CHAPTER 3

PHYSICAL FUNDAMENTALS OF HYDRAULICS

The purpose of this chapter is to define some physical properties of hydraulic fluids, and to discuss the fundamental laws and equations of fluid motion, types of flow, and the flow through orifices and valves. It should be mentioned that the intention of this chapter is not to present the complete theoretical basics of fluid motion in hydraulics (or fluid mechanics); it rather summarises those equations and concepts which will be required in the subsequent chapters. For more details and information the reader is always referred to the corresponding literature given throughout the whole chapter.

The writing of this chapter was largely influenced by the extensive discussions on flow through orifices and valves in the standard textbooks by Merritt (1967), and more recently the fine work by Beater (1999). Analyses and discussions on the material of this chapter can also be found, *e.g.*, in Findeisen and Findeisen (1994) and in the recent work by Will *et al.* (1999).

3.1 Physical Properties of Fluids

Fluids (liquids and gases) are bodies without their own shape; they can flow, *i.e.*, they can undergo great variations of shape under the action of forces; the weaker the force, the slower the variation (Lencastre, 1987).

The normal tension on the surface element of a fluid is called *pressure*. It is, at a given point, identical in all directions. Pressure can be calculated as

$$p = \frac{\text{Force}}{\text{Area}} = \frac{F}{A} \tag{3.1}$$

and thus has the dimensions of force per unit area (N/m^2).

3.1.1 Viscosity and Related Quantities

The coefficient of dynamic viscosity, η, is the parameter that represents the existence of tangential forces in liquids in movement. Suppose two plates (or fluid layers) are moving at a distance apart of dy, and at a relative velocity dv_x (see Figure 3.1.), then the shear stress

$$\tau = \frac{\text{Shear Force}}{\text{Area}} = \eta \frac{\mathrm{d}v_x}{\mathrm{d}y} \qquad \text{(Newton)} \tag{3.2}$$

arises. Thereby, η is a proportionality factor and is called *dynamic viscosity*.

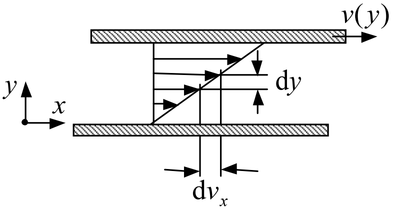

Figure 3.1. Couette flow; definition of shear stress

The coefficient of *kinematic viscosity*, μ, is the ratio of the coefficient of dynamic viscosity to the fluid density, *i.e.*,

$$\mu = \frac{\text{Dynamic Viscosity}}{\text{Density}} = \frac{\eta}{\rho} \tag{3.3}$$

The dynamic viscosity of liquids varies considerably with the temperature:

$$\eta_\vartheta = \eta_0 \, \mathrm{e}^{-\lambda_1(\theta-\theta_0)} \tag{3.4}$$

where η_0 is the dynamic viscosity at reference temperature θ_0. The viscosity–temperature coefficient λ_1 should be determined by experiments for the fluid considered. For mineral oils, it lies between 0.036 and 0.057 K^{-1} (Ivantysyn and Ivantysynova, 1993).

The influence of pressure is given by

$$\eta = \eta_0 \, \mathrm{e}^{\alpha p} \tag{3.5}$$

where α is the viscosity–pressure coefficient that depends on the temperature; see Table 3.1 for the mineral oil HLP 32. For HFC fluids and HFD fluids the values of $\alpha = 0.35\ \mathrm{Pa}^{-1}$ and $\alpha = 2.2\ \mathrm{Pa}^{-1}$ can be used respectively (Ivantysyn and Ivantysynova, 1993). The effect of pressure on viscosity is not so important in practice.

Table 3.1. Viscosity–pressure coefficient for the mineral oil HLP 32 (Ivantysyn and Ivantysynova, 1993)

θ [°C]	α [$10^{-2}\mathrm{Pa}^{-1}$]
0	3.268
10	2.900
20	2.595
30	2.339
40	2.121
50	1.933
60	1.770
70	1.626
80	1.499
90	1.385
100	1.283

3.1.2 Mass Density, Bulk Modulus and Related Quantities

Mass density, ρ, or simply *density* is the mass contained in a unit volume:

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{\mathrm{d}m}{\mathrm{d}V} \tag{3.6}$$

The density of hydraulic fluids normally lies between 0.85 and 0.91 kg/dm^3. Indeed, the density of hydraulic fluids is a function of both pressure and temperature, *i.e.*, $\rho = \rho(p,\theta)$. The first three terms of a Taylor's series for two variables may be used as an approximation (Merritt, 1967):

$$\begin{aligned}\rho &\approx \rho_0 + \left(\frac{\partial \rho}{\partial p}\right)_\theta (p - p_0) + \left(\frac{\partial \rho}{\partial \theta}\right)_p (\theta - \theta_0) \\ &= \rho_0 \left[1 + \frac{1}{E}(p - p_0) - \alpha(\theta - \theta_0)\right]\end{aligned} \tag{3.7}$$

where ρ, p and θ are respectively the mass density, pressure, and temperature of the fluid about the initial values ρ_0, p_0 and θ_0. Equation 3.7 is the linearised equation of state for fluids. In hydraulic phenomena, the usual assumption of constant temperatures reduces the linearised state Equation 3.7 of fluids to the simple form (Merritt, 1967)

$$\rho = \rho_\mathrm{i} + \frac{\rho_\mathrm{i}}{E} p \tag{3.8}$$

where ρ_i is the mass density at zero pressure.

The quantity

$$E \equiv \rho_0 \left(\frac{\partial p}{\partial \rho}\right)_\theta = -V_0 \left(\frac{\partial p}{\partial V}\right)_\theta \tag{3.9}$$

is the change in pressure divided by the fractional change in volume at a constant temperature. It is called the modulus of elasticity, also termed the *isothermal bulk modulus* or simply *bulk modulus* of the liquid. It significantly influences the dynamics of hydraulic servo-systems. For mineral oils, and for common pressures and temperatures ($\theta \in [-40,120]$°C, $p \leq 450$ bar), one may assume a mean value for the bulk modulus, typically

$$E_{\text{mineral oil}} = (14\text{–}16)\times 10^3 \text{ bar} = (1.4\text{–}1.6)\times 10^9 \text{ N/m}^2 = (1400\text{–}1600) \text{ MPa} \tag{3.10}$$

However, from a practical viewpoint, this is a very rough approximation, as the bulk modulus varies considerably with pressure, for instance, according to

$$E_\mathrm{isen} = E + K_p p \tag{3.11}$$

Typical values are $E = 16500$ bar and $K_p = 9.558$. The influence of temperature is negligible.

The quantity

$$\alpha \equiv -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial \theta}\right)_p = \frac{1}{V_0}\left(\frac{\partial V}{\partial \theta}\right)_p \tag{3.12}$$

is the fractional change in volume due to a change in temperature at constant pressure. It is called the *cubical expansion coefficient* of the liquid.

