$. Hence the quadratic form $x \mapsto u(x) \cdot x$ is identically zero on E, so that all scalar products $u(x) \cdot y$ are zero. For every $x \in E$, $u(x)$ is thus orthogonal to all vectors of E, whence $u = 0$. $\square$

We now explain a way to calculate with projections by means of matrices. We still consider a pair of a basis $\mathcal{B}$ and its dual basis $\mathcal{B}^*$, but we express the (orthogonal) projections with respect to the bases $\mathcal{B}^*$ and $\mathcal{B}$, for exchanging $\mathcal{B}$ and $\mathcal{B}^*$ simplifies some formulae. We have $p_x(e_j^*) = \frac{x \cdot e_j^*}{N(x)} x$ whence (using the components x_i of x in $\mathcal{B}$)

$$N(x)\, p_x(e_j^*) = \Big(\sum_k x_k\, e_k \cdot e_j^* \Big) \Big(\sum_i x_i\, e_i \Big) = \sum_i x_i x_j\, e_i\,.$$

We are thus led to the following definition:

Definition 3.1.12. *Let x be a nonzero vector in E, whose components $x_1, \ldots, x_n$ in some basis $\mathcal{B}$ are represented by the column-matrix X. We denote by P_x or by P_X the matrix $X\,{}^t X$ (a symmetric matrix of order n).*

Explicitly, we have

$$P_X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1 x_2 & \ldots & x_1 x_n \\ x_2 x_1 & x_2^2 & \ldots & x_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n x_1 & x_n x_2 & \ldots & x_n^2 \end{pmatrix}.$$

The proof of the following proposition results immediately from the calculation we have done before Definition 3.1.12:

Proposition 3.1.13. *Let $\mathcal{B}$ be a basis for E and let $x \in E$ be represented by the column-matrix X. Then the matrix of $N(x)\,p_x$ with respect to the bases $\mathcal{B}^*$ and $\mathcal{B}$ is $P_X = X\,{}^t X$.* [Warning: note the inversion between $\mathcal{B}$ and $\mathcal{B}^*$ with respect to the usual ordering.] $\qquad\square$

Let x be a nonzero element in E. The matrices P_X are of rank 1 and the corresponding quadratic forms are

$$Y \mapsto {}^t Y \left(X\,{}^t X \right) Y = {}^t({}^t XY)\,({}^t XY),$$

or denoting by $y_1, y_2, \ldots, y_n$ the components of y,

$$(y_1, y_2, \ldots, y_n) \mapsto (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2.$$

When x runs through $E \smallsetminus \{0\}$, we find in this way all rank-1 positive quadratic forms, and two such forms are equal if and only if they correspond to equal or opposite vectors.

To study the Hermite invariant we shall use either of the following two convexity results with which we end Section 3.1.

Proposition 3.1.14. *Let v be a nonzero symmetric endomorphism, let I be an interval such that the function $t \mapsto 1 + \lambda t$ is positive for all $t \in I$ and for all eigenvalues λ of v, and for $t \in I$, let $u_t = \mathrm{Id} + t\,v$. Then u_t is a positive definite symmetric endomorphism, the map $t \mapsto \det(u_t)$ is strictly logarithmically concave, and the map $t \mapsto \dfrac{1}{\det(u_t)}$ is strictly convex.*

Proof. Since both functions we consider are C^∞ on I, we just have to show that the second derivative of the logarithm of the first function (resp. of the second function) is strictly negative (resp. positive) on I. As a function of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of v, we have $\det(u_t) = \prod_i (1 + \lambda_i t)$.

For the first function, we find

$$\frac{d^2}{dt^2} \log \prod_i (1 + \lambda_i t) = -\sum \frac{\lambda_i^2}{(1 + \lambda_i t)^2},$$

and this expression is strictly negative, since at least one of the eigenvalues is nonzero.

The result for the second function follows from the previous one by applying the following identity, valid for any strictly positive twice differentiable function:

$$\left(\frac{1}{f}\right)'' = -\frac{1}{f}(\log f)'' + \frac{f'^2}{f^3}. \quad \square$$

3.2 Linear Forms on Spaces of Endomorphisms

In this section, we denote by $\mathcal{T}$ a subspace of $\operatorname{End}^s(E)$; $\operatorname{Tr}(u)$ stands for the trace of the endomorphism u.

Definition 3.2.1. *For all $x \in E$, let φ_x be the linear form $u \mapsto u(x) \cdot x$ on $\operatorname{End}^s(E)$; for every line $D \subset E$, let $\varphi_D = \varphi_x$ for any unit vector $x \in D$. When there is no risk of confusion, we use the same notation to denote the restrictions to the subspace $\mathcal{T}$ of the linear forms above; this convention applies in particular to Definition 3.2.2 below.*

The equality $\varphi_{\lambda x} = \lambda^2 \varphi_x$ justifies the notation φ_D. In practice, we consider φ_x only up to a positive factor, so that the choice of any nonzero $x \in D$ would have been suitable.

[*However, for the notion of strong eutaxy defined below, one must be careful and avoid repeated lines or vectors, or non-constant numbers of vectors defining the same line.*]

Definition 3.2.2. *Let $\mathcal{F}$ be a finite family of nonzero vectors in E and let $\mathcal{D}$ be a finite family of lines in E.*

1. *We say that $\mathcal{F}$ (resp. $\mathcal{D}$) is $\mathcal{T}$-perfect if the forms φ_x, $x \in \mathcal{F}$ (resp. the forms φ_D, $D \in \mathcal{D}$) span the dual space $\mathcal{T}^*$ of $\mathcal{T}$. More generally, the rank r of the system φ_x (or φ_D) is called the $\mathcal{T}$-perfection rank of the family, and the codimension $\dim \mathcal{T} - r$ its $\mathcal{T}$-perfection corank.*
2. *We say that real numbers $\rho_x, x \in \mathcal{F}$ (resp. $\rho_D, D \in \mathcal{D}$) are $\mathcal{T}$-eutaxy coefficients for $\mathcal{F}$ (resp. for $\mathcal{D}$) if there exists a relation $\operatorname{Tr} = \sum_{x \in \mathcal{F}} \rho_x \varphi_x$ (resp. $\operatorname{Tr} = \sum_{D \in \mathcal{D}} \rho_D \varphi_D$) between the restrictions to $\mathcal{T}$ of the forms $\operatorname{Tr}, \varphi_x, \varphi_D$.*
3. *We say that the family $\mathcal{F}$ (resp. $\mathcal{D}$) is $\mathcal{T}$-weakly eutactic if it possesses $\mathcal{T}$-eutaxy coefficients, that it is $\mathcal{T}$-semi-eutactic (resp. $\mathcal{T}$-eutactic, resp. strongly eutactic) if it is weakly eutactic with positive (resp. strictly positive, resp. equal) eutaxy coefficients.*

   [Warning: For *strong eutaxy*, we must consider either distinct lines, or vectors of equal norm such that no two of them are collinear.]

When $\mathcal{T} = \operatorname{End}^s(E)$, one simply says perfect, weakly eutactic, semi-eutactic, eutactic, *and* strongly eutactic.

To say that $\mathcal{F}$ (or $\mathcal{D}$) is $\mathcal{T}$-perfect amounts to the same thing as saying that the intersection of the kernels of the restriction to $\mathcal{T}$ of the forms φ_x (or φ_D) is reduced to $\{0\}$.

Note that perfect families are those which have corank 0, and that they are always weakly eutactic; note also that strongly eutactic families are actually eutactic: let $s = |\mathcal{F}|$; applying a given eutaxy relation $\mathrm{Tr} = \sum_{x \in \mathcal{F}} \rho_x \varphi_x$ to the identity, we obtain the new relation

$$\sum_{x \in \mathcal{F}} N(x)\, \rho_x = n \qquad (\mathbf{3.2.2'})$$

from which we deduce that equal eutaxy coefficients have the common positive value

$$\rho = \frac{1}{N(x)}\, \frac{n}{s}\,. \qquad (\mathbf{3.2.2''})$$

The notions of a perfect or of a eutactic family $\mathcal{F}$ of vectors solely depend on the set of lines which contain them; it is possible to suppress a vector in $\mathcal{F}$ whenever $\mathcal{F}$ contains another one which is proportional to it, and hence to restrict oneself to families of vectors with a given norm. The natural notions are thus those of a family (one also says *configuration*) of perfect or eutactic lines. However, we shall have to work with vectors rather than with lines, and shall essentially apply Definition 3.2.2 to families of vectors.

The following trivial proposition is nevertheless very useful:

Proposition 3.2.3.   *1. If $\mathcal{T}'$ is a subspace of $\mathcal{T}$, any $\mathcal{T}$-perfect (resp. $\mathcal{T}$-eutactic) configuration is $\mathcal{T}'$-perfect (resp. $\mathcal{T}'$-eutactic).*
2. Any $\mathcal{T}$-perfect configuration has cardinality at least $\dim \mathcal{T}$.   □

We now consider more closely the case where $\mathcal{T}$ is the whole space $\mathrm{End}^s(E)$.

Lemma 3.2.4. *For any nonzero $x \in E$, the duality defined by $\mathrm{Tr}(u^2)$ (see Proposition 3.1.10) transforms $N(x)\, p_x$ into φ_x.*

Proof. This is just a reformulation of Proposition 3.1.11.   □

Theorem 3.2.5. *Let $\mathcal{F}$ be a finite family of nonzero vectors in E, let $\mathcal{B} = (e_1, \ldots, e_n)$ be a basis for E, with Gram matrix $A = (a_{i,j})$ (i.e., we have $a_{i,j} = e_i \cdot e_j$), and let $(a^*_{i,j})$ be the Gram matrix of $\mathcal{B}^*$ (i.e., we have $(a^*_{i,j}) = A^{-1}$). For all $x \in E$, denote by $c_1(x), \ldots, c_n(x)$ the components of x in $\mathcal{B}$. The following conditions relative to a family ρ_x, $x \in \mathcal{F}$ of real numbers are then equivalent:*
1. The ρ_x are eutaxy coefficients for $\mathcal{F}$.
2. We have the identity $\mathrm{Id}_E = \sum_{x \in \mathcal{F}} \rho_x N(x)\, p_x$.
3. For all $y \in E$, $N(y) = \sum_{x \in \mathcal{F}} \rho_x\, (y \cdot x)^2$.

3'. For all y, $z \in E$, $y \cdot z = \sum_{x \in \mathcal{F}} \rho_x (y \cdot x) (z \cdot x)$.
4. For all i and all j, $a_{i,j}^* = \sum_{x \in \mathcal{F}} \rho_x c_i(x) c_j(x)$.
5. We have the identity $A^{-1} = \sum_{x \in \mathcal{F}} \rho_x P_x$.

Proof. The equivalence of (1) and (2) is an immediate consequence of Lemma 3.2.4, that of (3) and (3') results from the correspondence between quadratic and bilinear forms, and that of (4) and (5) is obvious, as (4) is just an explicit formulation of (5). The remainder of the proof is organized along the logical scheme (2) $\Rightarrow$ (3') $\Rightarrow$ (4) $\Rightarrow$ (2).

If (2) is satisfied, we have for all $y \in E$ the relation

$$y = \sum_{x \in \mathcal{F}} \rho_x N(x) p_x(y) = \sum_{x \in \mathcal{F}} \rho_x (x \cdot y) x$$

and (3') follows, as one sees by performing the scalar product of both sides with z.

One obtains (4) from (3') by putting $y = e_i$ and $z = e_j$.

Finally, if (4) is satisfied, we recover the preceding formula by setting $y = \sum \lambda_i e_i^*$ and $z = \sum \mu_j e_j^*$: we actually have

$$y \cdot z = \sum_{i,j} \lambda_i \mu_j \sum_x \rho_x (x \cdot e_i^*)(x \cdot e_j^*) = \sum_x \rho_x \left(x \cdot \sum_i \lambda_i e_i^* \right) \left(x \cdot \sum_j \mu_j e_j^* \right)$$

$$= \sum_x \rho_x (x \cdot y)(x \cdot z)$$

and the relation $y = \sum_x \rho_x (x \cdot y)$ follows, since both sides have the same scalar product with any $z \in E$. $\square$

Corollary 3.2.6. *Any weakly eutactic family spans E; in particular any perfect family spans E.*

Proof. Let $\mathcal{F}$ be a weakly eutactic family, and let $y \in E$ be orthogonal to all vectors in $\mathcal{F}$. Then, by condition (2) above, we have

$$y = \sum \rho_x N(x) (x \cdot y) y = 0. \ \square$$

Remark 3.2.7. *When $\mathcal{T}$ is no longer the whole space $\mathrm{End}^s(E)$, Assertion (2) above still holds, with the p_x replaced by their orthogonal projections ω_x onto $\mathcal{T}$ (in $\mathrm{End}^s(E)$, for the scalar product $\langle . , . \rangle$).*

We now return to the general case of an arbitrary subspace $\mathcal{T}$ of $\mathrm{End}^s(E)$ and give the definitions we need for lattices.

Definition 3.2.8. *We say that a lattice is $\mathcal{T}$-perfect (resp. weakly $\mathcal{T}$-eutactic, resp. $\mathcal{T}$-semi-eutactic, resp. $\mathcal{T}$-eutactic, resp. strongly $\mathcal{T}$-eutactic) if the set of its minimal vectors constitutes a $\mathcal{T}$-perfect (resp. a weakly $\mathcal{T}$-eutactic,*

resp. a $\mathcal{T}$-semi-eutactic, resp. a $\mathcal{T}$-eutactic, resp. a strongly $\mathcal{T}$-eutactic)
family. When $\mathcal{T}$ is the whole space $\mathrm{End}^s(E)$, *one simply says* perfect, weakly
eutactic, semi-eutactic, eutactic *and* strongly eutactic. *We define as in*
Definition 3.2.2 the $(\mathcal{T})$-perfection rank *and* corank *of a lattice.*

Let Λ be a lattice. To a basis $\mathcal{B}$ for Λ, we can attach a symmetric matrix
A, namely the Gram matrix of $\mathcal{B}$, whence also a quadratic form Q on $\mathbb{R}^n$.
By Theorem 2.5, the properties of perfection or eutaxy of Λ are equivalent
to the corresponding properties for Q (or for A) as defined below. (For the
sake of simplicity, we ignore the case of a general subspace $\mathcal{T}$ of $\mathrm{End}^s(E)$.)

Definition 3.2.9. *Let Q be a positive definite quadratic form on $\mathbb{R}^n$ with*
matrix $A \in \mathrm{Sym}_n$. We say that Q (or A) is perfect *if the matrices P_X,*
$X \in S(Q)$ span Sym_n, *and that it is* weakly eutactic *if there exists a relation*

$$A^{-1} = \sum_{X \in S(Q)} \rho_X \, P_X \, .$$

We define similarly semi-eutaxy, eutaxy *and* strong eutaxy *for forms and*
symmetric matrices. [Warning: the coefficients ρ_X coincide with those of Theorem 3.2.5
only *up to the factor* $N(X) = {}^t X X$.]

