1 The Discrete Case

In this first chapter, we bring together results concerning both the valuation of financial assets and equilibrium models, in a discrete framework: there are two dates, and the asset prices only take a finite number of values. We have chosen to introduce in the context of very simple models, concepts that will be developed further on in the book, in the hope of easing the reader's task.

1.1 A Model with Two Dates and Two States of the World

Here we study a financial market with two dates, time 0 and time 1, in the very simple case of two possible *states of the world* at time 1. Obviously, this situation is not very realistic. It is a textbook case, which will allow us to draw out concepts (such as hedging portfolios, arbitrage and the risk-neutral measure), which will be useful for dealing with more sophisticated models, describing more realistic situations.

1.1.1 The Model

The financial market that we are studying is made up of one stock, and one riskless investment (such as a savings account).

At time 0 (today), the stock is worth S euros. At time 1 (tomorrow, or in six months' time), the stock will be worth either S_u euros or S_d euros with $S_d < S_u$, depending on whether its price goes up or down. The outcome is not known at time 0. We usually say that the stock is worth S_u or S_d depending on the "state of the world".

The riskless investment has a rate of return equal to r $(r > 0)$: one euro invested today will yield $1 + r$ euros at time 1 (whatever the state of the world). This is why the investment is called riskless.

We now consider a *call option* (an option to buy). A call option is a financial instrument: the buyer of the option pays the seller an amount q (the premium) at time 0, in return for the right, but not the obligation, to buy the stock at time 1, and at a price K (the *exercise price* or *strike*), which is set when the contract is signed at time 0. At the time when the buyer decides whether or not to buy the stock, he knows its price, which here is either S_u or S_d. If the price of the stock at time 1 is greater than K, the option holder buys the stock at the agreed price K and immediately sells it on, so making a gain; otherwise, he does not buy it.

A *put option* (an option to sell) gives the right to its buyer to sell a stock at a price K, which is agreed upon when the contract is signed.

The profit linked to a call is unlimited, and the losses are limited to q. For a put, the profit is limited, and the losses are unlimited.

The *valuation* of an option consists in determining the price q of the option under normal market conditions.

1.1.2 Hedging Portfolio, Value of the Option

Call Options First, we consider the case where $S_d \leq K \leq S_u$. The other two cases are not as interesting: if $K < S_d$, the option holder will gain at least $S_d - K$ whatever the state of the world, and the seller will always make a loss (and the opposite is true when $S_u < K$).

Suppose then that $S_d \leq K \leq S_u$. We will see how to build a "portfolio" with the same payoff as the option at time 1. A *portfolio* is made up of a pair (α, β), where α is the amount, in euros, invested in the riskless asset, and β is the number of stocks the investor holds (α and β can be of any sign: one can sell stocks one does not hold[1], and borrow money). If (α, β) is the portfolio held at time 0, its value in euros is $\alpha + \beta S$. At time 1, this same portfolio is worth:

$\alpha(1+r) + \beta S_u$ if we are in the first state of the world, the high state (the stock price has risen),

$\alpha(1+r) + \beta S_d$ if we are in the second state of the world, the low state.

We say that a portfolio *replicates* the option if it has the same payoff at time 1 as the option, and this whatever the state of the world. In other words, the two following equalities must hold:

$$\begin{aligned} \alpha(1+r) + \beta S_u &= S_u - K \\ \alpha(1+r) + \beta S_d &= 0 \,. \end{aligned}$$

By solving the linear system above, we can easily obtain a pair (α^*, β^*):

$$\alpha^* = -\frac{S_d(S_u - K)}{(S_u - S_d)(1+r)}; \qquad \beta^* = \frac{S_u - K}{S_u - S_d} \,.$$

[1] A short sale: we can short the stock, or have a short position in the stock. Having a long position means holding the stock.

The price of the option is the value at time 0 of the portfolio (α^*, β^*), that is

$$q \;=\; \alpha^* + \beta^* S \;=\; \frac{S_u - K}{S_u - S_d}\left(S - \frac{S_d}{1+r}\right) . \qquad (1.1)$$

This is a "fair price": with the amount q received, the option seller can buy a portfolio (α^*, β^*), which generates the gain $S_u - K$ if prices rise, and which will then cover (or hedge) his losses (we call this a *hedging portfolio*[2]). As to the option buyer, he is not prepared to pay more than q, because otherwise he could use the money to build a portfolio which would yield more than the option, for example using the same β^* and an α that is larger than α^*.