In the literature, there are many formulae for the calculation of the density of hydraulic fluids as a function of temperature and pressure. For example, the density at atmospheric pressure (1 bar = 10^5 N/m^2) and variable temperature θ is given by

$$\rho_\theta = \frac{\rho_0}{1+\beta_\theta(\theta-\theta_0)} \tag{3.13}$$

where ρ_0 is the density at reference temperature θ_0 (say 15°C), and β_θ denotes the heat expansion factor, *e.g.*, 0.65×10^{-3} K^{-1} for mineral oils, 0.7×10^{-3} K^{-1} for HFC-fluids, and 0.75×10^{-3} K^{-1} for HFD-fluids (Matthies, 1995). The density of hydraulic fluids following a change in pressure can be expressed as

$$\rho_p = \frac{\rho_0}{1-\kappa_p(p-p_0)} \tag{3.14}$$

where κ_p is the compressibility factor

$$\kappa_p = \frac{1}{V_0}\left(\frac{\partial V}{\partial p}\right)_\theta \tag{3.15}$$

For variable temperature *and* variable pressure the density can be calculated by

$$\rho = \rho_\theta\left(1+\kappa_p \Delta p\right) \tag{3.16}$$

Typical values for κ_p are: 0.7×10^{-4} bar^{-1} for mineral oils, 0.3×10^{-4} bar^{-1} for HFC-fluids, and 0.35×10^{-4} bar^{-1} for HFD-fluids (Matthies, 1995). From these values, it can be concluded that the effect of pressure on the fluid density is minor, and thus negligible in practice.

3.1.3 Effective Bulk Modulus

The bulk modulus of a liquid is substantially lowered by entrained gas and mechanical compliance. According to Merritt (1967), estimates of entrapped air in hydraulic systems run as high as 20% when the fluid is at atmospheric pressure. As the pressure is increased, much of this air dissolves into the liquid and does not affect the bulk modulus. The major source of mechanical compliance may be the hydraulic lines connecting valves and pumps to actuators.

3.1.3.1 Influence of Entrained Air

Some work has been done on determining the bulk modulus of liquid–air mixtures and that of containers due to mechanical compliance. Backé und Murrenhoff (1994:103) proposed the following formulae for the isentropic bulk modulus of liquid–air mixtures (see also Beater, 1999:26):

$$E'_{\text{isen}}(p) = E_{\text{isen}} \frac{1 + r_V}{1 + \left(\frac{p_0}{p}\right)^{1/\kappa} r_V \frac{E}{\kappa p}} \qquad r_V \equiv \frac{V_{\text{G0}}}{V_{\text{L0}}} \tag{3.17}$$

with

E_{isen}	isentropic bulk modulus of the liquid (without entrained air),
V_{G0}	volume of gas entrained in the liquid at atmospheric pressure,
V_{L0}	volume of the liquid at atmospheric pressure,
p_0	atmospheric pressure $(p_0 = 1 \text{ bar})$,
p	liquid pressure, and
κ	isentropic exponent $(\kappa = 1.4)$.

Figure 3.2 shows traces of the ratio $E'_{\text{isen}}(p)/E_{\text{isen}}$ for some values of the volume ration r_V. In Figure 3.3, specific pressure-dependence of bulk modulus is plotted.

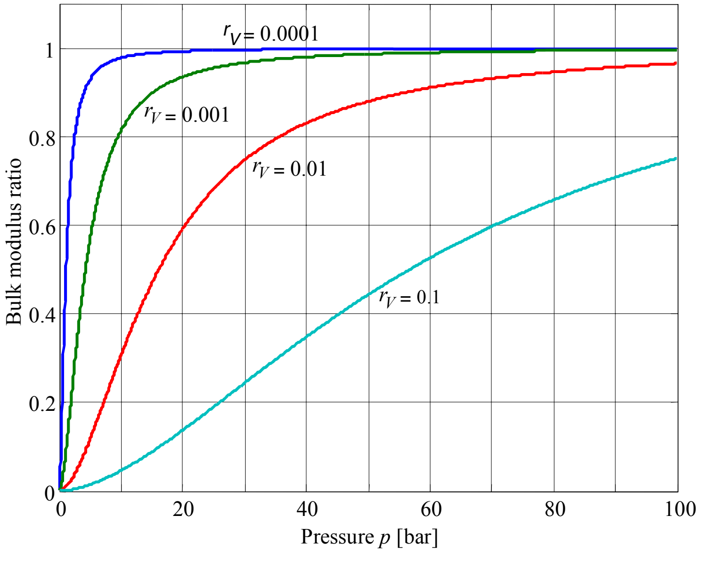

Figure 3.2. Influence of entrained air volume on the isentropic bulk modulus

Especially in low-pressure regions (say, $p \leq 100$ bar), the influence of entrained gas on the bulk modulus is substantial. At a pressure of about 0.6 bar, entrained air can explode (so-called Diesel effect). This effect cause highly undesired erosion defects, power losses, pressure peaks, and noise. This phenomenon is better known as cavitation (sudden implosion of gas bubbles, when fluid pressure decreases under vapour pressure) (Lemmen, 2002).

Note that the expressions given above require the accurate determination of many quantities (for example, of the volume of gas entrained in the liquid), and thus may be difficult to use in practice.

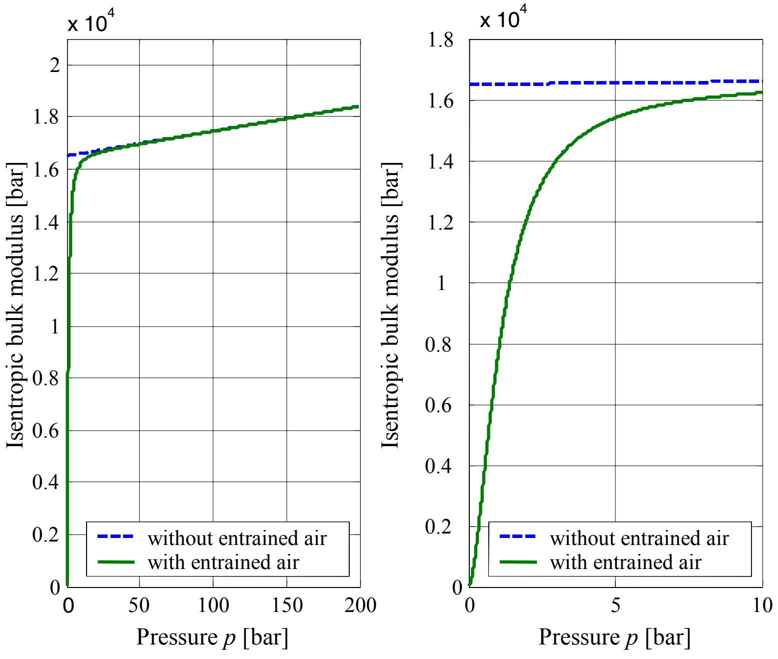

Figure 3.3. Typical trace of isentropic bulk modulus ($r_V = 0.0001$)

3.1.3.2 Influence of Mechanical Compliance

The bulk modulus of cylindrical pipelines can be calculated as (Theißen, 1983)

$$E' = E \frac{1}{1 + \frac{E}{E_\mathrm{p}} w} \tag{3.18}$$

where E_p is the bulk modulus of the (steel) pipeline. For thick-walled pipelines the coefficient w is given by

$$w = \frac{2\left(\frac{d_\mathrm{o}}{d_\mathrm{i}}\right)^2 (1+\nu) + 3(1-2\nu)}{\left(\frac{d_\mathrm{o}}{d_\mathrm{i}}\right)^2 - 1} \tag{3.19}$$

with

d_o	outer diameter,
d_i	inner diameter, and
ν	Poisson's number, $\nu = 0.3$ for steel.

For thin-walled pipelines with the wall thickness s, ($s/d_o < 0.1$), Equation 3.19 approximates to

$$w = \frac{d_i}{s} \tag{3.20}$$

Table 3.2 shows some results found by experiment. It can be seen that especially in the case of high-pressure rubber-based hoses, reinforced with interwoven metal threads, the influence of elasticity is considerable.