More generally, when going from lattices to matrices (or to quadratic
forms), we attach to $\mathcal{T}$ a subspace $\mathcal{T}_{\mathrm{mat}}$ (or $\mathcal{T}_{\mathrm{quad}}$) of Sym_n (or of the space
$\mathcal{Q}_n$ of quadratic forms on $\mathbb{R}^n$), which we shall often simply denote by $\mathcal{T}$. Def-
inition 3.2.9 applies to this situation, provided that Sym_n should be replaced
by $\mathcal{T}$ and the P_X by their projections Ω_X onto $\mathcal{T}$.

The following statement, which only involves the usual notion of per-
fection, is an easy consequence of the mere definition of perfection. Due to
its great importance, we state it as a theorem, which actually characterizes
perfect lattices or forms:

Theorem 3.2.10. (Korkine and Zolotareff.)

1. *A perfect lattice with given norm m is well defined up to isometry by the*
   components in some basis of its minimal vectors.
2. *A perfect quadratic form with given minimum m is uniquely defined by*
   the set of its minimal vectors in $\mathbb{Z}^n$.

Proof. The dictionary quadratic forms–lattices shows that the two state-
ments above are equivalent. Let us consider the case of quadratic forms.

Let Q_1 and Q_2 be two perfect quadratic forms with the same set $S \subset \mathbb{Z}^n$
of minimal vectors, and let Q be the form $Q_1 - Q_2$. For all $X \in S$, we have
$Q_1(X) = Q_2(X) = m$, hence $Q(X) = 0$. The matrix A of Q is thus such that
$\forall X \in S$, ${}^t X A X = 0$, a condition equivalent to $\forall X \in S$, $\langle A, P_X \rangle = 0$. Since
the P_X, $X \in S$ generate Sym_n, this implies $A = 0$. $\square$

From the theorem above, we now deduce another result which also goes back to Korkine and Zolotareff, and which accounts for the crucial rôle that integral lattices play in the theory of the Hermite constant:

Proposition 3.2.11. *A perfect lattice is proportional to an integral lattice.*

Proof. Let Λ be a perfect lattice. We may rescale Λ so as to give it the norm 1. Let A be the Gram matrix of some basis for Λ. The corresponding quadratic form is determined by its minimal vectors, and these vectors have, moreover, integral components. Thus the entries of A are the solution of a linear system with integer coefficients, from which we can extract a Cramer system (because A is uniquely determined by this system). These entries are thus rational numbers. Consequently, $\sqrt{m}\,\Lambda$ is an integral lattice for any m such that $m\,A$ has entries in $\mathbb{Z}$. $\square$

3.3 Linear Inequalities

We prove here a theorem which presently belongs to the so-called linear programming theory. We shall make use of it in the next section to characterize extreme lattices.

Theorem 3.3.1. (Stiemke, [Sti].) *Let V be a real vector space and let $\varphi_1, \ldots, \varphi_r$ be linear forms on V. The following conditions are equivalent:*

1. *Every $x \in V$ which is a solution of the system of linear inequalities $\varphi_i(x) \geq 0$ for $i = 1, 2, \ldots, r$ is a solution of the linear system $\varphi_i(x) = 0$ for $i = 1, 2, \ldots, r$.*
2. *There exist strictly positive real numbers $\rho_1, \rho_2, \ldots, \rho_r$ such that $\rho_1\varphi_1 + \rho_2\varphi_2 + \cdots + \rho_r\varphi_r = 0$.*

Proof. The implication $(2) \Rightarrow (1)$ is obvious, and we now prove the other one (the only one which will be useful). There is nothing to prove if $V = \{0\}$ or if $r \leq 1$. We thus suppose $V \neq \{0\}$ and $r \geq 2$, and prove the theorem by induction on r. Since Assertions (1) and (2) are invariant under replacement of V by the intersection $W = \cap_i \ker \varphi_i$, we may and shall assume that the linear forms φ_i generate $V^* = \mathcal{L}(V, \mathbb{R})$.

Let $m \leq r$ be the largest integer such that there are m forms among the φ_i with the following property: there exists $x \in V$ on which these forms take values which are positive but not all zero. After permutation of the indices, we may assume that these forms are the first m forms.

Let us first show that $\varphi_1, \ldots, \varphi_m$ span V^*. If it were not the case, we could find a nonzero element $y \in \cap_{1 \leq i \leq m} \ker \varphi_i$ and an index $k > m$ with $\varphi_k(y) \neq 0$. Since such a k would contradict the definition of m, we certainly have $m = r$. But this contradicts the fact that V^* is spanned by the r linear forms φ_i.

We now distinguish two cases.

Case 1: $r > m + 1$. The fact that m is as large as possible allows us to apply the induction hypothesis to the $r - 1 > m$ forms $\varphi_1, \ldots, \varphi_{r-1}$: there exists a relation $\rho'_1 \varphi_1 + \cdots + \rho'_{r-1} \varphi_{r-1} = 0$ with strictly positive coefficients ρ'_i. We can write φ_r as a linear combination $\lambda_1 \varphi_1 + \cdots + \lambda_{r-1} \varphi_{r-1}$. Choose $\rho_r > 0$ and sufficiently small in order that $\rho_i = \rho'_i - \rho_r \lambda_i$ be positive for all $i < r$. We then have the equality $\sum_{1 \le i \le r} \rho_i \, \varphi_i = 0$, which completes the proof in this case.

Case 2: $r = m + 1$. We apply the induction hypothesis to $H = \mathrm{Ker}\, \varphi_r$. Let $x \in H$. Since $\varphi_r(x) = 0$, the inequalities $\varphi_i(x) \ge 0$ for all $i < r$ imply that we have $\varphi_i(x) = 0$ for all $i < r$.

On the one hand, there exist $\rho_1, \ldots, \rho_{r-1} > 0$ such that the linear form $\sum_{i<r} \rho_i \, \varphi_i$ is zero on H. On the other hand, the definition of m shows that there exists $v \in V$ with $\varphi_r(v) < 0$ and $\varphi_i(v) \ge 0$ for all $i < r$, with at least one of the $v_i \ne 0$. The inequalities $\sum_{i<r} \rho_i \, \varphi_i(v) > 0$ and $\varphi_r(v) < 0$ show that there exists $\rho_r > 0$ such that the linear form $\sum_{1 \le i \le r} \rho_i \, \varphi_i$ is zero at v. Since this form is also zero on H, it is zero on the whole space V. □

Here is an equivalent statement which shows up the convexity property which lies behind Stiemke's theorem:

Theorem 3.3.2. *Let K be the convex polytope defined by the inequalities $\varphi_i(x) \ge 0$. Then K reduces to the intersection of the kernels of the φ_i if and only if there exist strictly positive real numbers ρ_i such that $\sum_i \rho_i \varphi_i = 0$.* □

3.4 A Characterization of Extreme Lattices

We keep in this section the notation of Sections 3.1 and 3.2, but the results of Section 3.2 will be used only for $\mathcal{T} = \mathrm{End}^s(E)$. Given a lattice $\Lambda \subset E$, recall (Chapter 1, Definition 1.2.1 and Chapter 2, Definition 2.2.5) that $S = S(\Lambda)$ is the set of minimal vectors in Λ and $\gamma(\Lambda) = N(\Lambda) \det(\Lambda)^{-1/n}$ is its Hermite invariant.

Definition 3.4.1. *We say that Λ is* extreme *if the Hermite invariant attains a local maximum on Λ (for the topology on the set $\mathcal{L}$ of lattices in E defined in Section 1.1 of Chapter 1), and that it is* critical *(or* absolutely extreme*) if the Hermite invariant attains the absolute maximum on Λ.*

Equivalently, the density of the sphere packing attached to Λ (see Definition 1.8.1) is a local maximum. Clearly, the notion of extremality solely depends on the similarity class of Λ.

To test whether Λ is extreme, we study the family $u(\Lambda)$ when u runs through some neighbourhood of the identity of $\mathrm{GL}(E)$. The topology on

$\mathrm{End}(E)$ is defined by an arbitrary norm (all norms are equivalent). A classical choice is $\|u\| = \sup_{x \in E \setminus \{0\}} \frac{\|u(x)\|}{\|x\|}$; one can as well use the norm $\sqrt{\mathrm{Tr}({}^t uu)}$.

Thanks to Theorem 3.1.7, we could restrict ourselves to symmetric u, and even assume that u preserves the norm of Λ, since extremality is invariant under similarity. The way the determinant of Λ transforms is obvious: we have $\det(u(\Lambda)) = \det(\Lambda) \det(u)^2$. To evaluate the norm of Λ, we shall need the following lemma:

Lemma 3.4.2. *There exists a neighbourhood $\mathcal{V}$ of the identity in $\mathrm{GL}(E)$ such that, for $u \in \mathcal{V}$, the minimal vectors in $u(\Lambda)$ are the images of some minimal vectors in Λ; in other words, we have $S(u(\Lambda)) \subset u(S(\Lambda))$ for all $u \in \mathcal{V}$.*

Proof. Set $N_1 = N(\Lambda)$ and $N_2 = \min_{x \in \Lambda, N(x) > N(\Lambda)} N(x)$, and let $\mathcal{V} = \left\{ u \in \mathrm{GL}(E) \mid \|u^{-1}\| < \sqrt{N_2/N_1} \right\}$. We clearly have $N(u(y)) > N(u(x))$ for all $u \in \mathcal{V}$ whenever x is minimal and $N(y)$ is larger than N_2. $\qquad \square$

To see how $u(\Lambda)$ varies with u, we shall make use of a series expansion of $\gamma(u(\Lambda))$ in an appropriate neighbourhood of the identity. However, the calculations are greatly simplified by making use of ${}^t uu$ rather than u. Writing ${}^t uu = \mathrm{Id} + v$, we transform by $v \mapsto {}^t uu$ a fundamental system of neighbourhoods of 0 in $\mathrm{End}^s(E)$ into a fundamental system of neighbourhoods of Id in $\mathrm{GL}(E)$. Recall (Definition 3.2.1) that $\varphi_x(v) = v(x) \cdot x$ for $x \in E$ and $v \in \mathrm{End}^s(E)$.

Lemma 3.4.3. *There exists a neighbourhood $\mathcal{V}$ of 0 in $\mathrm{End}^s(E)$ such that $N(u(\Lambda)) = N(\Lambda)$ if and only if $\min_{x \in S} \varphi_x(v) = 0$.*

Proof. By Lemma 3.4.2, provided $\mathcal{V}$ is sufficiently small, we have

$$N(u(\Lambda)) = \min_{x \in S} N(u(x)),$$

and the equalities

$$N(u(x)) = u(x) \cdot u(x) = {}^t uu(x) \cdot x = \varphi_x(\mathrm{Id}) + \varphi_x(v) = N(x) + \varphi_x(v)$$

yield

$$N(u(\Lambda)) = N(\Lambda) + \min_{x \in S} \varphi_x(v). \quad \square$$

We now look at the determinant of u.

Lemma 3.4.4.   1. *There exists a neighbourhood $\mathcal{V}$ of 0 in $\mathrm{End}^s(E)$ such that, for $v \in \mathcal{V}$ with $\mathrm{Tr}(v) \le 0$ and $u \in \mathrm{GL}(E)$ such that ${}^t uu = \mathrm{Id} + v$, we have $u \in \mathrm{O}(E)$ or $\det(u) < 1$.*
  2. *Let $\mathcal{C}$ be a closed cone in $\mathrm{End}^s(E)$ such that $\mathrm{Tr}(v)$ is strictly positive for every nonzero $v \in \mathcal{C}$. There exists then $\alpha > 0$ such that*

$$v \in \mathcal{C} \quad \text{and} \quad 0 < \|v\| < \alpha \implies \det(\mathrm{Id} + v) > 1.$$

Proof. (1) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the (real) eigenvalues of v; those of $\mathrm{Id} + v$ are $1 + \lambda_1, 1 + \lambda_2, \ldots, 1 + \lambda_n$; they are strictly positive on any sufficiently small neighbourhood $\mathcal{V}$ of 0. We can thus consider on the interval $[0, 1]$ the two functions $t \mapsto \varphi_v(t) = \det(\mathrm{Id} + tv)$ and $\psi_v = \log \varphi_v$. The derivative of the second one is $\sum_{i=1}^{n} \dfrac{\lambda_i}{1 + \lambda_i t}$, and we thus have

$$\psi_v'(0) = -\sum_{i=1}^{n} \lambda_i = -\operatorname{Tr}(v) \leq 0\,.$$

If $v = 0$, u is an isometry.

If $v \neq 0$, ψ_v is strictly concave by Proposition 3.1.14, and this implies the inequality $\psi_v(t) < 0$ on $[0, 1]$. In particular, we have $\psi_v(1) < 0$, hence $\det(u) = (\varphi_v(1))^{1/2} < 1$.

(2) Let $\Sigma = \{w \in \mathrm{End}^s(E) \mid \|w\| = 1\}$ be the unit sphere of $\mathrm{End}^s(E)$ and let $w \in \mathcal{C} \cap \Sigma$. The function ψ_w is defined in any sufficiently small neighbourhood of 0, and we have by our hypothesis $\operatorname{Tr}(w) > 0$, hence $\psi_w'(0) > 0$; there thus exists $t_w > 0$ such that $\psi_w(t_w) > 0$. Since the function $w' \mapsto \psi_{w'}(t_w)$ is continuous on $\mathcal{C} \cap \Sigma$, there exists an open neighbourhood $\mathcal{V}(w)$ of w in $\mathcal{C} \cap \Sigma$ such that $\psi_{w'}$ is positive at t_w for all $w' \in \mathcal{V}(w)$, and also by convexity on the whole interval $(0, t_w]$.

From the covering $\bigcup_{w \in \mathcal{C} \cap \Sigma} \mathcal{V}(w)$ of the compact set $\mathcal{C} \cap \Sigma$, we can extract a finite covering $\bigcup_{1 \leq i \leq r} \mathcal{V}(w_i)$. Let $\alpha = \min(t_{w_1} \cdots t_{w_r})$, let $v \in \mathcal{C}$ such that $0 < \|v\| < \alpha$, and let $w = \frac{v}{\|v\|} \in \Sigma$. There exists i, $1 \leq i \leq r$, such that $w \in \mathcal{V}(w_i)$. Hence $f_w(t)$ is strictly positive on the interval $(0, \alpha) \subset (0, t_{w_i}]$, and in particular for $t = \|v\|$. This is equivalent to the inequality $\det(\mathrm{Id} + v) > 1$. $\qquad \square$

The calculation of the Hermite invariant by means of Lemmas 3.4.3 and 3.4.4, with $\mathcal{C} = \{v \in \mathrm{End}^s(E) \mid \forall\, x \in S(\Lambda),\ \varphi_x(v) \geq 0\}$ in 3.4.4 (2), immediately yields:

Theorem 3.4.5. (Korkine and Zolotareff) *A lattice Λ is extreme if and only if the following implication holds:*

$$v \in \mathrm{End}^s(E)\,, \quad \min_{x \in S(\Lambda)} \varphi_x(v) = 0 \quad and \quad \operatorname{Tr}(v) \leq 0 \implies v = 0\,.$$

Moreover if Λ is extreme, there exists a neighbourhood of Λ on which the Hermite invariant of any lattice which is not similar to Λ is strictly smaller than that of Λ. $\qquad \square$

We have previously defined (Definition 3.2.8) the notions of a perfect and of a eutactic lattice. They allow the following characterization of extreme lattices, which is the central result of this section:

Theorem 3.4.6. (Voronoi.) *A lattice is extreme if and only if it is both perfect and eutactic.*

Proof. The proof relies on the characterization of extreme lattices we have given in Theorem 3.4.5.