To obtain the option pricing formula without assuming $S_d \leq K \leq S_u$, we use the same method. We look for a pair (α^*, β^*) such that

$$\alpha^*(1+r) + \beta^* S_u \;=\; \max(0,\, S_u - K) := \; C_u \;,$$
$$\alpha^*(1+r) + \beta^* S_d \;=\; \max(0,\, S_d - K) := \; C_d \;.$$

We find $\beta^* = \dfrac{C_u - C_d}{S_u - S_d}$. Notice that $\beta^* \geq 0$. In other words, the hedging portfolio of a call is a long position in the stock.

Moreover,

$$q := \; \alpha^* + \beta^* S \;=\; \frac{1}{1+r}\,(\pi C_u + (1-\pi) C_d) \;, \qquad (1.2)$$

where

$$\pi := \; \frac{1}{S_u - S_d}\;((1+r)S - S_d) \;. \qquad (1.3)$$

Put Options Similarly, we can show that the price P of a put option satisfies

$$P := \; \frac{1}{1+r}\,(\pi P_u + (1-\pi) P_d) \;,$$

where $P_u = \max(0,\, K - S_u)$; $P_d = \max(0,\, K - S_d)$.

Of course, call options (options to buy) and put options (options to sell) can themselves be either bought or sold.

The valuation principle employed here is very general, and can be applied to other contingent claims. The cost of replicating a cash flow of H_u in the high state and of H_d in the low state is $\frac{1}{1+r}(\pi H_u + (1-\pi) H_d)$.

[2] The hedging portfolio covers the losses whatever the state of the world.

1.1.3 The Risk-Neutral Measure, Put–Call Parity

The Risk-Neutral Probability Measure Let us comment on the formulae in (1.2) and (1.3).

If $S_d < (1+r)S < S_u$, then $\pi \in]0,1[$. We can interpret (1.2) in terms of "neutrality with respect to risk". Equation (1.3) can be written

$$(1+r)S \ = \ \pi S_u + (1-\pi)S_d \ . \tag{1.4}$$

The left-hand side of (1.4) is the gain obtained by putting S euros into a riskless investment, the right-hand side is the expected gain attained by buying a stock at a price of S euros, if the probability of the high state of world occurring is π, and if the low state of the world has probability $(1-\pi)$. Equality (1.4) translates the fact we are in a model that is *"neutral with respect to risk"*: the investor would be indifferent to the choice between the two possibilities for investment (the riskless one and the risky one) as his (expected) gain remains the same. It is "as if" there were a probability π attached to the states of the world, and under which the investor were neutral with respect to risk.

Proposition 1.1.1. *The price of a contingent claim (for example an option) is the discounted value of the expected gain with respect to the "risk-neutral" probability measure.*

Proof. For a call option, the realized gain is equal to C_u or to C_d, depending on the state of the world. As the present value of 1 euro at time 1 is $\frac{1}{1+r}$ euros at time 0, so the present values of the realized gains are $\frac{1}{1+r}\,C_u$ and $\frac{1}{1+r}\,C_d$. The fair price of the option being given by (1.2), the result follows. □

There is another interpretation of this result: let S_1 be the price of the asset at time 1, and let P be the risk-neutral probability measure defined by $P(S_1 = S_u) = \pi, \ P(S_1 = S_d) = 1-\pi$. The price of a call option is the expectation, under this probability measure, of $(S_1 - K)^+/(1+r)$. Similarly, we can show that the price of a put is the expectation under P of $(K - S_1)^+/(1+r)$.

Put–Call Parity It is obvious that we have $(S_1 - K)^+ - (K - S_1)^+ = S_1 - K$. Hence, taking present values and expectations under the risk-neutral measure, and noticing also that the expectation of $S_1/(1+r)$ is equal to S (property (1.4)), we obtain

$$C = P + S - K/(1+r) \tag{1.5}$$

where C is the price of the call and P is that of the put. This formula, which we will later generalize, is known as the "put–call parity".

Remark 1.1.2. It would be tempting to model the situation by introducing the probability of the event "the price goes up". However, the proof above shows that this probability does not come into the valuation formulae.