Table 3.2. Values for bulk modulus E' (Viersma, 1980)

Nominal pressure [Mpa]	E' [MPa] for steel pipeline $R_i = 6.25$ mm; $R_o = 8$ mm	E' [MPa] for high-pressure hose $R_i = 6.25$ mm
5	1460	500
9	1510	537
13	1570	568
22.5	1890	

3.1.3.3 Empirical Effective Bulk Modulus

Other researchers have derived empirical formulae for the calculation of the *effective bulk modulus* E', including the effects of entrained air and mechanical compliance, based on direct measurements. The commonly used equation for calculation of the bulk modulus E' for hydraulic cylinders in German literature is that of Lee (1977):

$$E'(p) = a_1 E_{\max} \log\left(a_2 \frac{p}{p_{\max}} + a_3 \right) \tag{3.21}$$

with the parameters $a_1 = 0.5$, $a_2 = 90$, $a_3 = 3$, $E_{max} = 18000$ bar, and $p_{max} = 280$ bar.

Hoffmann (1981) proposed the formula

$$E'(p) = E_{\max}\left[1 - \exp\left(-0.4 - 2\times10^{-7} p\right)\right] \tag{3.22}$$

with the pressure p in pascals.

According to Eggerth (1980), the effective bulk modulus can be expressed as

$$E'(p) = \frac{1}{k_1 + k_2 \left(p/p_0\right)^{-\lambda}} \tag{3.23}$$

with the parameters k_1 and k_2 in Table 3.3; p_0 is assumed to be 10 bar.

Table 3.3. Parameters of Eggerth's formula (Beater, 1999)

Temperature [°C]	k_1 [10^{-10} m^2/N]	k_2 [10^{-10} m^2/N]	λ
20	4.943	1.9540	1.480
50	5.469	3.2785	1.258
90	5.762	4.7750	1.100

The relations for effective bulk modulus E' are plotted in Figure 3.4. Although these formulae are approximate, they are sufficient for design purposes. However, experimental data are always preferable.

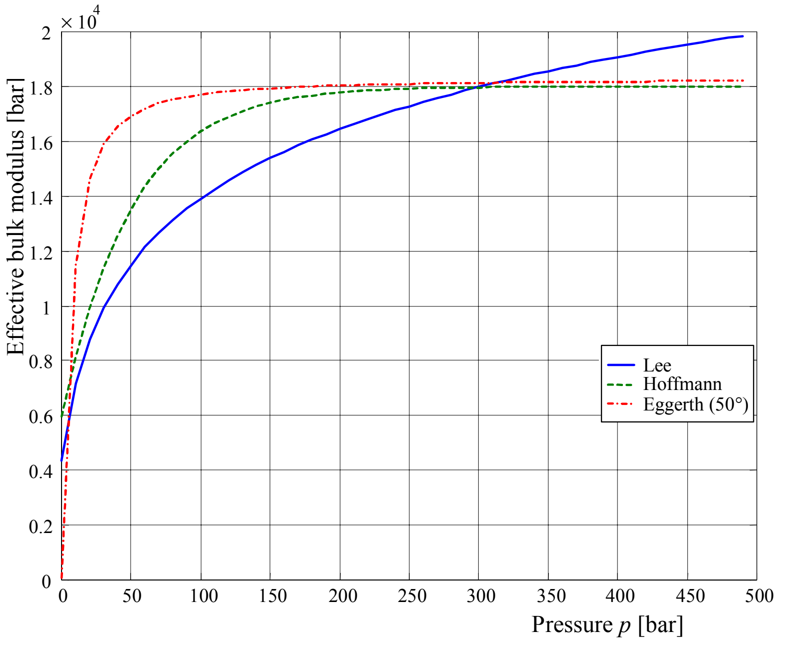

Figure 3.4. Comparison of different formulae for the calculation of E'

3.1.4 Section Summary

The most important three physical properties (viscosity, density and bulk modulus) have been introduced and discussed. The following concluding statements are important from a practical viewpoint:

- Density can be considered constant.
- Viscosity of fluids varies markedly with temperature (Equation 3.4), and to a much lesser degree with pressure.
- Bulk modulus essentially depends on pressure, entrained air and mechanical compliance. Empirical formulae, such as Equations 3.21, 3.22, and 3.23, are recommended for the calculation of the effective bulk modulus. However, some parameter adjustments may be necessary in practice.

3.2 General Equations of Fluid Motion

In this section, the basic principles of conservation and laws governing fluid flow and associated phenomena will be briefly summarised. More detailed derivations can be found in a number of standard textbooks on fluid mechanics (*e.g.*, Slattery, 1972; White, 1986; Lencastre, 1987; Spurk, 1996; Oertel, 1999). Detailed theoretical development and discussions about the conservation laws in fluid mechanics with a special "nice" section (4.3) on pipeline hydraulics can be found in the excellent book by Truckenbrodt (1996).

Conservation laws can be derived by considering a given quantity of matter, control mass or control volume, and its extensive properties, such as mass, momentum and energy. In fluid mechanics, there are several ways to present the conservation equations, such as the control mass approach, the control volume approach and the control tube approach.

3.2.1 Continuity Equation and Pressure Transients

Consider a control tube as depicted in Figure 3.5. The integral form of the mass conservation (continuity) equation can be formulated as (Truckenbrodt, 1996)

$$\int_{(1)}^{(2)} \frac{\partial(\rho A)}{\partial t} \mathrm{d}s + \rho_2 v_2 A_2 - \rho_1 v_1 A_1 = 0 \qquad \text{(control tube)} \qquad (3.24)$$

where the density $\rho = \rho(t,s)$ is, in general, not constant.

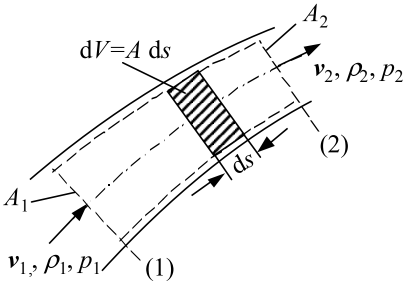

Figure 3.5. Definition of control tube

For incompressible fluids, *i.e.*, $\rho = \text{const.}$ (which is a standard assumption in hydraulics), Equation 3.24 can be reduced to

$$v_1(t)A_1 = v_2(t)A_2 \qquad (\rho = \text{const.}) \quad (3.25)$$

or more generally

$$Q = v(t)A = \text{const.} \qquad \text{(volume flow)} \qquad (3.26)$$

For steady flow, the continuity equation can be expressed as

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2 \qquad (\rho \neq \text{const.}) \qquad (3.27)$$

or in the general form

$$\dot{m} = \rho v A = \text{const.} \qquad \text{(mass flow rate)} \qquad (3.28)$$

Next, the mass conservation equation is written in the differential coordinate-free form (for a control tube element of length ds; see Figure 3.5)

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{v}) = 0 \qquad (3.29)$$

Again, two special cases can be given:

$$\operatorname{div} \boldsymbol{v} = 0 \qquad (\rho = \text{const.}) \tag{3.30}$$

$$\operatorname{div}(\rho \boldsymbol{v}) = 0 \qquad \text{(steady flow)} \tag{3.31}$$

Given a coordinate system (Cartesian, cylindrical or spherical), Equation 3.29 can take a specific form by providing the expression for the divergence (div) operator in that coordinate system. The expression of the continuity equation in a cylindrical coordinate system (r,φ,x) is given by (Truckenbrodt, 1996)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left(\frac{\partial(\rho r v_r)}{\partial r} + \frac{\partial(\rho v_\varphi)}{\partial \varphi}\right) + \frac{\partial(\rho v_x)}{\partial x} = 0 \tag{3.32}$$