Let us first show that a eutactic and perfect lattice Λ is extreme. Let $v \in \mathrm{End}^s(E)$ such that $\min_{x \in S} \varphi_x(v) = 0$ and $\mathrm{Tr}(v) \leq 0$. In particular, we have $\varphi_x(v) \geq 0$ and $\mathrm{Tr}(v) \leq 0$, so that the definition of eutaxy implies that $\varphi_x(v)$ is zero for all $x \in S$. Since Λ is perfect, v must be zero.

Conversely, let Λ be an extreme lattice. Let us first show that Λ is perfect. Consider an element $v \in \mathrm{End}^s(E)$ such that $\varphi_x(v) = 0$ for all $x \in S$. We must verify that $v = 0$. Replacing if need be v by $-v$, we may assume that $\mathrm{Tr}(v) \leq 0$. Since $\min_{x \in S} \varphi_x(v) = 0$, Theorem 3.4.5 shows that v is zero.

To prove that Λ is also eutactic, we apply Stiemke's Theorem 3.3.1 to the set of linear forms φ_x, $x \in S$ and $-\mathrm{Tr}$ on $V = \mathrm{End}^s(E)$. We are reduced to prove that the inequalities $\varphi_x(v) \geq 0$ and $\mathrm{Tr}(v) \leq 0$ imply the equalities $\varphi_x(v) = 0$ and $\mathrm{Tr}(v) = 0$. For $k \in \mathbb{R}$, let $v' = v - k\,\mathrm{Id}$. We have $\mathrm{Tr}(v') = \mathrm{Tr}(v) - kn$ and $\varphi_x(v') = \varphi_x(v) - kN(\Lambda)$, hence $\min_{x \in S} \varphi_x(v') = \min_{x \in S} \varphi_x(v) - kN(\Lambda)$. Choose k in order that this minimum be zero. We clearly have $k \geq 0$ and $\mathrm{Tr}(v') \leq \mathrm{Tr}(v) \leq 0$. Theorem 3.4.5 shows that v' is zero, hence that v is multiplication by k. From $\varphi_x(v) = -kN(\Lambda) \leq 0$ and $\mathrm{Tr}(v) = kn \geq 0$, we deduce that $k = v = 0$, and this proves in particular that $\mathrm{Tr}(v) = 0$ and that the equalities $\varphi_x(v) = 0$ hold for all $x \in S$.   □

Corollary 3.4.7. (Korkine and Zolotareff.) *The n-th power of the Hermite constant is a rational number.*

Proof. A lattice whose Hermite invariant is maximum is extreme, hence in particular perfect, and we may apply Proposition 3.2.11.   □

3.5 Perfect Configurations

We keep the notation of the previous sections, and denote by S a configuration of vectors in E (S is thus a finite set of nonzero vectors in E). In practice, S is the set of minimal vectors of some lattice Λ; this is the reason why we work with vectors rather than with the family of lines that they define, though this point of view would have been more natural.

We study the perfection of S in the usual sense, disregarding the possible extensions of this notion to various subspaces $\mathcal{T}$ of $\mathrm{End}^s(E)$ other than $\mathrm{End}^s(E)$ itself, even when this does not cause any difficulty, as is the case for the following trivial (but useful) result:

Proposition 3.5.1. *Any configuration which contains a perfect configuration is perfect.*   □

Theorem 3.5.2. *1. A necessary and sufficient condition for a configura-*
  tion to be perfect is that it should not be contained in any quadratic cone.
  2. A perfect configuration is not contained in the union of two hyperplanes;
  in particular, a perfect configuration cannot be the union of two ortho-
  gonal strict subsets.
  3. The vectors of a perfect configuration span E.

  Proof. (2) results from (1) applied to the degenerated cones which are
the union of two hyperplanes and (3) is the particular case of (2) in which
the two hyperplanes collapse to a single one. We now prove (1) in terms of
matrices.

  To say that a configuration $S \subset \mathbb{R}^n$ is not perfect amounts to saying that
the matrices $X {}^t X$, $X \in S$ are contained in some hyperplane H inside Sym_n.
Since H is the subspace of Sym_n orthogonal to some nonzero symmetric ma-
trix A for the Voronoi scalar product $\langle U, V \rangle = \mathrm{Tr}(UV)$ of Proposition 3.1.10,
we have for every $X \in S$ the equivalences

$$X {}^t X \in H \Longleftrightarrow \langle A, X {}^t X \rangle = 0 \Longleftrightarrow \mathrm{Tr}(A {}^t X X) = 0$$
$$\Longleftrightarrow \mathrm{Tr}({}^t X A X) = 0 \Longleftrightarrow {}^t X A X = 0 \,,$$

and the last equality means that the elements of S belong to the cone defined
by the equation ${}^t X A X = 0$. $\qquad\square$

  The forthcoming perfection criterion, though far from being a general one,
is nevertheless very often useful, see e.g. the case of root lattices (Chapter 4,
Section 4.7). This is a kind of converse to the second assertion of the pre-
ceding theorem, as it shows that when S is not contained in the union of
two hyperplanes and possesses a perfect hyperplane section, it is then itself
perfect.

Proposition 3.5.3. *1. Let H be a hyperplane of E such that $S \cap H$ is a*
  perfect configuration in H. Then S is perfect if and only if the vectors
  in S which lie outside H span E.
  2. If there exist three distinct hyperplanes H_1, H_2, H_3 of E such that $S \cap H_i$
  is for every i a perfect configuration in H_i, S is then perfect.
  3. In order that a lattice Λ in E possessing a perfect hyperplane section with
  the same norm as Λ be perfect, it is necessary and sufficient that $S(\Lambda)$
  should contain n independent vectors outside this hyperplane section.
  4. A lattice Λ possessing three distinct hyperplane perfect sections having
  the same norm as Λ is perfect.

  Proof. (1) The necessity of the condition follows immediately from Theo-
rem 3.5.2 (the fact that $H \cap S$ be perfect is irrelevant here). To show the suf-
ficiency of the condition, we consider the space $\mathcal{P}_H = \{ p_H \circ u \mid u \in \mathrm{End}^s(E) \}$
(p_H stands for the orthogonal projection onto H), which has codimension n
in $\mathrm{End}^s(E)$. Choose a vector $e \neq 0$ in $H^\perp$ and a basis $\mathcal{B}$ for E consisting of n

vectors $e_1, \dots, e_n$ which lie in S but not in H. It suffices to show that there cannot exist a relation $\sum_i \lambda_i \, p_{e_i} \in \mathcal{P}_H$ with real not all zero λ_i. We now have the equivalences

$$\sum_i \lambda_i \, p_{e_i} \in \mathcal{P}_H \Longleftrightarrow \sum_i \lambda_i \, p_e \circ p_{e_i} = 0 \Longleftrightarrow \forall \, y \in E, \; \sum_i \lambda_i \, (e_i \cdot y)(e_i \cdot e) = 0 \,.$$

When one takes $y = e_j^*$, the last equality reduces to $\lambda_j \, (e_j \cdot e) = 0$, which implies $\lambda_j = 0$ since e_j does not lie in H.

(2). Since it is perfect, $S \cap H_2$ contains $n-1$ independent vectors outside $H_1 \cap H_2$. Since these vectors do not lie in H_1, (1) shows that it suffices to prove that there exists in $S \cap H_3$ a vector which does not lie in $H_1 \cup H_2$, a result which follows from Theorem 3.5.2 (2) applied to the hyperplanes $H_1 \cap H_3$ and $H_2 \cap H_3$ of H_3.

Finally, (3) and (4) are immediate consequences of (1) and (2). $\square$

We shall return to the question of relative perfection in Chapter 12, contenting ourselves in this section with the case of codimension 1 just dealt with above, for which the result can be expressed using only geometric properties of vectors in E. We now prove an important finiteness theorem:

Theorem 3.5.4. (Voronoi.) *For a given dimension n, the number of similarity classes of perfect lattices and the number equivalence classes up to proportionality of perfect quadratic forms are finite.*

Proof. The two statements are clearly equivalent. We shall prove the first one by showing the finiteness assertion up to isometry for norm 1 lattices.

Let thus Λ be a perfect lattice of norm 1. Since the minimal vectors in Λ span E, we have $\det(\Lambda) \le 1$ by the Hadamard inequality (Theorem 2.1.1). The Hermite inequality (Theorem 2.2.1) then shows that there exists a basis $\mathcal{B} = (e_1, \dots, e_n)$ for Λ for which the norms of the basis vectors are bounded from above by some constant c_n which solely depends on the dimension.

For $x = \sum_i a_i \, e_i \in \Lambda$, we have

$$|a_i|^2 = \frac{\det(\langle e_1, \dots, e_{i-1}, x, e_{i+1}, \dots, e_n \rangle)}{\det(\langle e_1, \dots, e_{i-1}, e_i, e_{i+1}, \dots, e_n \rangle)} \,.$$

Replacing the numerator by the upper bound given by the Hadamard inequality and the denominator by the lower bound given by the Hermite inequality, we obtain the upper bound

$$|a_i|^2 \le \gamma_n^n c_n^{n-1} N(x) \,,$$

which bounds $|a_i|^2$ for $1 \le i \le n$ and all $x \in \Lambda$ by $C_n N(x)$, where the constant C_n depends only on n.

The components in $\mathcal{B}$ of any $x \in S(\Lambda)$ are thus bounded from above by $\sqrt{C_n}$. Hence the set of the components of the minimal vectors of a perfect

norm 1 lattice in a basis which satisfies the Hermite inequality is a subset of a finite subset of $\mathbb{Z}^n$, and our claim now follows from Theorem 3.2.10.   □

We end this section with applications of the notion of perfection to the study of the kissing number.

Theorem 3.5.5. *An n-dimensional lattice for which the kissing number is maximum among lattices of dimension n is perfect.*

Proof. Let Λ be such a lattice. If it is not perfect, there exists a nonzero $v \in \mathrm{End}^s(E)$ such that $\varphi_x(v) = 0$ for all $x \in S(\Lambda)$ (see Definitions 3.2.1 and 3.2.2). For $\lambda \geq 0$ and sufficiently small for the square root to exist, let $u_\lambda = \sqrt{\mathrm{Id} - \lambda v}$. Lemma 3.4.3 shows that the lattice $u_\lambda(\Lambda)$ has the same norm as Λ for sufficiently small λ and satisfies the property $S(u_\lambda(\Lambda)) = u_\lambda(S(\Lambda))$. Since $\det(u_\lambda(\Lambda)) = \det(u_\lambda) \det(\Lambda)$ strictly decreases and tends to 0 on the interval of definition of u_λ, the norm of $u_\lambda(\Lambda)$ becomes smaller than that of Λ for all sufficiently large λ. Let μ be the supremum of the λ for which $N(u_\lambda(\Lambda)) = N(\Lambda)$ and $S(u_\lambda(\Lambda)) = u_\lambda(S(\Lambda))$. The lattice $u_\mu(\Lambda)$ has clearly the same norm as Λ but at least one pair of minimal vectors outside $u_\lambda(S(\Lambda))$. We thus have $s(u_\mu(\Lambda)) > s(\Lambda)$.   □

Finer results have been obtained for dimensions 8 and 24 for the proof of which we refer the reader to [C-S], Chapter 14; the lattice $\mathbb{E}_8$ (resp. the Leech lattice Λ_{24}) is defined in Section 4.4 (resp. in Sections 5.7 and 8.7):

Theorem 3.5.6. (Bannai and Sloane.) *In dimension 8 (resp. 24), the configuration of the centres of $240 = 120 \times 2$ (resp. $196560 = 98280 \times 2$) spheres of radius 1 in contact with the unit sphere of E is similar to that of the minimal vectors in $\mathbb{E}_8$ (resp. in Λ_{24}).*   □

From this theorem, we first deduce that the maximum of s for dimension 8 (resp. 24) is equal to 120 (resp. to 98280), a result first proved by Odlyzko and Sloane; see [C-S], Chapter 13, then that it is attained only on $\mathbb{E}_8$ (resp. on Λ_{24}), and finally that these two lattices are perfect. Actually, for a given dimension n, if an inequality $s(\Lambda) \geq s_0$ has only finitely many solutions up to similarity, then all lattices with $s(\Lambda) \geq s_0$ are perfect.

Some other examples of lattices for which a characterization can be obtained via their kissing numbers will be seen in Chapter 6.

3.6 Eutactic Configurations and Extreme Lattices

We keep the notation of the preceding sections. The aim of this section is to obtain methods which will allow us to prove that some configurations are eutactic without finding explicitly the eutaxy coefficients. We shall also give

a theoretical characterization of eutaxy. For the sake of simplicity, we do not consider $\mathcal{T}$-eutaxy for $\mathcal{T} \neq \mathrm{End}^s(E)$.

Proposition 3.6.1. *A union of weakly eutactic (resp. semi-eutactic, resp. eutactic) configurations is weakly eutactic (resp. semi-eutactic, resp. eutactic); a union of* disjoint *strongly eutactic configurations is strongly eutactic.*

Proof. We consider line configurations. An induction argument shows that it suffices to prove the proposition for a union $S = S_1 \cup S_2$ of two configurations. By weak eutaxy, there exist two eutaxy relations $\mathrm{Id} = \sum_{x \in S_1} \rho_D \, \varphi_D$ and $\mathrm{Id} = \sum_{D \in S_2} \pi_D \, \varphi_D$.

Let $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Then the eutaxy relation

$$\mathrm{Id} = \sum_{D \in S_1} \alpha \rho_D + \sum_{\Delta \in S_2} \beta \rho_\Delta$$

holds for S. This proves the proposition for (weak, semi-) eutaxy. In the case where strong eutaxy holds for S_1 and S_2, the eutaxy relations for S_1 and S_2 take the forms $\mathrm{Id} = \rho \sum_{D \in S_1} \varphi_D$ and $\mathrm{Id} = \pi \sum_{D \in S_2} \varphi_D$ and yield a eutaxy relation for S with the two coefficients $\alpha \rho$ and $\beta \pi$, which are equal when choosing $\alpha = \frac{\pi}{\rho + \pi}$ and $\beta = \frac{\rho}{\rho + \pi}$. [Variant for strong eutaxy: since the partial sums $\sum p_x$ are proportional to the identity, so is the global sum.] $\qquad \square$

Recall (Corollary 3.2.6) that the vectors of a weakly eutactic configuration span E.

Theorem 3.6.2. *A configuration which contains a perfect and eutactic configuration is itself perfect and eutactic.* [However, without the perfection hypothesis, only weak eutaxy is invariant by extension; see Exercise 3.6.1.]

Proof. Let S be a perfect and eutactic configuration, and let S' be a configuration which contains S. By induction on $|S'| - |S|$, we restrict ourselves to the case where S' is of the form $S \cup \{D'\}$. By Proposition 3.5.1, S' is perfect; let us now prove that it is also eutactic. By hypothesis, there exists a relation $\mathrm{Id} = \sum_{D \in S} \rho_D \, p_D$ with strictly positive coefficients ρ_D. Since S is perfect, we can express $p_{D'}$ as a linear combination of the p_x, $x \in S$: there exists a relation $p_{D'} = \sum_{D \in S} \lambda_D \, p_D$ with real coefficients λ_D. This yields for S' the new eutaxy relations

$$\mathrm{Id} = \mu p_{D'} + \sum_{D \in S} (\rho_D - \mu \lambda_D) p_D$$

with arbitrary $\mu \in \mathbb{R}$. For sufficiently small strictly positive $\mu > 0$, this is a eutaxy relation with strictly positive coefficients. $\qquad \square$

We now easily deduce from Theorem 3.6.2 an extremality criterion for lattices:

Theorem 3.6.3. *Let Λ be a lattice. The following conditions are equivalent:*

1. *Λ is extreme.*
2. *The lattice Λ_S generated by the minimal vectors in Λ is extreme.*
3. *Λ contains an extreme lattice Λ' with the same norm.*

Proof. (1) $\Rightarrow$ (2). Let Λ_S be the subgroup of Λ generated by $S(\Lambda)$. Theorem 3.5.2 shows that Λ_S is a sublattice of Λ. Since $S(\Lambda_S) = S(\Lambda)$, Λ_S is both perfect and eutactic, hence extreme.

(2) $\Rightarrow$ (3). Take $\Lambda' = \Lambda_S$.

(3) $\Rightarrow$ (1). Since Λ' and Λ have the same norm, we have the inclusion $S(\Lambda') \subset S(\Lambda)$. Since Λ' is extreme, the configuration $S(\Lambda')$ is both perfect and eutactic. By Theorem 3.6.2, the same properties hold for $S(\Lambda)$, and Λ is thus extreme, since it is both perfect and eutactic. $\square$

We now look at connections which exist between the eutaxy property and automorphisms. We denote by S a configuration of vectors in E. As usual, the definitions and results which follow can easily be translated in terms of line configurations.

Definition 3.6.4. *An automorphism of S is an orthogonal transformation which preserves S.*

When S generates E, $\mathrm{Aut}(S)$ is a finite group. The orthogonal group $\mathrm{O}(E)$ acts on $\mathrm{End}^s(E)$ by the rule $s \cdot u = s \circ u \circ s^{-1}$.

For all nonzero $x \in E$ and for all $s \in \mathrm{O}(E)$, we have $s \circ p_x \circ s^{-1} = p_{sx}$, because of the following equalities, valid for all $y \in E$:

$$\left(s \circ p_x \circ s^{-1}\right)(y) = \frac{x \cdot s^{-1}(y)}{x \cdot x}\, sx = \frac{sx \cdot y}{x \cdot x}\, sx = p_{sx}(y)\,.$$

The linear forms φ_x behave in a similar way under the action of the orthogonal group: we indeed have

$$\varphi_x(s \cdot u) = \left(s^{-1} \circ u \circ s\right)(x) \cdot x = u(sx) \cdot sx = \varphi_{sx}(u)$$

for all $x \in E$, $u \in \mathrm{End}^s(E)$, and $s \in \mathrm{O}(E)$.

Given a finite subgroup G of $\mathrm{O}(E)$ and an orbit ω of G on S, we define $p_\omega \in \mathrm{End}^s(E)$ by $p_\omega = \frac{1}{|G|} \sum_{x \in \omega} p_x$ and $\varphi_\omega \in (\mathrm{End}^s(E))^*$ by $\varphi_\omega = \sum_{x \in \omega} \varphi_x$.

Proposition 3.6.5. *Let $\mathcal{O}$ be the set of orbits of G acting on S. The following conditions are equivalent:*

1. *The configuration S is weakly eutactic (resp. eutactic).*
2. *There exists a relation $\mathrm{Tr} = \sum_{\omega \in \mathcal{O}} \rho_\omega\, \varphi_\omega$ with real (resp. with strictly positive) coefficients ρ_ω.*

Proof. Suppose that S is weakly eutactic, and let

$$\mathrm{Tr} = \sum_{x \in S} \lambda_x \, \varphi_x$$

be a eutaxy relation with real coefficients λ_x. For all $u \in \mathrm{End}^s(E)$ and all $s \in G$, we have

$$\mathrm{Tr}(u) = \mathrm{Tr}(s^{-1} \circ u \circ s) = \sum_{x \in S} \lambda_x \, \varphi_{sx}(u) \Longleftrightarrow \mathrm{Tr} = \sum_{x \in S} \lambda_x \, \varphi_{sx} \, ,$$

whence by summation on the elements of G:

$$|G| \, \mathrm{Tr} = \sum_{x \in S} \lambda_x \sum_{s \in G} \varphi_{sx} \, .$$

Taking $\rho_\omega = \sum_{x \in \omega} \lambda_x$, we obtain a relation of the form we want, and the coefficients ρ_ω are plainly strictly positive when the λ_x are.

The converse statements are obvious.   □

To test whether a given lattice Λ is eutactic, it is thus interesting to find a subgroup of $\mathrm{Aut}(\Lambda)$ which is as large as possible. The following two theorems list examples for which we can directly prove the eutaxy property, for a lattice or for its dual. (The complete statement of the second theorem anticipates Section 3.8.)

Theorem 3.6.6. *Let G be a subgroup of $\mathrm{Aut}(S)$. Suppose that one of the following conditions holds:*

1. *The trace form belongs to the span in $\mathrm{End}^s(E)$ of the φ_x, $x \in S$, and G acts transitively on S.*
2. *G acts irreducibly on E.*

Then S is a strongly eutactic configuration.

[One says that G acts *irreducibly* or is *irreducible* on E if the only invariant subspaces of E are E itself and $\{0\}$; this means that the representation of G afforded by the embedding $\mathrm{O}(E) \subset \mathrm{GL}(E)$ is irreducible *over* $\mathbb{R}$.]

Theorem 3.6.7. *Let Λ be a perfect lattice. Suppose that one of the following conditions holds:*

1. *G acts transitively on $S(\Lambda)$.*
2. *G acts irreducibly on E.*

Then Λ is extreme, and both Λ and Λ^* are strongly eutactic (and dual-extreme in the sense of Definition 3.8.1 below).

Proof of 3.6.6 under hypothesis (1). The existence of a eutaxy relation with a single eutaxy coefficient is an immediate consequence of Proposition 3.6.5 (2). (This unique coefficient is equal to $\frac{s}{n}$, see Formula (2) in Theorem 3.2.5.)

The proof under hypothesis (2) will rely on the following lemma:

Lemma 3.6.8. *If G is irreducible, the set of symmetric endomorphisms of E which commute with G is the 1-dimensional subspace of E generated by the identity.*

Proof of 3.6.6 under hypothesis (2). Let us take provisionally for granted the lemma above, and consider the orbit ω of some element $x \in S$. The endomorphism $p_\omega = \sum_{s \in G} p_{sx}$ commutes with G, and is thus of the form $\lambda \operatorname{Id}$ for an appropriate $\lambda \in \mathbb{R}$. This shows that the orbit ω is a strongly eutactic configuration, and so is S by Proposition 3.6.1, since it is a disjoint union of orbits. $\qquad \square$

Proof of 3.6.8. Let $u \in \operatorname{End}^s(E)$ which commutes with G. Let x be an eigenvector for u, with corresponding eigenvalue λ (which exists in $\mathbb{R}$ since u is symmetric). For all $s \in G$, we have

$$ u(sx) = u \circ s(x) = s \circ u(x) = s(\lambda \, x) = \lambda \, (sx) \, . $$

Since G is irreducible, the sx, $s \in G$ span E. Hence u is the homothetic transformation $x \mapsto \lambda \, x$. $\qquad \square$

Proof of 3.6.7. By Theorem 3.4.6 (and Corollary 3.8.6 below), it suffices to prove that both Λ and Λ^* are strongly eutactic. Under assumption (2), this is a direct consequence of Theorem 3.6.6, which also applies to Λ^*, since $\operatorname{Aut}(\Lambda) = \operatorname{Aut}(\Lambda^*)$. Moreover, since Λ is perfect, assumption (1) in Theorem 3.6.6 is satisfied, which shows that Λ is strongly eutactic.

To prove that Λ^* is strongly eutactic under assumption (1), consider the sum $u \in \operatorname{End}^s(E)$ of all orthogonal projections onto the directions of minimal vectors in Λ^*. By the perfection of Λ, u is a linear combination of orthogonal projections onto the directions of minimal vectors in Λ. Taking the average under G of both sides of this equality, we obtain a relation of the form

$$ \sum_{x \in S(\Lambda)/\{\pm 1\}} p_x = \lambda \sum_{y \in S(\Lambda^*)/\{\pm 1\}} p_y \, , $$

$\lambda \in \mathbb{R}$. Since Λ is strongly eutactic, the left–hand side is proportional to the identity. The same result thus holds for the right–hand side. $\qquad \square$

Remark 3.6.9. *One can generalize Theorem 3.6.6 to $\mathcal{T}$-eutactic configurations such that $\mathcal{T}$ is invariant under G.*

Following Coxeter and Hadwiger, we shall now prove a geometrical characterization of the eutaxy condition. We consider line configurations with s elements, with which we associate one of the vector configurations, denoted by S, obtained by choosing arbitrarily a unit vector on each line.

Theorem 3.6.10. *1. If S is weakly eutactic, then for all subspaces F of E, the lines in S which do not lie inside F span a subspace of E which contains the orthogonal of F.*
2. Any weakly eutactic configuration contains at least n lines which span E.
3. Weakly eutactic configurations which contain exactly n lines are the configurations of n orthogonal lines, and they are strongly eutactic.

Proof. (1) A eutaxy relation for S can be written in the form

$$\mathrm{Id} = \sum_{x \in S \cap F} \lambda_x \, p_x + \sum_{y \in S \smallsetminus F} \lambda_y \, p_y \,.$$

Applying both sides to a vector $e \in F^{\perp}$, we obtain $e = \sum_{y \in S \smallsetminus F} \lambda_y p_y(e)$.

(2) This result, previously proved as Corollary 3.2.6, follows immediately from (1).

(3) By induction on n, we see that among the configurations of n lines, only those which constitute an orthogonal system may be weakly eutactic. That these configurations are actually strongly eutactic results from the identity $\mathrm{Id} = \sum_{i=1}^{n} p_{\varepsilon_i}$ which holds for any orthogonal basis $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ for E. $\qquad\square$

Proposition 3.6.11. *Any orthogonal projection of a (weakly, semi-) eutactic configuration is a (weakly, semi-) eutactic configuration.*
[For this statement to make sense, one must of course remove the lines or vectors which project to $\{0\}$.]

Proof. Let S be a (weakly) eutactic configuration of unit vectors in E and let F be a subspace of E. Since for $E \supset F \supset G$, the orthogonal projection onto G can be obtained by performing first the orthogonal projection onto F and then the orthogonal projection *in F* onto G, we may assume by induction that F is a hyperplane of E.

Let e be a unit vector in $F^{\perp}$ and let S' be the projection of S onto F (after having removed the vector 0 when e or $-e$ belongs to S). Any vector $x \in S$ can be written in a unique way as a sum $x = x' + \lambda e$ with $x' \in S' \cup \{0\}$ and $\lambda \in \mathbb{R}$. There exist coefficients $\rho_x \in \mathbb{R}$ such that

$$\forall \, y \in E, \ y = \sum_{x \in S} \rho_x \, (x \cdot y) \, x \,.$$

For $y \in F$, this also reads

$$y = \sum_{x \in S} \rho_x \, (x' \cdot y) \, x' + \sum_{x \in S} \rho_x \lambda_x \, (x' \cdot y) \, e \,.$$

Since the left–hand side and the first sum in the equality above belong to F, the second sum is zero, and we obtain (after removing the vector 0 if need be) a eutaxy relation for S', whose coefficients $\rho'_x = \rho_x (x' \cdot y)$ are (strictly) positive whenever the ρ_x are. $\square$

Theorem 3.6.12. (Hadwiger.) *For a configuration S of s lines in E to be eutactic, it is necessary and sufficient that it should be the orthogonal projection on E of an orthogonal configuration of s lines in an s-dimensional Euclidean space F containing E.*

Proof. We know by Theorem 3.6.10 and Proposition 3.6.11 that we have $s \geq n$ and that our claim is true if $s = n$. Suppose now $s > n$. By induction on $s - n$, we are reduced to show that S is the projection onto E of some eutactic configuration lying in an $(n+1)$-dimensional space F containing E.

We thus embed E into Euclidean space F of dimension $n+1$ which we decompose as an orthogonal sum $F = E \perp \mathbb{R}e$ for some unit vector $e \in F$. We then lift all vectors $x \in S$ to vectors $x' = x + \lambda_x e \in F$, where the λ_x are real numbers to be chosen later. We obtain in this way a set S' of s vectors in F.

Let ρ_x, $x \in S$ be the eutaxy coefficients for S. We shall now show that for a suitable choice of the λ_x, we obtain a decomposition the identity of F of the form $\mathrm{Id} = \sum_{x \in S} \rho_x N(x') p_{x'}$. The existence of such an equality is equivalent to the condition

$$\forall\, y \in F\,,\ \ y = \sum_{x \in S} \rho_x (x' \cdot y)\, x + \sum_{x \in S} \rho_x \lambda_x (x' \cdot y)\, e \qquad (*)$$

which we are going to check for $y \in E$ and for $y = e$.

If $y \in E$, we have $x' \cdot y = x \cdot y$ and thus

$$\sum_{x \in S} \rho_x (x' \cdot y)\, x = \sum_{x \in S} \rho_x (x \cdot y)\, x = y\,,$$

so that condition $(*)$ is equivalent to the equality $\sum_{x \in S} \rho_x \lambda_x (x \cdot y) = 0$. Since $s > n$, there exists between the vectors in S a non-trivial relation $\sum_{x \in S} \alpha_x\, x = 0$. We then satisfy condition $(*)$ in this case by taking $\lambda_x = \lambda \frac{\alpha_x}{\rho_x}$ where $\lambda \in \mathbb{R}$ can be arbitrarily chosen.

If $y = e$, we have

$$\sum_{x \in S} \rho_x (x' \cdot y)\, x = \sum_{x \in S} \rho_x \lambda_x\, x = \lambda \sum_{x \in S} \alpha_x\, x = 0\,,$$

and condition $(*)$ reduces to

$$\lambda = \left(\sum_{x \in S} \frac{\alpha_x^2}{\rho_x} \right)^{-1/2}\,,$$

a condition which it is possible to ensure since the various terms in the sum above are positive or zero, but not all zero.   □

The eutaxy condition for a configuration S of vectors in E can be expressed in terms of *eutactic stars*: these are vector configurations T for which the identity $N(y) = \sum_{x' \in T} (x' \cdot y)^2$ holds on E. Starting from a eutactic configuration S and making use of condition (3) in Theorem 3.2.5, we transform S into a eutactic star S' by multiplying each vector in S by its eutaxy coefficient.