Consider again the mass conservation equation for a control volume V and let the accumulated or stored mass of fluid inside be m with a mass density of ρ. Since all fluid must be accounted for, as the medium is assumed continuous, the rate at which mass is stored must equal the incoming mass flow rate minus the outgoing mass flow rate. Therefore, we can write

$$\sum \dot{m}_{\text{in}} - \sum \dot{m}_{\text{out}} = \frac{\mathrm{d}(\rho V)}{\mathrm{d}t} = \rho \dot{V} + V\dot{\rho} \tag{3.33}$$

Taking into account Equation 3.8 and dividing Equation 3.33 by ρ leads to

$$\sum Q_{\text{in}} - \sum Q_{\text{out}} = \dot{V} + \frac{V}{E}\dot{p} \tag{3.34}$$

If the volume is fixed ($V = V_0$), Equation 3.33 becomes

$$\dot{p} = \frac{E}{V_0}\left(\sum Q_{\text{in}} - \sum Q_{\text{out}}\right). \tag{3.35}$$

This equation is fundamental for the description of the pressure dynamics in hydraulic compartments.

3.2.2 Navier–Stokes Equation

The momentum conservation equation is known as the *Navier–Stokes equation* (in differential form) (Lencastre, 1987)

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \rho \boldsymbol{g} - \operatorname{grad} p + \eta \left[\operatorname{div}(\operatorname{grad} \boldsymbol{v}) + \frac{1}{3}\operatorname{grad}(\operatorname{div} \boldsymbol{v})\right] \tag{3.36}$$

where

- $\rho \boldsymbol{g}$ represents the body forces.
- $\rho\, \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ the inertial forces.
- $\operatorname{grad} p$ is the vector of components $\partial p/\partial x_i$. It corresponds to the derivative or inclination of the pressure in the direction of the flow.
- The term $\eta \operatorname{div}(\operatorname{grad} \boldsymbol{v})$ represents diffusion of the vector $\boldsymbol{v}$ within the flow. It represents the action of one particle on the others owing to the effect of viscosity.
- The term $1/3\eta \operatorname{grad}(\operatorname{div} \boldsymbol{v})$ represents the influence of compressibility and vanishes in the case of incompressible liquids.

Letting the fluid be incompressible and dividing Equation 3.36 throughout by ρ leads to

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{g} - \frac{1}{\rho}\operatorname{grad} p + \mu \operatorname{div}(\operatorname{grad} \boldsymbol{v}) \tag{3.37}$$

Given the hypothesis of the external forces being derived from a potential ξ, then $\boldsymbol{g} = \operatorname{grad} \xi$. Thus, in the case of incompressible liquids, Equation 3.37 becomes

$$-\operatorname{grad} \xi + \frac{1}{\rho}\operatorname{grad} p = -\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \mu \operatorname{div}(\operatorname{grad} \boldsymbol{v}) \tag{3.38}$$

If the potential is that of gravity, *i.e.*, $\xi = -gz$, then dividing throughout by g gives:

$$\operatorname{grad}\left(z + \frac{p}{\gamma}\right) = -\frac{1}{g}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \frac{\eta}{\gamma}\operatorname{div}(\operatorname{grad} \boldsymbol{v}) \tag{3.39}$$

with $\gamma = \rho g$. In the case of a perfect or ideal liquid, *i.e.*, $\eta = 0$, which does not exist in reality, Equation 3.39 becomes

$$\operatorname{grad}\left(z + \frac{p}{\gamma}\right) = -\frac{1}{g}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \tag{3.40}$$

For a fluid element along the path line, Equation 3.39 can be rearranged, taking into account $\eta = \rho\mu$ and $\gamma = \rho g$, to give *Euler's equation*

$$\frac{\partial}{\partial s}\left(z + \frac{p}{\gamma}\right) = -\frac{1}{g}\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mu}{g}\operatorname{div}(\operatorname{grad} v) \qquad v = v_s \tag{3.41}$$

Let us now return to the Navier–Stokes Equation 3.37 and write the system in cylindrical coordinates (Truckenbrodt, 1996):

$$\frac{\mathrm{d}v_r}{\mathrm{d}t} = g_r - \frac{1}{\rho}\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \varphi^2} + \frac{\partial^2 v_r}{\partial x^2} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi} - \frac{v_r}{r^2}\right] \tag{3.42}$$

$$\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} = \frac{1}{r}\left(g_\varphi - \frac{1}{\rho}\frac{\partial p}{\partial \varphi}\right) + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial \varphi^2} + \frac{\partial^2 v_\varphi}{\partial x^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r^2}\right] \tag{3.43}$$

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_x}{\partial \varphi^2} + \frac{\partial^2 v_x}{\partial x^2}\right] \tag{3.44}$$

3.2.3 Bernoulli's Theorem

Considering Equation 3.41, and bearing in mind that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s} \qquad \text{(substantial acceleration)}$$

$$v\frac{\partial v}{\partial s}=\frac{\partial\left(v^2/2\right)}{\partial s} \tag{3.45}$$

it follows that

$$\frac{\partial}{\partial s}\left(z+\frac{p}{\gamma}+\frac{v^2}{2g}\right)=-\frac{1}{g}\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{\mu}{g}\mathrm{div}(\mathrm{grad}\,v) \tag{3.46}$$

The first element of the equation has an essentially global energy significance. It represents the variation in the total energy discharged per unit weight of a particle along its trajectory.

If the viscosity terms are removed from the equation, *i.e.*, the flow may resemble a perfect fluid, we have

$$\frac{\partial}{\partial s}\left(gz+gp/\gamma+v^2/2\right)=-\frac{1}{g}\frac{\partial v}{\partial t} \tag{3.47}$$

In the case of steady flow, $\partial v/\partial t=0,$ energy conservation holds:

$$E=z+\frac{p}{\gamma}+\frac{v^2}{2g}=\text{const.} \tag{3.48}$$

this being the expression that represents *Bernoulli's theorem* for one-dimensional steady flows. In the case of an incompressible liquid in steady flow, in which the friction forces and, consequently, energy losses can be disregarded, the total energy of a particle is maintained along its trajectory.

3.2.4 Section Summary

From the variety of formulae presented in this section, those most important and those most often needed in practice are the continuity equations for incompressible fluids (Equations 3.25–3.28) and Bernoulli's Equation 3.48 (or other equivalent variants of it). Another often-used equation is the fundamental Equation 3.35 for the description of the pressure dynamics in hydraulic compartments.

The general continuity equation and Navier–Stokes equations are only interesting for the analysis of pipeline dynamics, see Section 4.2.5.

3.3 Flow Through Passages

Two distinct types of fluid flow through passages can occur:

- *Laminar* or *viscous* flow, in which each fluid particle describes a well-defined trajectory, with a velocity only in the direction of the flow.
- *Turbulent* or *hydraulic* flow (this being the most usual in hydraulic phenomena), in which each particle, apart from the velocity in the direction of the flow, is animated by fluctuating cross-current velocities.

The Reynolds number, Re, defined by

$$\mathrm{Re} = \frac{\rho v d_{\mathrm{h}}}{\eta} = \frac{v d_{\mathrm{h}}}{\mu} \tag{3.49}$$

is the characteristic parameter: for lower values of Re, the flow is laminar; for higher values the flow is turbulent. Thereby, v is the average velocity of flow. d_{h} represents the hydraulic diameter, which is defined by

$$d_{\mathrm{h}} = \frac{4A}{S} \tag{3.50}$$

where A is the flow section area and S is the flow section perimeter. For each flow case, the characteristic length is agreed upon and empirical values are obtained for the Reynolds number which describes transition from that viscosity- to inertia-dominated flows.