Among the various sufficient conditions for eutaxy to hold, we quote the following one, which is particularly useful to handle dual-extreme lattices, a notion to be defined in Section 3.8.

Theorem 3.6.13. *Suppose that E is an orthogonal direct sum of subspaces $E_1, \ldots, E_r$. For all i, let S_i be a configuration (say, of vectors) in E_i and let Λ_i be a lattice in E_i. Let S be the union $S = \cup_i S_i$ and let Λ be the (orthogonal) sum $\Lambda = \Lambda_1 + \cdots + \Lambda_r$. Then:*

1. *S is (weakly, semi-) eutactic if and only if all S_i are (weakly, semi-) eutactic.*
2. *S is strongly eutactic if and only if all S_i are strongly eutactic and the ratio $\dfrac{|S_i|}{\dim E_i}$ is independent of i.*
3. *Λ is (weakly, semi-) eutactic if and only if Λ_i is (weakly, semi-) eutactic for all i and all Λ_i have the same norm.*
4. *Λ is strongly eutactic if and only if Λ_i is strongly eutactic for all i, all Λ_i have the same norm, and the ratio $\dfrac{s(\Lambda_i)}{\dim \Lambda_i}$ is independent of i.*

Proof. (1) For any i and any nonzero $x \in E_i$, we have $p_{E_i} \circ p_x = p_x$. Thus S is eutactic in E, S_i is eutactic in E_i for all i. Conversely, a eutaxy relation for S_i can be written in the form $p_{E_i} = \sum_{x \in S_i} \rho_x \, p_x$, and the equality $\mathrm{Id} = \sum_i p_{E_i}$ transforms the set of eutaxy relations for the S_i into a eutaxy relation for S, with appropriate signs of the coefficients.

(2) From a strong eutaxy relation $\mathrm{Id} = \rho \sum_{x \in S} p_x$, we deduce the equalities $p_{E_i} = \rho \sum_{x \in S_i} p_x$ which are strong eutaxy relations for the S_i in which moreover $\dfrac{|S_i|}{\dim E_i} = \rho$ is constant. Conversely, the eutaxy relation obtained using the identity $\mathrm{Id} = \sum_i p_{E_i}$ is a relation of strong eutaxy whenever the coefficients of strong eutaxy for the S_i do not depend on i.

(3) and (4) We have $N(\Lambda) = \min_i N(\Lambda_i)$. Hence $S(\Lambda)$ does not span E if the lattices Λ_i do not have the same norm. Conversely, if all Λ_i have the same norm, then $S(\Lambda) = \cup_i S(\Lambda_i)$, and (3) and (4) are direct consequences of (1) and (2) respectively.   □

3.7 The Lamination Process

Let $n_0 \geq 0$ be an integer, let Λ_0 be a lattice in a Euclidean space E_0 of dimension n_0, and let $N_0 = N(\Lambda_0)$. Embed E_0 as a hyperplane in a Euclidean space E_1 of dimension $n_0 + 1$. We are interested in lattices $\Lambda_1 \subset E_1$ with norm N_0 and such that $\Lambda_1 \cap E_0 = \Lambda_0$.

Definition 3.7.1. *We say that $\Lambda_1 \subset E_1$ is* extreme relative to Λ_0 *(resp.* critical relative to Λ_0*) if its Hermite constant is a local (resp. an absolute) maximum on the set of lattices in E_1 with norm N_0 whose intersection with E_0 is Λ_0.*

Proposition 3.7.2. *Let Λ_0 be a lattice in E_0 and let $\Lambda_1 \subset E_1$ be extreme relative to Λ_0. Then the set of minimal vectors in Λ_1 lying outside E_0 spans E_1.*

Proof. Otherwise, the minimal vectors in E_1 lie in some hyperplane $H \neq E_0$ in E_1. Let $\alpha \in (0, \frac{\pi}{2})$ be the angle of E_0 and H (non-oriented angle of two unit vectors e_0 and e orthogonal to E_0 and to H respectively). For $0 < \theta < \frac{\pi}{2}$, let e_θ be the vector of the plane $\langle e_0, e \rangle$ whose angle with e_0 (resp. e) is $\alpha - \theta$ (resp. θ), and let H_θ be the hyperplane of E_1 which is orthogonal to e_θ. Consider the rabatment u_θ which maps H onto H_θ (u_θ is the identity on E_0 and transforms e into $(\sin \theta)\, e + (\cos \theta)\, e_0$).

For all $x \in H$, we have $N(u_\theta(x)) = N(x)$. Since the vectors in $S(\Lambda_1)$ lie in $E_0 \cup H$, the image under u_θ of $S(\Lambda_1)$ is made of vectors of norm N_0. Since u_θ is continuous and $u_0 = \mathrm{Id}$, the images under u_θ of vectors in Λ_1 which do not lie in $E_0 \cup H$ all have for sufficiently small θ a norm $N > N_0$. The equalities

$$\det(u_\theta(\Lambda_1)) = \det(u_\theta)^2 \cdot \det(\Lambda_1) = (\cos^2 \theta)\, \det(\Lambda_1)$$

show that the determinant of $u_\theta(L_1)$ is strictly smaller than that of Λ_1 for $\theta > 0$, and this contradicts the hypothesis that the Hermite invariant of Λ_1 is a local maximum. $\qquad\square$

Applying Proposition 3.5.3, we obtain:

Corollary 3.7.3. *A lattice which is extreme relative to a perfect lattice is perfect.* $\qquad\square$

Remark 3.7.4. *We shall prove in Chapter 12 (in any codimension) a characterization à la Voronoi of relatively extreme lattices as relatively perfect and relatively eutactic lattices, and also show that a lattice which is perfect relative to a perfect lattice is perfect. In this section, we have restricted ourselves to the case of codimension 1 (for which the statement above reduces to Proposition 3.5.3).*
However, there exist non-extreme lattices which are extreme relative to an

extreme lattice: it may actually happen that a deformation of Λ_1 does not preserve the isometry class of Λ_0, and examples in codimension 1 have been given by Barnes ([Bar5], I, p. 64); an explicit example is provided by the lattices $L_{2r}^r \subset L_{2r+1}^r$ of Definition 8.4.4.

Definition 3.7.5. *Let Λ_0 be a lattice of dimension n_0, and let $n \geq n_0$ be an integer.*

1. *We say that an n-dimensional lattice Λ is a* weakly laminated *lattice above Λ_0 if there exists a sequence $L_0 = \Lambda_0, L_1, \ldots, L_{n-n_0} = \Lambda$ of lattices of dimensions $n_0, n_0 + 1, \ldots, n$, each term of which is critical relative to the preceding one.*
2. *We say that Λ is* (strongly) laminated *above L_0 if its Hermite invariant is maximal among all n-dimensional weakly laminated lattices.*

Laminated lattices above a given lattice Λ_0 have the same norm as Λ_0, and those which are strongly laminated moreover share the same value for their determinant. By Corollary 3.7.3, laminated lattices (strongly or weakly) above a perfect lattice are perfect. We say *laminated lattices* without any other precision for the particular case considered by Conway and Sloane:

Definition 3.7.6. *An n-dimensional laminated lattice is an n-dimensional strongly laminated lattice above the trivial lattice $\{0\}$ to which is given the norm 4. We denote by Λ_n any n-dimensional laminated lattice, with possibly a superscript when uniqueness does not hold in dimension n.*

For $n \geq 1$, they are laminated above $2\mathbb{Z}$, but to consider dimension 0 has some importance to take advantage of various symmetries. Conway and Sloane ([C-S], Chapter 6) have determined the determinants of the laminated lattices up to dimension 48 and found all laminated lattices up to dimension 25. These determinants can be calculated by simple symmetry rules once they are known in dimensions 1 to 4. For the proofs, see [C-S], Chapter 6.

Theorem 3.7.7. *(Conway and Sloane.) The values λ_n of the determinants of the laminated lattices Λ_n in dimensions $n \leq 48$ satisfy the following rules, which determine them:*

1. *$\lambda_0 = 1$, $\lambda_1 = 4$, $\lambda_2 = 12$, $\lambda_3 = 32$, $\lambda_4 = 64$.*
2. *$\lambda_n = 2^{2n-8}\lambda_{8-n}$ for $0 \leq n \leq 8$.*
3. *$\lambda_n = 2^{16-n}\lambda_{n-8}$ for $8 \leq n \leq 16$.*
4. *$\lambda_n = \lambda_{24-n}$ for $0 \leq n \leq 24$.*
5. *$\lambda_n = 2^{24-n}\lambda_{n-24} = 2^{24-n}\lambda_{48-n}$ for $24 \leq n \leq 48$.* □

Laminated lattices are integral up to dimension 24, and are unique except in dimensions 11, 12 and 13 where there exist respectively 2, 3 and 3 isometry

classes, which have different kissing numbers and which we distinguish by superscripts min, mid, max, according to the following table:

Table 3.7.7′. Laminated lattices in dimensions 11-13

$n = 11$	$s(\Lambda_{11}^{\min}) = 216$		$s(\Lambda_{11}^{\max}) = 219$
$n = 12$	$s(\Lambda_{12}^{\min}) = 312$	$s(\Lambda_{12}^{\mathrm{mid}}) = 316$	$s(\Lambda_{12}^{\max}) = 324$
$n = 13$	$s(\Lambda_{13}^{\min}) = 444$	$s(\Lambda_{13}^{\mathrm{mid}}) = 445$	$s(\Lambda_{13}^{\max}) = 453$

[We shall construct in Chapter 8, Sections 8.6 to 8.8, all laminated lattices up to dimension 24 as well as a "principal series" between dimensions 25 and 48, whose members become integral when rescaled to norm 8 (the smallest possible norm for $25 \leq n \leq 48$, see Exercise 3.7.2).]

It should be noticed that among strongly laminated lattices above a given lattice, there may exist dead-ends. This is the case for $\Lambda_{13}^{\mathrm{mid}}$: laminated lattices above $\Lambda_{13}^{\mathrm{mid}}$ have a determinant larger than that of Λ_{14}. This is the only example among the usual laminated lattices of dimension up to 24.

Modifications of the lamination process, involving algebraic structures (given automorphism groups, module structures over various orders of number fields or quaternion skew-fields) have been considered. Note, however, that lattices which can be constructed in this way are not *a priori* perfect, at least for the notion of perfection defined in this chapter.

Besides the previous laminations that they name "geometrical laminations", Plesken and Pohst ([Pl-P1], [Pl-P2]), have defined "arithmetical laminations", in both a weak and a strong sense. One starts with an integral lattice Λ_0 of dimension n_0 and norm m_0, and then consider integral lattices of norm m_0 which contain Λ_0 as a codimension 1 section and which, moreover, has a basis obtained by adjoining a new minimal vector to a basis for Λ_0. These are the *weak arithmetical laminations over* Λ_0, and one obtains the *strong arithmetical laminations over* Λ_0 by keeping only those lattices with the largest possible Hermite invariant. (As previously, a discrepancy may appear from dimension $n_0 + 2$ onwards.) There is no reason for perfection to be preserved. It seems likely that the weak laminated lattices in the geometrical sense above $\Lambda_{13}^{\mathrm{mid}}$ lie among the arithmetical ones, which in particular allows the embedding of $\Lambda_{13}^{\mathrm{mid}}$ into Λ_{17} (the embedding in Λ_{16} is not possible; see Chapter 8).

3.8 Dual-Extreme Lattices

We still consider a Euclidean vector space of dimension n. Recall (Chapter 2, Section 2.8) that the "dual Hermite invariant" of a lattice $\Lambda \subset E$ is

$$\gamma'(\Lambda) = \|\Lambda\| \cdot \|\Lambda^*\| = (N(\Lambda)\,N(\Lambda^*))^{1/2} = (\gamma(\Lambda)\,\gamma(\Lambda^*))^{1/2}\,.$$

We have $\gamma'(\Lambda) = \gamma'(\Lambda^*) = (\gamma(\Lambda)\,\gamma(\Lambda^*))^{1/2} \le \gamma_n$, which proves the existence of $\gamma'_n = \sup_\Lambda \gamma'(\Lambda)$.

Definition 3.8.1. *We say that Λ is* dual-extreme *if the invariant γ' attains a local maximum on Λ.*

Notice that Λ and Λ^* play symmetric rôles in this definition, so that a lattice is dual-extreme if and only if its dual is.

We now prove a first necessary condition for a lattice to be dual-extreme, which is of a great practical importance, and which constitutes a fundamental lemma for the proof of the characterization of dual-extreme lattices given in Theorem 3.8.4 below.

Proposition 3.8.2. *Each of the sets of minimal vectors of a dual-extreme lattice and of its dual spans E.*

Proof. Let Λ be a lattice which does not satisfy the condition above, and let $\mathcal{V}$ be a neighbourhood of Λ in the space $\mathcal{L}$ of lattices. We shall show that $\mathcal{V}$ contains a lattice whose invariant γ' is strictly larger than that of Λ. Exchanging if need be Λ and Λ^*, we may assume that $S(\Lambda)$ spans a subspace $F \ne E$ of E. We shall use the following obvious generalization of Lemma 3.4.2: there exists in $\mathrm{End}(E)$ a neighbourhood the identity on which all minimal vectors in $u(\Lambda)$ and in $u(\Lambda)^*$ are images of minimal vectors in Λ and Λ^* respectively.

For nonzero $\lambda \in \mathbb{R}$, let $u_\lambda \in \mathrm{GL}(E)$ be the map which is the identity on F and multiplication by λ on $F^\perp$, and let $\Lambda' = u_\lambda(\Lambda)$.

Since $S(\Lambda)$ is contained in F, it is invariant under u_λ. For λ sufficiently close to 1, Λ' belongs to $\mathcal{V}$, and the norms of the vectors in Λ' which do not belong to F remain strictly larger than that of Λ. The equalities $\Lambda'^* = {}^t u_\lambda^{-1}(\Lambda^*) = u_{\lambda^{-1}}(\Lambda^*)$ show that u_λ strictly increases the norms of the elements of (Λ^*) which do not belong to F. If $S(\Lambda^*) \cap F = \emptyset$, we have $N(\Lambda'^*) > N(\Lambda^*)$, and Λ is not dual-extreme.