3.3.1 Flow Establishment in Pipelines

One basic element of hydraulic systems is cylindrical pipelines, in which flow may be *laminar* or *turbulent*. The characteristic length to be used for the Reynolds number is inside pipeline diameter d, *i.e.*,

$$\mathrm{Re} = \frac{vd}{\mu} \tag{3.51}$$

The transition from laminar to turbulent flow has been observed experimentally to occur in the range $2000 < \mathrm{Re}_{\mathrm{crit}} < 4000$, typically $\mathrm{Re}_{\mathrm{crit}} = 2300$. Below $\mathrm{Re} = 2300$ the flow is always laminar; above $\mathrm{Re} = 4000$ the flow is usually, but not always, turbulent. It is possible to have laminar flow at Reynolds number considerably above 4000 if extreme care is taken to avoid disturbances which would lead to turbulence. However, these instances are exceptional, and the upper limit of 4000 is a good rule (Viersma, 1980).

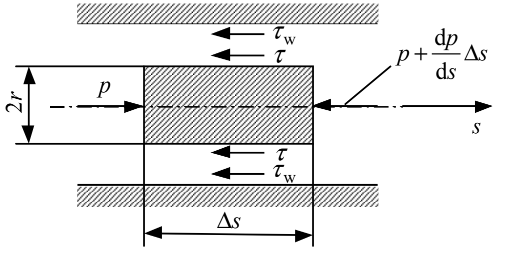

Figure 3.6. Force equilibrium of fluid elements in cylindrical pipelines

3.3.1.1 Hagen–Poiseuille Law

Consider a cylindrical pipeline of radius $r \le R$ and let the flow be steady and laminar. The starting point is the force equilibrium in the axial direction (Figure 3.6); that means

$$\tau = -\frac{r}{2}\frac{\mathrm{d}p}{\mathrm{d}x} \qquad \tau_\mathrm{w} = -\frac{R}{2}\frac{\mathrm{d}p}{\mathrm{d}x} \qquad \frac{\tau}{\tau_\mathrm{w}} = \frac{r}{R} \tag{3.52}$$

where τ_w is the shear stress at the pipeline wall (*i.e.*, at $r = R$). On the other hand, the shear stress Equation 3.2 can be written as ($v \equiv v_x$, $\mathrm{d}y \equiv -\mathrm{d}r$)

$$\tau = -\eta\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{r}{2}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{3.53}$$

Equations 3.52 and 3.53 can then be combined to obtain

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{r}{2\eta}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{3.54}$$

the relationship for calculating the velocity profile for laminar flows in cylindrical pipelines. In fact, the integration of Equation 3.54 (with $\mathrm{d}p/\mathrm{d}x$ = const., and $v(R) = 0$) yields the velocity profile (Figure 3.7)

$$v(r) = -\frac{1}{4\eta}(R^2 - r^2)\frac{\mathrm{d}p}{\mathrm{d}x} \tag{3.55}$$

and leads to the maximum velocity and the mean velocity

$$v_\mathrm{max} = v(0) = -\frac{R^2}{4\eta}\frac{\mathrm{d}p}{\mathrm{d}x} \qquad \bar{v} = -\frac{R^2}{8\eta}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{3.56}$$

respectively.

Finally, the continuity Equation 3.26 and Equation 3.56 are combined to give the so-called *Hagen–Poiseuille equation*

$$Q = A\bar{v} = -\frac{\pi R^4}{8\eta}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{3.57}$$

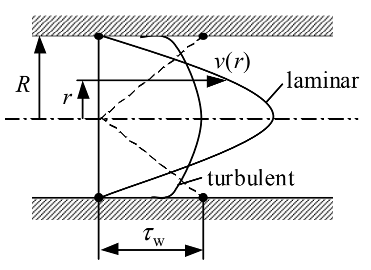

Figure 3.7. Velocity profiles for laminar and turbulent flows in a cylindrical pipeline

3.3.2 Flow Through Orifices

Orifices are sudden restrictions of short length (ideally zero length for a sharp-edged orifice) in the flow passage and may have a fixed or variable area (see Figure 3.8). Orifices are generally used to control flow, or to create a pressure differential (valves). Two types of flow regime exist, depending on whether inertia or viscous forces dominate. The flow velocity through an orifice must increase above that in the upstream region to satisfy the law of continuity. At high Reynolds numbers, the pressure drop across the orifice is caused by the acceleration of the fluid particles from the upstream velocity to the higher jet velocity. At low Reynolds numbers, the pressure drop is caused by the internal shear forces resulting from fluid viscosity.

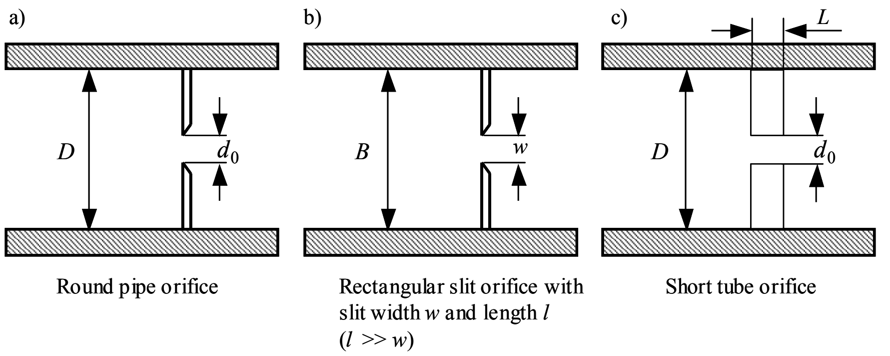

Figure 3.8. Round, slit-type and short tube orifices

3.3.2.1 Orifice Equations for Turbulent Flow

Since most orifice flows occur at high Reynolds numbers, this region is of major importance. Such flows are often referred to as "turbulent" (Figure 3.9b), but the term does not have quite the same meaning as in pipeline flow (Merritt, 1967). Referring to Figure 3.9a, the fluid particles are accelerated up to the jet velocity between sections 1 and 2. The flow between these sections is streamline or potential flow, and experience justifies the use of Bernoulli's theorem in this region.

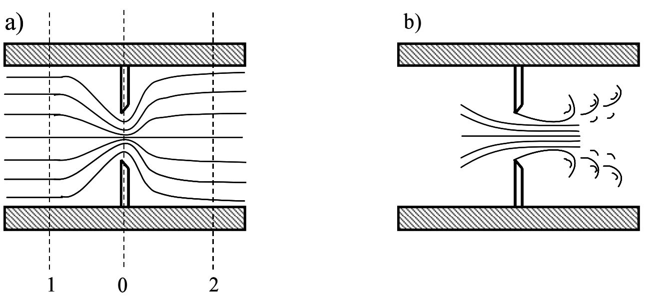

Figure 3.9. Flow through an orifice: (a) laminar flow; (b) turbulent flow

According to Bernoulli's theorem (Equation 3.48), the total energy losses of the hydraulic flow are derived from the energy degraded into heat by friction of the particles against one another and by friction of the particles against the walls of the conduit. The energy dissipated due to friction between sections 1 and 2 will be equal to

$$\Delta p_{\mathrm{ls}} = \left(p_1 + \rho v_1^2/2 + \rho g z_1\right) - \left(p_2 + \rho v_2^2/2 + \rho g z_2\right) \tag{3.58}$$

It is common to use the dimensionless pressure loss factor ζ, which is defined as

$$\zeta = \frac{\Delta p_{\mathrm{ls}}}{\rho v_1^2/2} \quad \text{or} \quad \Delta p_{\mathrm{ls}} = \zeta \rho v_1^2/2 \tag{3.59}$$

The factor ζ depends on the geometry of the conduit and on the Reynolds number which can be approximated by

$$\zeta(\mathrm{Re}) = \frac{k_1}{\mathrm{Re}} + k_2 \tag{3.60}$$

Taking into account that at a point far from the orifice

$$v_1 = v_2 = v \quad \text{and} \quad A_1 = A_2 = A = \frac{\pi}{4} d^2 = \text{const.} \tag{3.61}$$

we get the flow as the product of conduit area and the speed, *i.e.*,

$$Q = Av = A\sqrt{\frac{2}{\rho\zeta}\left(p_1 - p_2\right)} \tag{3.62}$$

Instead of Equation 3.62, it is common in the field of hydraulics to use the modified orifice equation

$$Q = \alpha_{\mathrm{d}} A \sqrt{\frac{2}{\rho}\Delta p} \tag{3.63}$$

where α_{d} is the discharge coefficient. Theoretically, $\alpha_{\mathrm{d}} = \pi/(\pi+2) = 0.611$ (von Mises, 1917). This can be used for all sharp-edged orifices regardless of the particular geometry, if the flow is turbulent and $A_0 << A$.

3.3.2.2 Discharge Coefficient for Turbulent Flow

Sharp-edged orifices (Figure 3.8) are desirable for their predictable characteristics and insensitivity to temperature changes. However, cost frequently prohibits their use, especially as fixed restrictors, and orifices with length (Figure 3.8c) are often employed instead. An average discharge coefficient for such short tube orifices can be expressed as (Merritt, 1967); see Figure 3.10:

$$\begin{aligned} \alpha_{\mathrm{d}} &= \left(2.28 + 64\frac{2L}{d_0 d}\right)^{-1/2} && \text{for } \frac{d_0 d}{2L} \le 50 \\ \alpha_{\mathrm{d}} &= \left[1.5 + 13.74\left(\frac{2L}{d_0 d}\right)^{1/2}\right]^{-1/2} && \text{for } \frac{d_0 d}{2L} > 50 \end{aligned} \tag{3.64}$$

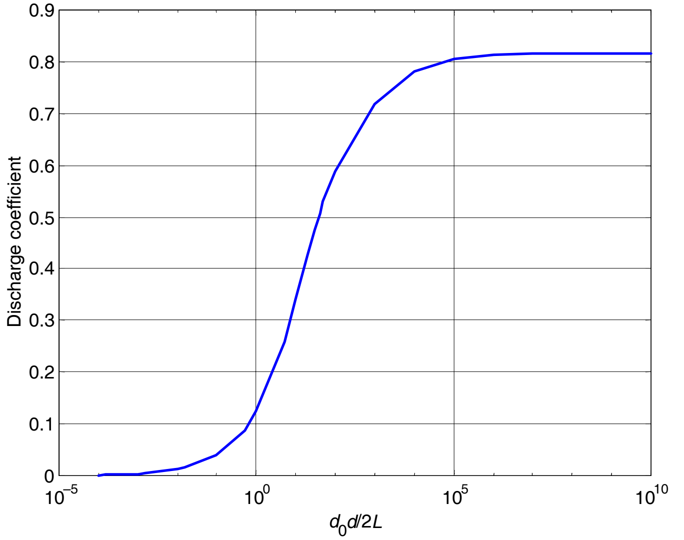

Figure 3.10. Discharge coefficient α_d for short tube orifices according to Equation 3.64

According to Lichtarowicz *et al.* (1965), the average discharge coefficient can be estimated using (see Figure 3.11)

$$\frac{1}{\alpha_{\mathrm{d}}} = \sqrt{1-\beta^4}\left[\frac{1}{\alpha_{\mathrm{d,max}}} + \frac{20}{\mathrm{Re_h}}\left(1+2.25\frac{L}{d_{\mathrm{h}}}\right) - \frac{0.005\dfrac{L}{d_{\mathrm{h}}}}{1+7.5(\log \mathrm{Re_h} - 3.824)^2}\right] \tag{3.65}$$

where

$$\mathrm{Re_h} = \sqrt{\frac{2\Delta p}{\rho}}\frac{d_{\mathrm{h}}}{\mu} = \mathrm{Re}\frac{\sqrt{1-\beta^4}}{\alpha_{\mathrm{d}}} \qquad \beta = \frac{d_0}{d} \tag{3.66}$$

$$\alpha_{\mathrm{d,max}} = 0.827 - 0.0085\frac{L}{d_{\mathrm{h}}} \tag{3.67}$$

3.3.2.3 Discharge Coefficient for Turbulent–Laminar Flow

The formulae proposed above are only valid if turbulent flow occurs. Turbulence is ensured only at "large enough" Reynolds numbers:

$$\mathrm{Re} = \frac{2vh}{\mu} \tag{3.68}$$

where h is the smallest dimension of the (rectangular) orifice. At low temperatures, low orifice pressure drops, and/or small orifice openings, the Reynolds number may become sufficiently low to permit laminar flow.

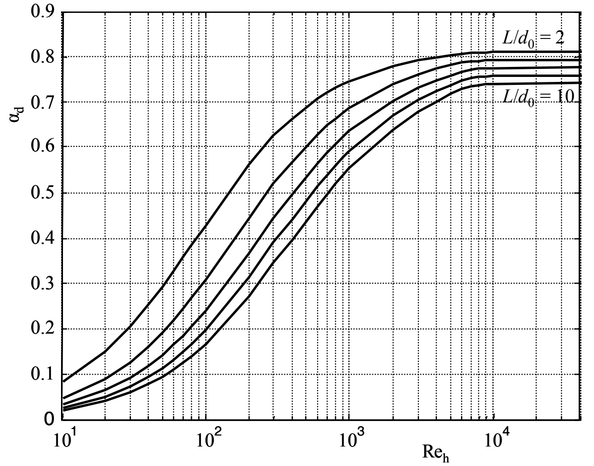

Figure 3.11. Discharge coefficient α_d for short tube orifices according to Equation 3.65

Experiments carried out by Viersma (1980) proved that at very sharp edges in narrow orifices the critical value Re_{crit} is as low as 20, whereas slightly rounded off edges increased Re_{crit} to 80 or higher. Thus, at very sharp edges, α_d may be assumed to be constant at $Re > Re_{crit} \approx 20$.

Although the analysis leading to Equation 3.63 is not valid at low Reynolds numbers, many attempts have been made to extend this equation to the laminar region by plotting the discharge coefficient as a function of Reynolds number, *i.e.*,

$$\alpha_d = \delta\sqrt{Re} \tag{3.69}$$

as pointed out by many investigators (Wuest, 1954; Viersma, 1980). The quantity δ depends on geometry and is called the laminar flow coefficient. Viersma (1980) found that

$$\delta = \frac{\alpha_{d,turb}}{\sqrt{Re_{crit}}} = \frac{0.611}{\sqrt{10}} \approx 0.1932 \qquad \text{hypothetical} \tag{3.70}$$

$$\delta = \frac{\alpha_{d,turb}}{\sqrt{Re_{crit}}} = \frac{0.611}{\sqrt{20}} \approx 0.1366 \qquad \text{for sharp edges} \tag{3.71}$$

$$\delta = \frac{\alpha_{d,turb}}{\sqrt{Re_{crit}}} = \frac{0.611}{\sqrt{80}} \approx 0.0683 \qquad \text{for slightly rounded-off edges} \tag{3.72}$$

The discharge coefficient can then be represented by the asymptotes shown in Figure 3.12 in the laminar region and $\alpha_d = 0.611$ in the turbulent region.