Suppose now that $S(\Lambda^*) \cap F$ is not empty. The minimal vectors in Λ' and in Λ'^* then all belong to F. Let w be an endomorphism of E which is zero on F and which maps $F^\perp$ into F. Its transpose ${}^t w$ is zero on $F^\perp$ and maps F into $F^\perp$. Let us choose w in such a way that the kernel of ${}^t w$ does not contain any vector in $S(\Lambda'^*)$. For any $\mu \in \mathbb{R}$, let $v_\mu = \mathrm{Id} + \mu\,w$. We have $({}^t w)^2 = w^2 = 0$, hence ${}^t v_\mu^{-1} = \mathrm{Id} - \mu\,{}^t w$. For μ sufficiently close to 0, $v_\mu(\Lambda')$ belongs to $\mathcal{V}$, and the norms of the elements in $v_\mu(\Lambda')$ (resp. in $v_\mu(\Lambda'^*)$) which do not belong to F are strictly greater that $N(\Lambda')$ (resp. than $N(\Lambda'^*)$). Since the restriction of v_μ to F is the identity, we have $S(v_\mu(\Lambda')) = S(\Lambda') = S(\Lambda)$, whence $N(v_\mu(\Lambda')) = N(\Lambda)$. For $x \in F$, ${}^t v_\mu$ transforms x into $x + x_\mu$ where $x_\mu = \mu\,{}^t w(x)$ is orthogonal to x. We thus have $N(v_\mu(x)) = N(x) + N(x_\mu) > N(x)$ for $x \notin \mathrm{Ker}\,w$, which shows that the norm of the dual of $v_\mu(\Lambda')$ is strictly larger than that of Λ'^*.

To summarize, for the choices above of u_λ and of v_μ, the lattice $v_\mu \circ u_\lambda(\Lambda)$ belongs to $\mathcal{V}$ and has the same norm as Λ, but the norm of its dual is strictly larger than that of Λ^*. Hence the lattice Λ is not dual-extreme. $\qquad\square$

To characterize dual-extreme lattices, we need analogues to the notions of perfection and eutaxy. These are provided by the following definition:

Definition 3.8.3. *We say that a pair (S, S') of configurations of lines (or of vectors) in E is* dual-perfect *if $S \cup S'$ is a perfect configuration, i.e. if the projections p_x, $x \in S \cup S'$ span $\mathrm{End}^s(E)$, and that it is* dual-eutactic *if there are two families ρ_x, $x \in S$ and ρ'_y, $y \in S'$ of strictly positive real numbers such that $\sum_{x \in S} \rho_x\, p_x = \sum_{y \in S'} \rho'_y\, p_y$. We say that a lattice Λ is* dual-perfect *(resp.* dual-eutactic*) if the pair $(S(\Lambda), S(\Lambda^*))$ is dual-perfect (resp. dual-eutactic).*

Given a basis $\mathcal{B}$ for Λ with Gram matrix A, expressing the projections onto minimal vectors with respect to the pair $(\mathcal{B}, \mathcal{B}^*)$ of bases for E shows that Λ is dual eutactic if and only if there exists a relation

$$\sum_{\pm X} \rho_X\, X\,{}^t X = A^{-1} \left(\sum_{\pm Y} \rho'_Y\, Y\,{}^t Y \right) A^{-1} \qquad (\mathbf{3.8.3'})$$

with strictly positive coefficients ρ_X, ρ'_Y, where X (resp. Y) runs through the column-matrices of the components in $\mathcal{B}$ (resp. in $\mathcal{B}^*$) of the vectors in $S(\Lambda)$ (resp. in $S(\Lambda^*)$).

Definition 3.8.3 is clearly invariant under the exchange of S and S', and Definitions 3.8.1 and 3.8.3 are invariant under the exchange of Λ and Λ^*.

Definition 3.8.3 possesses the following dual version, which makes use as usual of linear forms φ_x on $\mathrm{End}^s(E)$, that we give in the case of a lattice: Λ is dual-perfect if and only if the forms φ_x, $x \in S(\Lambda) \cup S(\Lambda^*)$ span $\mathrm{End}^s(E)^*$, and dual-eutactic if and only if there exists a relation of the form $\sum_{x \in S(\Lambda)} \rho_x\, \varphi_x = \sum_{y \in S(\Lambda^*)} \rho'_y\, \varphi_y$ with strictly positive coefficients ρ_x, ρ'_y.

The following theorem is an analogue to Korkine and Zolotareff's characterization of extreme lattices (Theorem 3.4.5):

Theorem 3.8.4. *A necessary and sufficient condition for a lattice Λ to be dual-extreme is that the system of inequalities*

$$\varphi_x(v) \geq 0\,, x \in S(\Lambda) \quad and \quad \varphi_y(v) \leq 0\,, y \in S(\Lambda^*)$$

only have the solution $v = 0$ in $\mathrm{End}^s(E)$. Moreover, when these conditions are satisfied, there exists a neighbourhood of Λ on which any lattice Λ' such that $\gamma'(\Lambda') \geq \gamma'(\Lambda)$ is similar to Λ (and we thus have $\gamma'(\Lambda') = \gamma'(\Lambda)$).

Proof. Suppose first that there exists a nonzero $v \in \mathrm{End}^s(E)$ which satisfies the system of inequalities above. Let $u \in \mathrm{End}(E)$ such that ${}^t uu = \mathrm{Id} + \varepsilon\, v$

where $\varepsilon > 0$ has been chosen in such a way that the minimal vectors in $u(\Lambda)$ (resp. in $u(\Lambda)^* = {}^t u^{-1}(\Lambda^*)$) come from $S(\Lambda)$ (resp. from $S(\Lambda^*)$); see Lemma 3.4.2. Consider the formula $N(u(\Lambda)) = N(\Lambda) + \min_{x \in S(\Lambda)} \varphi_x(v)$ (see the proof of Lemma 3.4.3) and its analogue $N(u(\Lambda)^*) = N(\Lambda^*) + \min_{y \in S(\Lambda^*)} \varphi_y(w)$ for Λ^* in which w is defined by $\mathrm{Id} + w = (\mathrm{Id} + v)^{-1}$. Replacing if need be $\mathcal{V}$ by a smaller neighbourhood, we may suppose that the series expansion of $(\mathrm{Id} + v)^{-1}$ converges, and this yields for w the expansion

$$w = -v(\mathrm{Id} - v + v^2 + \cdots + (-1)^{m-1} v^m + \dots).$$

The fact that v is a solution to the inequalities of Theorem 3.8.4 shows that, provided that v is sufficiently close to 0, both inequalities $N(u(\Lambda)) \geq N(\Lambda)$ and $N(u(\Lambda)^*) \geq N(\Lambda^*)$ hold, whence $\gamma'(u(\Lambda)) \geq \gamma'(\Lambda)$. [One can make use of a convexity lemma similar to Lemma 3.4.4; we shall not give the details here, as a more general result will be proved in Chapter 10.]

If equality holds, we have in particular $N(u(\Lambda)) = N(\Lambda)$ which implies by Lemma 3.4.3 that $S(u(\Lambda)) = \{x \in S(\Lambda) \mid v(x) = 0\}$ is contained in some hyperplane of E. Proposition 3.8.2 then shows that every neighbourhood of $u(\Lambda)$ contains a lattice Λ_1 such that $\gamma'(\Lambda_1) > \gamma'(u(\Lambda))$, and thus that Λ is not dual-extreme.

Conversely, assuming that the system of inequalities $\forall\, x \in S(\Lambda), \varphi_x(v) \geq 0$ and $\forall\, y \in S(\Lambda^*)$, $\varphi_y(v) \leq 0$ only has the solution $v = 0$ in $\mathrm{End}^s(E)$, we show that every lattice Λ' sufficiently close to Λ, such that $\gamma'(\Lambda') \geq \gamma'(\Lambda)$, is similar to Λ.

Let $u \in \mathrm{End}(E)$ such that $\Lambda' = u(\Lambda)$; composing u with a suitable similarity, we may assume that it is symmetric and that $N(u(\Lambda^*)) = N(\Lambda^*)$, and we now prove that $u = \mathrm{Id}$. Otherwise, we could write $u^2 = \mathrm{Id} + v$ for some $v \neq 0$. For u sufficiently close to the identity, we would have $N(u(\Lambda)) \geq N(\Lambda)$, whence $v(x) \cdot x = \varphi_x(v) \geq 0$ for all $x \in S(\Lambda)$ and similarly $\varphi_y(v) \leq 0$ for all $y \in S(\Lambda^*)$, and this would contradict the hypothesis $u \neq \mathrm{Id}$. $\square$

Following [B-M1], we are now able to give for dual-extreme lattices a characterization à la Voronoi:

Theorem 3.8.5. (Bergé–Martinet.) *A lattice is dual-extreme if and only if it is both dual-perfect and dual-eutactic.*

Proof. Let $\mathcal{F} = \{\varphi_x, x \in S(\Lambda)\} \cup \{-\varphi_y, y \in S(\Lambda^*)\}$. Let us first show that a lattice Λ which is dual-perfect and dual-eutactic is dual-extreme. Let $v \in \mathrm{End}^s(E)$ such that $f(v) \geq 0$ for all $f \in \mathcal{F}$. Since Λ is dual-eutactic, there are coefficients $\rho_f > 0$ such that $\sum_{f \in \mathcal{F}} \rho_f\, f = 0$, from which we deduce $\sum_{f \in \mathcal{F}} \rho_f\, f(v) = 0$. We thus have $f(v) = 0$ for all $f \in \mathcal{F}$. Since Λ is dual-perfect, $\mathcal{F}$ generates the dual space of $\mathrm{End}^s(E)$. Consequently, v is zero, and Λ is dual-extreme by Theorem 3.8.4.

Conversely, let Λ be a dual-extreme lattice. Any element v of $\mathrm{End}^s(E)$ such that $f(v) = 0$ for all $f \in \mathcal{F}$ must be zero by Theorem 3.8.4. Hence

Λ is dual-perfect. To prove the eutaxy condition, we make use of Stiemke's theorem: if $v \in \text{End}^s(E)$ is such that $f(v)$ is ≥ 0 for all $f \in \mathcal{F}$, Theorem 3.8.4 shows that v is zero, thus in particular that $f(v) = 0$ on $\mathcal{F}$, and we conclude by Theorem 3.3.1.　　　　□

Corollary 3.8.6. *An extreme lattice whose dual is eutactic is dual-extreme. In particular, an extreme lattice is dual-extreme any time its automorphism group acts transitively on the set of its minimal vectors or irreducibly on E.* [The condition that the dual-lattice should be eutactic can be weakened: semi-eutaxy suffices; see Exercise 3.8.8.]

Proof. Let Λ be such a lattice. Since it is perfect, it is in particular dual-perfect. Moreover, since both Λ and Λ^* are eutactic, there are strictly positive coefficients ρ_x, $x \in S(\Lambda)$ and ρ'_y, $y \in S(\Lambda^*)$ such that

$$\text{Id} = \sum_{x \in S(\Lambda)} \rho_x \, p_x = \sum_{y \in S(\Lambda^*)} \rho'_y \, p_y \,,$$

and eliminating Id in these relations yields a relation of dual-eutaxy for Λ. The last assertions follow from Theorem 3.6.7.　　　　□

Corollary 3.8.7. *The inequality $s(\Lambda) + s(\Lambda^*) \geq \frac{n(n+1)}{2} + 1$ holds for any dual-extreme lattice Λ.*

Proof. The existence of a non-trivial linear relation between the p_x, $x \in S(\Lambda) \cup S(\Lambda^*)$ shows the lower bound $s + s^* > \dim \text{End}^s(E)$.　　　□

Some of the results we proved in Sections 3.5 and 3.6 generalize to dual-perfect or dual-eutactic configurations For instance, it is easily checked that dual-perfect lattices are irreducible. Similarly, Proposition 3.6.5. possesses the following counterpart, which is proved in the same way:

Proposition 3.8.8. *A lattice Λ is dual-eutactic if and only if there exists a relation $\sum_{x \in S(\Lambda)} \rho_x \, p_x = \sum_{x' \in S(\Lambda^*)} \rho_{x'} \, p_{x'}$ with coefficients ρ_x (resp. $\rho_{x'}$) which are strictly positive and constant on the orbits of $S(\Lambda)$ (resp. of $S(\Lambda^*)$) under $\text{Aut}(\Lambda)$.*　　　　□

The following proposition does not rely on perfection and eutaxy properties, and will be proved directly by means of suitable deformations:

Proposition 3.8.9. *Let Λ be a dual-extreme lattice possessing a hyperplane extreme section of the same norm as Λ which is orthogonal to a minimal vector in Λ^*. Then Λ is extreme.*

Proof. Let $x' \in S(\Lambda^*)$ such that $H = (\mathbb{R}x')^\perp$ is the hyperplane defining the section above of Λ. On a sufficiently small neighbourhood of the identity

of $\mathrm{GL}(E)$, we can construct a continuous map $u \mapsto \sigma$ in $\mathrm{O}(E)$ such that σ maps $u(x')$ into the line $\mathbb{R}x'$ (consider for instance a 2-dimensional rotation). Thus, replacing u by $\sigma \circ u$, we may assume that u preserves H, whence $u(\Lambda) \cap H = u(\Lambda \cap H)$. By Corollary 1.3.5, the determinants of Λ and $\Lambda \cap H$ satisfy the relation

$$\det(\Lambda \cap H) = \det(\Lambda) \, N(\Lambda^*) \, .$$

Since $\gamma(\Lambda)^n = N(\Lambda)^n \det(\Lambda)^{-1}$, $\gamma(\Lambda \cap H)^{n-1} = N(\Lambda)^{n-1} \det(\Lambda \cap H)^{-1}$, and $\gamma'(\Lambda)^2 = N(\Lambda) \, N(\Lambda^*)$, we have the further relation

$$\gamma(\Lambda)^n = \gamma(\Lambda \cap H)^{n-1} \, \gamma'(\Lambda)^2 \, .$$

For any $u \in \mathrm{GL}(E)$ sufficiently close to the identity, we have $\gamma'(u(\Lambda)) \leq \gamma'(\Lambda)$ and $\gamma(u(\Lambda \cap H)) \leq \gamma(\Lambda \cap H)$ since Λ is dual-extreme and $\Lambda \cap H$ is extreme, hence also $\gamma(u(\Lambda)) \leq \gamma(\Lambda)$.   □

Corollary 3.8.10. *A dual-extreme lattice possessing a critical hyperplane section with the same norm is extreme.*

Proof. The determinant of a critical section is in particular minimal among all hyperplane sections (of any norm) of Λ. Its orthogonal in Λ^* has thus also the smallest possible determinant, i.e. the smallest possible norm, since it is a 1-dimensional lattice.   □

It is worth noticing that, in contrast to the case of usual perfection, there are generally infinitely many similarity classes of dual-perfect lattices. Exercise 3.8.3 below shows the existence of a continuous one-parameter family in dimension 2, and further examples exist in larger dimensions (see Chapter 6, Section 6.3 and the corresponding exercises). A more restrictive notion of dual-perfection to be defined later yields a reasonable analogue to Voronoi's Theorem 3.5.4 (see Theorem 9.6.1). We can also prove that the set of similarity classes of dual-eutactic lattices in a given dimension whose minimal vectors span E is finite (see Proposition 10.5.3). This will justify the first assertion in following result, which we only state in this section:

Theorem 3.8.11. (A.-M. Bergé, [Ber1].)

1. *There are only finitely many similarity classes of dual-extreme lattices in a given dimension.*
2. *Dual-extreme lattices are proportional to algebraic lattices; in particular, the values of the invariant γ'_n on dual-extreme lattices are algebraic.*   □

These remarks suggest the following questions, reproduced from the French edition:

Questions 3.8.12.   *1. Do there exist dual-extreme lattices with irrational invariant γ'^2 ?*

2. *For what dimensions do there exist dual-extreme lattices which do not satisfy Corollary 3.8.6?*

3. *Can one improve for $n > 1$ on the inequality $s(\Lambda) + s(\Lambda^*) \geq \frac{n(n+1)}{2} + 1$ of Corollary 3.8.7?*

Since the publication of the French edition of this book, partial answers were found.

A 5-dimensional example – the smallest possible dimension – of an irrational dual-extreme lattice has been produced by A.-M. Bergé ([Ber4]). Her example shares with those of Martinet ([Mar2], which concern even dimensions $n \geq 8$) the following properties: none of the lattices Λ, Λ^* is perfect and at most one is eutactic, so that they cannot be proved to be eutactic using Corollary 3.8.6, and they satisfy the relation $s + s^* = \frac{n(n+1)}{2} + n$, giving the sum $s + s^*$ a smaller value that the smallest one previously known for $n \geq 2$, namely $\frac{n(n+1)}{2} + n + 1$, attained on $(\mathbb{A}_n, \mathbb{A}_n^*)$).

Examples of dual-extreme lattices for which $s + s^* = \frac{n(n+1)}{2} + 1$ have recently been obtained in [B-M6] for all $n \geq 8$ even, as cross-sections of some Coxeter's lattices $\mathbb{A}_n^r$ (defined in Section 5.2).

These partial results, however, are far from giving a complete answer to Questions 3.8.12 from dimension 5 onwards.

3.9 Exercises for Chapter 3

3.1.1. Let $u \in \mathrm{End}^s(E)^{++}$. Show that u possesses 2^n square roots if its eigenvalues are all different, and infinitely many otherwise.

3.1.2. Show that any endomorphism possesses left and right decompositions into positive symmetric and orthogonal components, but that only the symmetric ones are unique if u is not invertible.

3.1.3. 1. Show that $u \mapsto \mathrm{Tr}({}^t u u)$ is a positive definite quadratic form on $\mathrm{End}(E)$, with polar form $(u, v) \mapsto \frac{1}{2} \mathrm{Tr}({}^t u v + {}^t v u)$.

2. Show that $\mathrm{End}(E)$ is the orthogonal sum of $\mathrm{End}^s(E)$ and $\mathrm{End}^a(E)$ for each of the quadratic forms $u \mapsto \mathrm{Tr}(u^2)$ and $u \mapsto \mathrm{Tr}({}^t u u)$.

3. Show that $\mathrm{Tr}(u^2)$ has signature $(\frac{n(n+1)}{2}, \frac{n(n-1)}{2})$ on $\mathrm{End}(E)$.

3.1.4. Let $f = \sum_{p \geq 0} a_p T^p$ an entire complex series, let R be its convergence radius, let V be a complex vector space of finite dimension n and let $u \in \mathrm{End}(V)$. Show that the series $\sum a_p u^p$ converges if the eigenvalues of u have a modulus $< R$, and diverges if at least one modulus is strictly larger than 1. [Reduce first to the case of triangular matrices, then use the decomposition of such a matrix as a sum of a diagonal matrix D and an upper triangular matrix T such that $DT = TD$ and $T^n = 0$.]

3.1.5. For any $u \in \mathrm{End}(E)$, let $\exp(u) = \sum_{p=0}^{+\infty} \frac{u^p}{p!}$ be exponential map.

1. Verify that the series above converges everywhere in $\mathbb{C}$.

2. Show that the exponential map is a diffeomorphism of $\mathrm{End}^s(E)$ onto $\mathrm{End}^{s++}(E)$.

3. Define u^m on $\mathrm{End}^{s++}(E)$ for every real m.

3.1.6. Let e, $f \neq 0$ be two vectors in E. Prove the formula $\operatorname{Tr} p_f \circ p_e = \frac{(e \cdot f)^2}{(e \cdot e)(f \cdot f)}$.

3.2.1. Let Λ be a weakly eutactic lattice. Show that there exists a eutaxy relation $\operatorname{Id} = \sum_{x \in S(\Lambda)/\pm} \rho_x p_x$ with rational coefficients ρ_x if and only if Λ is rational, i.e. proportional to an integral lattice.
[To prove the "if" part, rescale Λ to a rational norm and use Theorem 3.2.5 (5); to prove the "only if" part, extract from S a basis for the span of the p_x, $x \in S$.]

3.2.2. (Venkov's index formula.) Preliminary questions:

(a) Show that the ranks of a finite system of vectors in E and of the corresponding Gram matrix are equal.

(b) Show that for an $s \times s$ (real) matrix with eigenvalues $\lambda_1, \ldots, \lambda_s$, the eigenvalues of $\overset{k}{\bigwedge} M$ are the products $\lambda_{i_1} \ldots \lambda_{i_k}$ with $i_1 < \cdots < i_k$.

1. Let Λ be a well–rounded lattice with set of minimal vectors $S = \{\pm x_1, \ldots, \pm x_s\}$, let M be the Gram matrix of $S' = (x_1, \ldots, x_s)$ and let $\lambda_1 \geq \cdots \geq \lambda_n > \lambda_{n+1} = \cdots = \lambda_s = 0$ be its eigenvalues. Show that $\operatorname{Tr}(\overset{n}{\bigwedge} M) = \lambda_1 \ldots \lambda_n$.

2. Show that $\operatorname{Tr}(\overset{n}{\bigwedge} M) = \det(\Lambda) \sum_X [\Lambda : \Lambda_X]^2$ where X runs through the subsets of n independent vectors in S' and Λ_X stands for the sublattice of Λ with basis X.

From now on, we suppose that Λ is strongly eutactic.

3. Show that $M^2 = \frac{s}{n} N(\Lambda) M$; show that the nonzero eigenvalues of M are equal to $\frac{s}{n} N(\Lambda)$.

4. Prove that any strongly eutactic lattice satisfies the *Venkov index formula*

$$\sum_X [\Lambda : \Lambda_X]^2 = \frac{s^n}{n^n} \gamma(\Lambda)^n.$$

3.3.1. Let φ_i, $0 \leq i \leq k$ be linear forms on a real vector space V. Consider the following three statements:
(1) $\varphi_i(x) \geq 0$, $0 \leq i \leq k \Rightarrow x = 0$.
(2) $\varphi_i(x) \geq 0$, $0 \leq i \leq k \Rightarrow \varphi_i(x) = 0$, $0 \leq i \leq k$.
(3) $\varphi_i(x) = 0$, $1 \leq i \leq k \Rightarrow x = 0$.
Prove the equivalence $(1) \Longleftrightarrow (2) + (3)$.
[To prove $(1) \Rightarrow (3)$, use the transformation $x \mapsto -x$.]

3.5.1. Prove directly that the set of minimal vectors of an extreme lattice is not contained in the union of two hyperplanes.
[Use a rabatment of one of the hyperplanes onto the other one.]

3.5.2. Show that three distinct lines in a plane constitute a perfect configuration.

3.5.3. For any subspace F of E, let p_F be the orthogonal projection onto F and let $\mathcal{P}_F$ be the image in $\operatorname{End}^s(E)$ of $u \mapsto p_F \circ u$.

1. Let F and F' be two subspaces of E and let $F'' = F \cap F'$. Prove the equality $\mathcal{P}_{F''} = \mathcal{P}_F \cap \mathcal{P}_{F'}$.

2. Let m, m', m'' be the respective dimensions of F, F', F''. Show that the dimension of $\mathcal{P}_F + \mathcal{P}_{F'}$ is $M = \frac{m(m+1)}{2} + \frac{m'(m'+1)}{2} - \frac{m''(m''+1)}{2}$.

3. Take for F and F' two hyperplanes of E. Show that if they are distinct (resp. if $F' = F$), $\mathcal{P}_F + \mathcal{P}_{F'}$ has codimension 1 (resp. n) in $\operatorname{End}^s(E)$.

4. Deduce from 3 a direct proof of statements (2) and (3) in Theorem 3.5.2.

5. Let Λ be a lattice possessing two perfect hyperplane sections with the same norm as Λ. Show that Λ is perfect if and only if it possesses a minimal vector which belongs to none of the two hyperplanes. [Other method: use Proposition 5.3.]

3.5.4. Let E_1 and E_2 be two Euclidean spaces of dimensions n_1 and n_2, and let $S_1 \subset E_1$ and $S_2 \subset E_2$ be two (finite) configurations.

1. Show that $\mathrm{End}^s(E_1) \bigotimes \mathrm{End}^s(E_2)$ has codimension $\frac{n_1 n_2 (n_1 - 1)(n_2 - 1)}{4}$ in $\mathrm{End}^s(E_1 \bigotimes E_2)$.

2. Prove the equality $p_{x \bigotimes y} = p_x \bigotimes p_y$ for all nonzero $x \in E_1$ and $y \in E_2$. [Consider bases (e_i) for E_1 and (f_j) for E_2 with $e_1 = x$, $f_1 = y$ and $e_i \cdot x = f_i \cdot y = 0$ otherwise.]

3. Deduce from 2 that the perfection rank of $S_1 \bigotimes S_2 = \{x_1 \otimes x_2 \mid x_i \in S_i\}$ is the product of the perfection ranks of S_1 and of S_2.

4. Show that the tensor product of two lattices of dimensions greater than 1 is not perfect whenever its minimal vectors are split tensors.

3.6.1. Let S be the 2-dimensional eutactic configuration consisting of two orthogonal lines. Show that a configuration S' of at least four lines containing S is eutactic if and only if the lines in $S' \smallsetminus S$ do not all lie in the same angular domain defined by the lines in S.

3.6.2. Let S be a 2-dimensional configuration consisting of three distinct lines.

1. Show that it is possible to choose an orientation of the plane and three unit vectors v_0, v_1, v_2 on these lines in such a way that the angles $\theta_1 = \widehat{v_1, v_0}$ and $\theta_2 = \widehat{v_0, v_2}$ both belong to the interval $\left(0, \frac{\pi}{2}\right)$.

2. Show that the eutaxy coefficients ρ_0, ρ_1, ρ_2 are given by the formulae

$$\rho_1 = \frac{\cos \theta_2}{\sin \theta_1 \sin(\theta_1 + \theta_2)} \ , \ \rho_2 = \frac{\cos \theta_1}{\sin \theta_2 \sin(\theta_1 + \theta_2)} \quad \text{and} \quad \rho_0 = 2 - \rho_1 - \rho_2.$$

3. Transform the condition $\rho_1 + \rho_2 < 2$ into
(1) $\sin 2\theta_1 + \sin 2\theta_2 < 4 \sin \theta_1 \sin \theta_2 \sin(\theta_1 + \theta_2)$;
(2) and then into $\cos(\theta_1 + \theta_2) < 0$.

4. Use the last two questions to deduce that three lines in a Euclidean plane constitute a eutactic configuration if and only if , once chosen an indexing D_1, D_2, D_3 of the lines such that the (non-oriented) angles $\widehat{D_1, D_2}$ and $\widehat{D_2, D_3}$ are acute and that D_2 lies between D_1 and D_3, the angle $\widehat{D_1, D_3}$ is then obtuse.

3.6.3. Write down a proof of Theorem 3.6.4 which does not rely on the Voronoi theorem. [Use Exercise 5.1 and observe that for a given pair $\Lambda' \subset \Lambda$ of lattices, the index $[u(\Lambda) : u(\Lambda')]$ does not depend on $u \in \mathrm{GL}(E)$.]

3.6.4. Let S be a configuration of lines containing a eutactic configuration S' such that the orthogonal projections on the lines of S' and on those of S generate the same subspace of $\mathrm{End}^s(E)$. Show that S is eutactic. [Argue as in the proof of Theorem 3.6.2.]

3.6.5. Let S be a configuration of lines in E such that the orthogonal projections on the lines in S are independent in $\mathrm{End}^s(E)$, and let S' be a configuration strictly contained

in S.

    1. Show that if S' is eutactic, then S is not.

    2. Let $\Lambda \simeq \mathbb{Z}^n \subset \mathbb{R}^n$ where $n = m^2$, $m \geq 3$, let $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ be its canonical orthonormal basis, let $e = \frac{1}{m}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)$, and let Λ' be the lattice generated by Λ and e. Show that $S(\Lambda')$ contains $S(\Lambda)$, but that Λ' is not eutactic.

3.6.6. Show that the tensor product of two (weakly, semi-, strongly) eutactic configurations is a (weakly, semi-, strongly) eutactic configuration.

3.7.1. Let Λ be a laminated lattice (in particular, Λ has norm 4), and let $n = \dim \Lambda$.

    1. Show that Λ is irreducible.

    2. Use the question above to prove that Λ is not an integral lattice for any $n > 24$. [Use Proposition 1.9.9.]

    3. Prove that $\det(\Lambda)$ is nevertheless an integer for $n \leq 32$, by showing that for $24 \leq n \leq 32$, they coincide with those of the lattices $\frac{1}{\sqrt{2}}\Lambda_n$, $0 \leq n \leq 8$, for which the determinants are $1, 2, 3, 4, 4, 4, 3, 2, 1$.

    4. Show that unimodular lattices of dimension 32 and norm 4 are not similar to laminated lattices, although they have the same Hermite invariant.
[Examples can be found in Section 6.7 and in [K-V].]

    5. Let Λ' be an odd unimodular lattice of norm 3 and dimension n equal to $27, 28$ or 29. Show that its sublattice of even norm vectors (of index 2 in Λ') has the same Hermite invariant as Λ_n, but is not a laminated lattice.
[These lattices have been classified for $n = 27, 28$ by Bacher and Venkov, see [Bc-V2] and also [Bc-V1]; one of them is $\Lambda' = \bigwedge^2 \mathbb{E}_8$, see Section 1.10.]

3.7.2. Let Λ be an integral lattice of dimension n, determinant d and norm m, similar to a laminated lattice Λ_n.

    1. Prove that $m^n = 4^n d \lambda_n^{-1}$; deduce from this that m is divisible by 4 for $24 \leq n \leq 48$.

    2. Show that m is divisible by 8 for $33 \leq n \leq 48$.

    3. Prove the inequality $m \geq 8$ for $25 \leq n \leq 48$.

3.8.1. Let E be a Euclidean plane and let $\mathcal{B} = (e_1, e_2)$ be a basis for E whose Gram matrix has determinant 1. Let $\mathcal{B}^* = (e_1^*, e_2^*)$ its dual basis.

    1. Prove that there exists an orthonormal basis $(\varepsilon_1, \varepsilon_2)$ for E in which $e_1 = a\varepsilon_1$ and $e_2 = c\varepsilon_1 + a^{-1}\varepsilon_2$ for some $a, c \in \mathbb{R}$ with $a > 0$ and $c \geq 0$.

    2. Prove that $e_1^* = a^{-1}\varepsilon_1 - c\varepsilon_2$ and $e_2^* = a\varepsilon_2$.

    3. Prove that a rotation of angle $\pm\frac{\pi}{2}$ maps $\mathcal{B}$ onto $(e_2^*, -e_1^*)$ or onto $(-e_2^*, e_1^*)$.