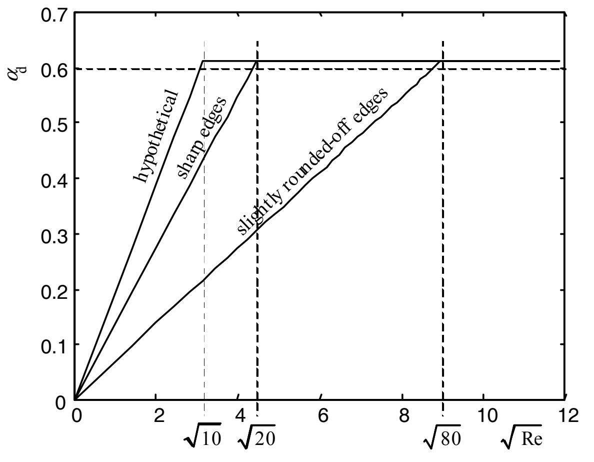

Figure 3.12. $\alpha_{\mathrm{d}} = f\left(\sqrt{\mathrm{Re}}\right)$ according to Viersma (1980)

3.3.2.4 Orifice Equations for Laminar Flow

Expressing the Reynolds number as

$$\mathrm{Re} = \frac{(Q / A_0) d_{\mathrm{h}}}{\mu} \tag{3.73}$$

and substituting Equations 3.73 and 3.69 into Equation 3.63 yields

$$Q = \frac{2\delta^2 d_{\mathrm{h}}}{\mu\rho} A_0 \Delta p \tag{3.74}$$

for low Reynolds numbers.

Wuest (1954) has theoretically determined expressions for laminar flow through sharp-edged circular orifices (in an infinite plane, *i.e.*, $d_0 \ll d$ in Figure 3.8a) as

$$Q = \frac{\pi d_0^3}{50.4 \mu\rho} \Delta p \tag{3.75}$$

and through sharp-edged rectangular slits (of height b_0 and width w in an infinite plane, *i.e.* , $b_0 \ll B$ in Figure 3.8b, with $w \gg b_0$)

$$Q = \frac{\pi b_0^2 w}{32 \mu\rho} \Delta p \tag{3.76}$$

Equating Equation 3.74 to Equations 3.75 and 3.76 gives $\delta = 0.2$ for a sharp-edged round orifice and $\delta = 0.157$ for a sharp-edged slit orifice.

3.3.3 Flow Through Valves

Flows through valving orifices (Figure 3.13) are usually described by the orifice Equation 3.63 with a linear relationship between the valve spool position x_v and the flow area (critical centre), *i.e.*,

$$Q = Q(x_v, \Delta p) = c_v x_v \sqrt{p_1 - p_2} = c_v x_v \sqrt{\Delta p} \tag{3.77}$$

with the flow coefficient

$$c_v = \pi d_v \alpha_d \sqrt{\frac{2}{\rho}} \tag{3.78}$$

for servo-valves (d_v: diameter of the valve spool), and

$$c_v = 4|x_v| \tan(\alpha/2) \alpha_d \sqrt{\frac{2}{\rho}} \tag{3.79}$$

for special proportional valves with triangular valve seats having groove angle α (see Köckemann *et al.*, 1991).

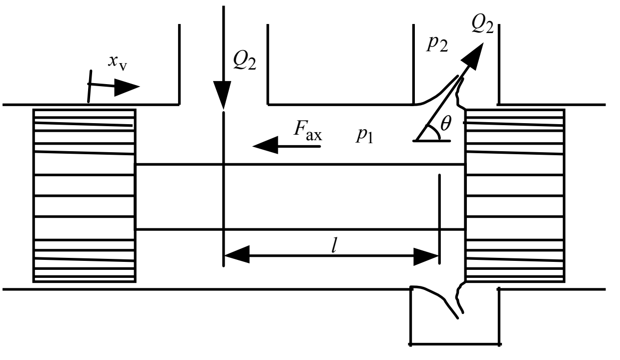

Figure 3.13. Axial flow force on spool due to unequal jet angles

Note that c_v is usually given in

$$\frac{\mathrm{dm}^3}{\min\sqrt{\mathrm{bar}}\ \mathrm{mm}} \quad \text{or equivenlently} \quad \frac{\mathrm{m}^3}{\mathrm{s}\sqrt{\mathrm{N}}} \tag{3.80}$$

Note also that Equation 3.77 can be written using the valve voltage u_v as

$$Q = Q(u, \Delta p) = c_v \frac{x_{v,\max}}{u_{\max}} u \sqrt{p_1 - p_2} \equiv c_{vu} u \sqrt{\Delta p} \tag{3.81}$$

This means that the value and the dimension of c_v have to be adapted to the signal used (which can be the valve stroke x_v, the valve voltage u_v, or the valve current I_v). In the rest of this book, only the symbol c_v will be used without regard to the nature of the valve signal (normalised or not).

In practice, the flow coefficient may best be determined experimentally, or it may be calculated using the catalogue data (Q_N, Δp_N, and $x_{v,max}$) of the valve manufacturer

$$c_v = \frac{Q_N}{\sqrt{\Delta p_N / 2}} \frac{1}{x_{v,max}} \tag{3.82}$$

where Q_N is the nominal flow, Δp_N the nominal pressure drop, and $x_{v,max}$ the maximum stroke of the valve. The corresponding discharge coefficient is

$$\alpha_d = \frac{Q_N}{A(x_{v,max})\sqrt{\Delta p_N / \rho}} \tag{3.83}$$

Since Equation 3.77 is not valid for low Reynolds numbers, Feigel (1987a) derived the following flow equation to be used for laminar–turbulent valve flow cases

$$Q = c_v x_v \left[\sqrt{\left(\frac{c_{lt}}{x_v}\right)^2 + \Delta p} - \frac{c_{lt}}{x_v} \right] \tag{3.84}$$

It is assumed that $A = \pi d x_v$, and the introduced laminar–turbulent flow coefficient is calculated by

$$c_{lt} = \frac{\delta \mu \sqrt{\rho}\,\mathrm{Re}_{crit}}{4\sqrt{2}\delta^2} \tag{3.85}$$

where δ is the slope of the curve $\alpha_d = f(\sqrt{\mathrm{Re}})$ according to Equation 3.69. As $x_v \to \infty$ Equation 3.84 becomes Equation 3.77, and thus may also be used for turbulent valve flow. A typical value of c_{lt} is 0.6 (Saffe, 1986).

Yet another approximation formula for the discharge coefficient that has been used by many researchers (*e.g.*, Klein, 1993), and which considers the dependence on the valve spool position x_v, is given by

$$\alpha_d = \alpha_d(x_v) = \alpha_{d0}\left(1 - K_{d,corr}\frac{|x_v|}{x_{v,max}}\right) \tag{3.86}$$

where α_{d0} is the basic discharge coefficient, $K_{d,corr}$ a correction factor, and $x_{v,max}$ the maximum spool displacement. Typical values are $\alpha_{d0} = 0.65$, and $K_{d,corr} = 0.32$.