    4. Deduce from this that any 2-dimensional lattice of determinant 1 is directly isometric to its dual.

    5. Show that any 2-dimensional lattice lattice is directly similar to its dual.

    6. Show that $s(\Lambda) = s(\Lambda^*) \in \{1, 2, 3\}$ for any 2-dimensional lattice Λ,

3.8.2. Let Λ be a 2-dimensional lattice. Prove that if (e_1, e_2) is a basis for Λ which represents its two minima (i.e, e_1 is minimal and e_2 is minimal among those vectors in Λ which are not proportional to e_1), then e_2^*, e_1^* are the two minima of Λ^*. [Use Question 5 in Exercise 8.1.]

3.8.3. The aim of this exercise is to classify 2-dimensional dual-perfect, dual-eutactic and dual-extreme lattices. Let Λ be a norm 1 lattice in $\mathbb{R}^2$, endowed with its canonical basis and the corresponding orientation.

1. Suppose that $s(\Lambda) \geq 2$. Show that Λ is the image under a rotation of a lattice with basis (e_1, e_2) such that $e_1 = \varepsilon_1$, $N(e_2) = 1$ and $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$, where $\theta = \widehat{e_1, e_2}$.

2. We consider in what follows a lattice Λ together with a basis (e_1, e_2) as above. Show that its dual basis (e_1^*, e_2^*) is defined by the equations $\|e_1^*\| = \|e_2^*\| = \frac{1}{\sin \theta}$ and $\widehat{e_1^*, e_1} = \widehat{e_2, e_2^*} = \frac{\pi}{2} - \theta$.

3. Show that the only dual-extreme 2-dimensional lattices are the hexagonal lattices; prove that $\gamma'^2 = \frac{4}{3}$.

4. Show that Λ is dual-perfect except if $\theta = \frac{\pi}{2}$. [Use Exercise 5.2.]

5. Show that Λ is dual-eutactic if and only if $\theta = \frac{\pi}{3}$ or $\theta = \frac{\pi}{2}$. [Notice that for $\frac{\pi}{3} < \theta < \frac{\pi}{2}$, the linear combinations $\lambda p_{e_1} + \mu p_{e_2}$ in $\mathrm{End}(E)$ with $\lambda, \mu > 0$ map e_1 inside the sector bounded by the half-lines generated by e_1 and e_2.]; use this to recover the result of Question 3.

6. Show that a 2-dimensional lattice Λ with $s(\Lambda) = 1$ cannot be dual-eutactic. [Use Exercise 8.2.]

3.8.4. Let S and S' be two finite families of nonzero vectors in E.

1. Show that for nonzero vectors $x, y \in E$, we have $\mathrm{Tr}(p_y \circ p_x) = \cos^2(\widehat{x, y})$.

2. Show that a vector $e \in E$ such that there exists a relation with strictly positive coefficients $\sum_{x \in S} \rho_x \, p_x(e) = 0$ is orthogonal to all vectors in S.

3. Show that if (S, S') is dual-eutactic, every hyperplane which contains S' also contains S.

4. Use this to deduce that if (S, S') is a dual-eutactic pair, then S and S' generate the same subspace of E.

3.8.5. Let $t \in \left(\frac{3}{4}, 1\right)$ and let Λ_t be the lattice $t\mathbb{A}_1 \perp \mathbb{A}_2$, whose Gram matrix in an appropriate basis (e_1, e_2, e_3) is $\begin{pmatrix} 2t & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

1. Show that $S(\Lambda) = \{e_1\}$ and $S(\Lambda^*) = \{e_1^*\}$.

2. Show that the lattices Λ_t are dual-eutactic.

3. Show that the equality $S(u(\Lambda)) = \{u(e_1)\}$ and $S(u(\Lambda^*)) = \{u(e_1)^*\}$ still holds for every $u \in \mathrm{GL}(E)$ sufficiently close to the identity, but that the lattices $u(\Lambda_t)$ are no longer dual-eutactic unless $u(e_1)$ is orthogonal to the plane $\langle u(e_2), (e_3) \rangle$.

3.8.6. Let $\mathcal{B} = (e_1, e_2, \ldots, e_n)$ be a basis for E. Denote by p_i (resp. p_i') the orthogonal projection onto e_i (resp. e_i^*). Let $\sum_j \lambda_j \, p_j = \sum_k \mu_k \, p_k'$ be a relation of dual-eutaxy with nonzero coefficients λ_j, μ_k.

1. Prove the equalities $\frac{1}{(e_i \cdot e_i)} \lambda_i e_i = \sum_k \mu_k \frac{e_i^* \cdot e_k^*}{e_k^* \cdot e_k^*} e_k^*$ $(i = 1, 2, \ldots, n)$. [Use projections onto the e_i^*.]

2. Deduce from 1 that the relations $\frac{e_i \cdot e_j}{e_i \cdot e_i} \lambda_i = \frac{e_i^* \cdot e_j^*}{e_j^* \cdot e_j^*} \mu_j$ hold for all i and all j.

3. Prove the equalities $\mu_i = \lambda_i$ and $\frac{(e_i \cdot e_j)^2}{(e_i \cdot e_i)(e_j \cdot e_j)} = \frac{(e_i^* \cdot e_j^*)^2}{(e_i^* \cdot e_i^*)(e_j^* \cdot e_j^*)}$.

4. Show that if the vectors of $\mathcal{B}$ and of $\mathcal{B}^*$ all have the same norm, then the Gram matrices of $\mathcal{B}$ and of $\mathcal{B}^*$ are proportional.

5. Show that if Λ is a dual-eutactic lattice such that $S(\Lambda)$ and $S(\Lambda^*)$ are up to their signs the vectors of a basis and of its dual basis, Λ is similar to $\mathbb{Z}^n$.

3.8.7. Show that the maximum of $s + s^*$ in a given dimension n is attained on dual-perfect lattices. (Compare Theorem 3.5.5.)

3.8.8. Let Λ be a perfect lattice.

1. Show that if there exists a relation $\sum_{x \in S(\Lambda)} a_x\, p_x = \sum_{y \in T} b_y\, p_y$ for some $T \subset S(\Lambda^*)$ with strictly positive coefficients a_x, b_y, then Λ is dual-eutactic. [Argue as in the proof of Theorem 3.6.2.]

2. Show that if Λ is extreme and if Λ^* is semi-eutactic, then Λ is dual-extreme.

3.10 Notes on Chapter 3.

The notion of an extreme quadratic form appears for the first time in Korkine and Zolotareff's 1873 article [K-Z2], where they moreover give a list of extreme forms, but proofs appear only in the 1877 paper [K-Z3] (Mathematische Annalen, volume **11**). The aim of this last article is to classify extreme forms up to dimension 5 and to derive from this classification the precise value of γ_5.

In [K-Z3], Korkine and Zolotareff give a characterization of extremality by means of inequalities which are equivalent to those of Theorem 3.4.5, they prove that extreme forms are perfect (without giving a name to perfection; the word "perfect" was used for the first time by Voronoi some thirty years later), and that perfect forms are well defined by the components in the canonical basis for $\mathbb{Z}^n$ of the set of their minimal vectors, a result stated in the form

> Toute forme extrême a au moins $\frac{n(n+1)}{2}$ représentations de son minimum qui déterminent complètement cette forme, en supposant que son minimum soit donné.

([K-Z3], p. 252). From this result, they deduce the remarkable fact that γ_n^n is a rational number. They however do not prove the finiteness theorem for perfect forms, but the way they wrote up their results makes it likely that they considered this theorem as true.

The lower bound $s \geq \frac{n(n+1)}{2}$ for the number of pairs of minimal vectors in any perfect lattice was extended in 1953 by Swinnerton-Dyer to lattices which are extreme for a bounded convex set (cf. [Cas2], Chapter V, Theorem VIII).

After [K-Z3], the theory of extreme forms falls asleep for thirty years until the publication in 1908 by Voronoi of his paper [Vo1]. The chief aim of Voronoi is the description of an algorithm which (at least theoretically) allows the classification of perfect forms in any given dimension (see the notes on Chapter 7). But Voronoi's paper also contains two results which we have proved in this chapter: the characterization of extreme forms as those which are perfect and eutactic (the notion of a eutactic form is due to Voronoi, who

gives no name to it; the word "eutaxy" was used for the first time in this setting by Coxeter in 1951 in [Cox2]), and the proof of the finiteness theorem for perfect forms.

The theory of perfect forms again falls asleep until Coxeter's 1951 paper quoted above, except for a paper [Hof] by Hofreiter dating back to 1933, which states an erroneous classification of 6-dimensional perfect forms.

Coxeter studies the notions of perfection and eutaxy for their own interest, and replaces (partially) for the first time forms by lattices. He met eutactic configurations in previous works by Schläfli and by Hadwiger on regular polytopes. His paper contains a description of new extreme lattices related to root lattices, to which we shall return later in Chapter 5, the characterization of eutactic configurations in terms of eutactic stars, and various conjectures, sometimes not well inspired (see Conway and Sloane's comments at the beginning of [C-S5]).

The study of regular polytopes is also the motivation of the joint paper [B-C] of 1940 with Richard Brauer (slightly generalized in [Cox2], no. 4.3). The proof involves a lot of calculations, and it is not clear that representations to be irreducible only on $\mathbb{R}$, not on $\mathbb{C}$. The short proof of Section 3.6 has been given to me by A.-M. Bergé.

A great deal of work on perfect forms took place in England from 1955 onwards. This yielded the discovery of numerous new perfect forms as well as classification theorems. As for what concerns the subject of this chapter, we must in particular quote Barnes's 1957 paper [Bar2] in which he brought to light Stiemke's theorem. The proof of Theorem 3.4.6 we have given essentially follows Barnes's. It should be noticed that Voronoi's proof is of the same nature, except that he had to prove Stiemke's theorem in the particular case of linear forms on spaces of symmetric matrices. At this stage, one must point out a difficulty in the proof of Theorem 3.4.5, related to uniformity problems in power series expansions in several variables. Thus many proofs (including one in which I was involved) cannot be considered as completely correct; a similar remark applies to Theorems 3.8.4 and 3.8.5. Kneser ([Kn1], 1955) was the first to publish a completely correct proof, in which he made use of convexity. Such a convexity argument is used in Conway and Sloane's [C-S5] (quite correct up to a forgotten logarithm). The proof I gave here is modelled on their proof. I would like to thank A.-M. Bergé and R. Coulangeon for having drawn my attention on the difficulty above, which we shall again consider in Chapter 10, where convexity underlies in the use of the exponential map which goes from the tangent space at the origin of a Lie group to the group itself.

The method of Proposition 3.5.3, which yields a characterization of perfect lattices possessing a perfect hyperplane section, was used by Barnes in [Bar5], I, pp. 64–67, where he proves the perfection of a certain family Q_n of forms (see Section 5.3) by making use of the perfection of its family P_{n-1}, previously proved in [Bar3].

Theorem 3.5.5 was pointed out to me by Oesterlé; Watson considers in [Wat5] that it was known to Voronoi.

Proposition 3.6.5 is Scott's Theorem 2.1 in [Sco1].

Conway and Sloane published the important paper *Laminated Lattices* in *Annals of Mathematics* (vol. **116**, 1982, pp. 593–620); it is reproduced as the basis for Chapter 6 of [C-S]. For the arithmetical laminations of Plesken and Pohst, see [Pl-P1], [Pl-P2] and the appendix to Chapter 6 of [C-S].

I know of no published proof of the extremality of laminated lattice at least up to dimension 24, for which it actually holds. (There is no reason for extremality to hold in any dimension.) These lattices are expected to be the (only) densest lattices up to dimension 25 except in dimensions 11, 12, 13. Extremality is proved explicitly for dimensions $n \leq 8$ in Chapter 4, and for $n \leq 24$ and $n \equiv 0$ or $1 \mod 4$ in Chapter 8 (where it is shown that the set of minimal vectors contains a configuration similar to $S(\mathbb{D}_n)$); Theorem 3.6.6 applies to some other dimensions, e.g. 23, 24.

Curiously, for $n \leq 24$, laminated lattices are the densest known lattices, except for the three dimensions for which they are not unique, were the densest lattices known are K_{11}, K_{12} and K_{13}. Lattices in the K_n series no longer represent the densest known lattices in dimensions $n \leq 10$ and $n \geq 14$, and it is precisely for dimensions 10 and 14 that a bifurcation to the lattices called K'_n in Chapter 8 appears!

The notion of a dual-extreme lattice was introduced in [B-M1], published in 1989. Most of the results of Section 3.8 come from this paper or from Bergé's [Ber1]. The motivation for [B-M1] was a paper by Zimmert on "twin classes" in number fields ([Zi], 1981; see also Oesterlé, [Oe1], where Zimmert's theory in explained in the setting of "Weil's explicit formulae"). The geometrical interpretation of Zimmert's inequalities (obtained via partial zeta functions) involves the product of the minima on a lattice and on its dual of a quadratic form Q depending on the signature of the field. This product reduces to $N(\Lambda) N(\Lambda^*)$ when one replaces Q by the Euclidean norm in $\mathbb{R}^n$.

Besides the various extremality problems which have been dealt with in this chapter or which we are going to handle later, the Humbert problem, that we shall not consider anywhere else in this book, deserves a special mention. It consists in estimating a Hermite like invariant $\gamma_{(K)}$ relative to a number field K (and a dimension n) and taking into account the $r_1 + r_2$ embeddings $\sigma_i : K \to \mathbb{C}$, with the usual notation, namely that the σ_i are real for $i \leq r_1$, and non-real, pairwise non-conjugates for $i > r_1$. Set $d_i = 1$ if $i \leq r_1$ and $d_i = 2$ if $i > r_1$.

Explicitly, a *Humbert form* is a system $\mathcal{A} = (A_1, \ldots, A_{r_1+r_2})$ of positive definite matrices, symmetric real for $i \leq r_1$, Hermitian complex for $i > r_1$. Given a column-vector $X \in \mathbb{Z}_K^n$, set $\mathcal{A}[X] = \prod_i ({}^t\sigma_i(X)A_i\sigma_i(X))^{d_i}$, and define the *minimum* and the *determinant* of the Humbert form by

$$\mu(\mathcal{A}) = \min\{\mathcal{A}[X] \mid X \in \mathbb{Z}_K^n \smallsetminus \{0\}\} \quad \text{and} \quad \det(\mathcal{A}) \prod_i \det(A_i)^{d_i}.$$

The *Hermite–Humbert invariant* is then

$$\gamma_{(K)}(\mathcal{A}) = \frac{\mu(\mathcal{A})}{\det(A)^{1/n}} \; .$$

(Hence $\gamma_{(\mathbb{Q})}$ is the usual Hermite invariant.)

The theory was founded by Humbert in his papers [Hmb1] and [Hmb2]. Convenient notions of perfection and eutaxy have been given by Coulangeon after work by Baeza and Icaza ([B-I], [I]), with which he was able to prove an analogue of Voronoi's Theorem 4.6 ([Cou6]). Some explicit computation have been done in [B-C-I-O] for quadratic fields. The Humbert problem has also been considered in the setting of linear algebraic groups by Watanabe; see [O-W], [Wata1], [Wata2].