Finally, a generalised expression for the flow through valve orifices reads:

$$Q = Q(x_v, \Delta p) = \alpha_d A(x_v)\sqrt{2/\rho}\sqrt{\Delta p} \tag{3.87}$$

where $A(x_v)$ is the area of the valve orifice. $A(x_v)$ depends on the orifice geometry (*i.e.*, geometrical form of the orifice and centre type), which varies from one manufacturer to another, especially for proportional valves.

3.3.4 Section Summary

Orifice flow is laminar for $\mathrm{Re} < \mathrm{Re}_{\mathrm{crit}}$ with flow rates directly related to pressure drop as given by Equation 3.74. In the vicinity of $\mathrm{Re}_{\mathrm{crit}}$, both inertia and viscosity are important. For $\mathrm{Re} > \mathrm{Re}_{\mathrm{crit}}$, the flow can be treated as turbulent and is described by the orifice Equation 3.63. Commonly, the orifice Equation 3.63 is most used for all situations with a total disregard for the types of flow that can be encountered. This is justified in the majority of cases, but it can lead to gross errors in certain instances. Typical and realistic values of α_{d} lie between 0.65 and 0.75.

The more practical way is to apply the orifice Equation 3.77 with the calculation of the flow coefficient according to Equation 3.82 (or equivalently Equation 3.83).

3.4 Spool Port Forces

Closely related to the flows through the spool ports is the axial force on the spool. This flow force is caused by the change of momentum of the flow, due to a difference in jet angles for the inlet flows and outlet flows, as depicted in Figure 3.13.

The steady-state axial flow force on the spool can be calculated by (Merritt, 1967; Lausch, 1990)

$$F_{\mathrm{ax,steady}} = 2\alpha_{\mathrm{d}}^{2} A(x_{\mathrm{v}}) \ \cos\theta \ \Delta p \tag{3.88}$$

The jet angle θ can be assumed constant, namely $\theta \approx 69°$, leading to $\cos\theta = 0.358$, which corresponds to the theoretical value (if there is no radial clearance between the valve spool and sleeve) derived by von Mises (1917).

Feigel (1992) proposed the following formulae for the calculation of steady-state flow forces on uncompensated spool valves:

$$F_{\mathrm{ax,steady}} = K_{\mathrm{f}} Q \sqrt{\Delta p} \tag{3.89}$$

with $K_{\mathrm{f}} = 0.077$ [N min/(dm^3 bar)] for one-edge valves, $K_{\mathrm{f}} = 0.054$ for two-edge valves, and $K_{\mathrm{f}} = 0.109$ for tetragonal valves. Equation 3.89 is combined with Equation 3.77 to give

$$F_{\mathrm{ax,steady}} = K_{\mathrm{f}} c_{\mathrm{v}} \Delta p x_{\mathrm{v}} \tag{3.90}$$

Thus far the discussion has considered only the steady-state flow force. If the slug of fluid in the valve chamber is accelerated, then a force is produced which reacts on the face of the spool valve lands. The magnitude of dynamic flow force is given by Newton's second law as

$$F_{\mathrm{ax,dyn}} = ma = \rho l \dot{Q} \tag{3.91}$$

With Equation 3.77, the dynamic flow force becomes

$$F_{\mathrm{ax,dyn}} = \rho l c_{\mathrm{v}} \Delta p \dot{x}_{\mathrm{v}} + \rho l c_{\mathrm{v}} x_{\mathrm{v}} \frac{\Delta \dot{p}}{2\sqrt{\Delta p}} \tag{3.92}$$

Therefore, the dynamic flow force is proportional to spool velocity and pressure changes. The velocity term is the more significant because it represents a damping force; the pressure rate term is usually neglected.

In practice, the axial spool forces do not seem to play any significant role for the valve manufacturers. Although several compensation techniques to reduce or eliminate these forces have been investigated (see Merritt, 1967, and Feigel, 1992), none has found wide acceptance by practitioners. The practical solution to this problem is to use a two-stage servo-valve, in which the pilot stage, usually a flapper-nozzle valve, provides an appropriate force to stroke the main-stage spool valve.

3.5 Electro-hydraulic Analogy

The principles of electro-hydraulic analogy are summarised in Figure 3.14.

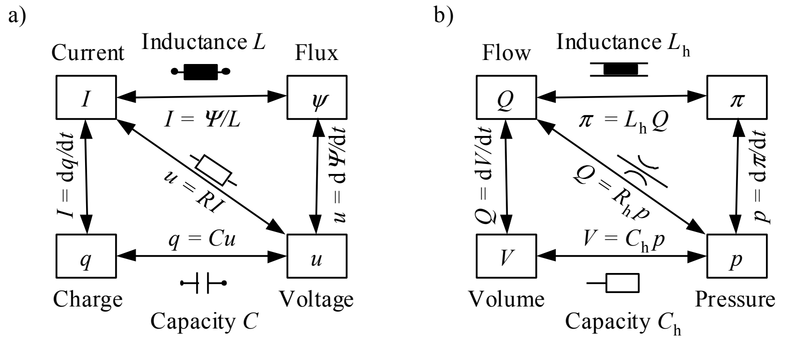

Figure 3.14. Relationships between variables in (a) electrical and (b) hydraulic systems (Beater, 1999)

3.5.1 Hydraulic Capacitance

Equation 3.35 can be written as

$$Q = \frac{V_0}{E}\dot{p} = C_h \dot{p} \tag{3.93}$$

The proportionality factor C_h is referred to as the *hydraulic capacitance*

$$C_h \equiv -\left(\frac{\partial V}{\partial p}\right)_\vartheta = \frac{V_0}{E} \tag{3.94}$$

in analogy to the capacitance of a capacitor in electrical circuits.

3.5.2 Hydraulic Resistance

The *hydraulic resistance* $R_{h,L}$ for laminar flow can, for instance, be determined from the Hagen–Poiseuille Equation (3.57)

$$Q = \frac{\pi R^4}{8\eta l}\Delta p \tag{3.95}$$

for cylindrical pipelines of radius R and length l. Equation 3.95 gives

$$R_{\mathrm{h,L}} = \frac{\Delta p}{Q} = \frac{8\eta l}{\pi R^4} \tag{3.96}$$

Since in this case the pressure drop is directly proportional to the flow, a resistance given by this equation is called a linear resistance to motion.

In general, however, the hydraulic resistance is non-linear to motion, *e.g.*, due to the square-root function in Equation 3.63, which can be rearranged to yield:

$$R_{\mathrm{h,t}} = \frac{\Delta p}{Q^2} = \frac{\rho}{2\alpha_{\mathrm{d}}^2 A^2} \tag{3.97}$$

or from Equation 3.77:

$$R_{\mathrm{h,t}} = \frac{\Delta p}{Q^2} = \frac{1}{c_{\mathrm{v}}^2 x_{\mathrm{v}}^2} \tag{3.98}$$

In practice, it is preferable to work with the Δp vs. Q characteristic (or flow–pressure function) established by measurements on the actual valve in question. The manufacturers of standardised valves usually present this characteristic in their catalogues.

3.5.3 Hydraulic Inductance

The combination of Newton's law

$$F_a = ma = Al\rho\, a \tag{3.99}$$

the continuity equation

$$Q = Av \Rightarrow a = \dot{v} = \frac{\dot{Q}}{A} \tag{3.100}$$

and

$$F_a = A\Delta p \tag{3.101}$$

leads to the hydraulic inductance

$$L_{\mathrm{h}} = \frac{\Delta p}{\dot{Q}} = \frac{l\rho}{A} \tag{3.102}$$