1.1.4 No Arbitrage Opportunities

An arbitrage opportunity occurs if, with an initial capital that is strictly negative, an agent can obtain a positive level of wealth at time 1, or if, with an amount capital that is initially zero, an agent can obtain a level of wealth that is positive and not identically zero. We generally make the assumption that no such opportunities exist.

Let us first show that there are no arbitrage opportunities (NAO) if and only if $S_d < (1+r)S < S_u$.

If $S_d < (1+r)S < S_u$, there exists $\pi \in]0,1[$ such that $(1+r)S = \pi S_u + (1-\pi)S_d$. Suppose that (α, β) satisfies

$$\alpha(1+r) + \beta S_u \geq 0, \qquad \alpha(1+r) + \beta S_d \geq 0$$

with at least one strict inequality. Then, multiplying the first inequality by π and the second by $1-\pi$, and by summing the two, we obtain $\alpha + \beta S > 0$. Similarly, if we have simply

$$\alpha(1+r) + \beta S_u \geq 0, \qquad \alpha(1+r) + \beta S_d \geq 0$$

then we deduce that $\alpha + \beta S \geq 0$. In neither case do we have an arbitrage opportunity.

Conversely, if $(1+r)S \leq S_d$, then the agent can, at time 0, borrow S at a rate of r, and buy the stock at price S. At time 1, he sells the stock for S_u or S_d, and repays his loan with $(1+r)S$. So he has made a gain of at least $S_d - (1+r)S \geq 0$. It is easy to apply an analogous reasoning to the case $S_u < (1+r)S$.

We can justify the option valuation formula using the assumption of no arbitrage opportunities. Let us assume that $S_d \leq K \leq S_u$. If the price of the option is $\overline{q} > q$, where q is defined as in (1.2), then there is an arbitrage opportunity:

- at time 0, we sell the option (even if we do not actually own it) at price $\overline{q}$. With q, we can build a hedging portfolio (α^*, β^*) as described previously, and we invest the remaining money $\overline{q} - q$ at a rate of r. We have:

$$\overline{q} \;=\; \alpha^* + (\overline{q} - q) + \beta^* S \;.$$

  The initial investment is zero.
- at time 1:

- if the price of the stock is S_u: the option is exercised by the buyer. We buy the stock at price S_u and hand it over to the option buyer as agreed, at a price of K; the portfolio $(\alpha^* + (\overline{q} - q), \beta^*)$ is worth $S_u - K + (\overline{q} - q)(1+r)$, and our final wealth is $K - S_u + [S_u - K + (\overline{q} - q)(1+r)] = (\overline{q} - q)(1+r)$, and is strictly positive,
- if the price of the stock is S_d: the option buyer does not exercise his right, and we are left with the portfolio, which is worth

$$(\overline{q} - q)(1 + r) > 0 \ .$$

Hence, we have strictly positive wealth in each state of the world with an initial funding of zero, that is, an arbitrage opportunity.

We can reason analogously in the case $\overline{q} < q$.

We will come back to the concept of no arbitrage opportunities repeatedly throughout this book.

Exercise 1.1.3. Show by reasoning in terms of no arbitrage opportunities that:

- the put–call parity formula holds,
- the price of a call is a decreasing function of the strike price,
- the price of a call is a convex function of the strike price.

We can turn to Cox–Rubinstein [71] for further consequences of no arbitrage opportunities.

1.1.5 The Risk Attached to an Option

In this section, we assume that investors believe that the stock will rise with probability p. The calculations here are carried out under this probability measure.

Risk Linked to the Underlying The rate of return on the stock is by definition $R = \dfrac{S_1 - S}{S}$. Its expectation is

$$m_S \ = \ \frac{pS_u + (1-p)S_d}{S} - 1 \ ,$$

where p is the probability of being in state of the world u.

The risk of the stock is usually measured by the variance of the rate of return of its price:

$$v_S^2 \ = \ p\left(\frac{S_u - S}{S} - m_S\right)^2 + (1-p)\left(\frac{S_d - S}{S} - m_S\right)^2 \ ,$$

i.e.,

$$v_S \ = \ \frac{S_u - S_d}{S}\,(p(1-p))^{1/2} \ .$$

We say that v_S is the *volatility* of the asset.

Risk Linked to the Option Let C be a call on the stock. The delta (Δ) of the option is the number of shares of the asset that are needed to replicate the option (it is the β of the hedging portfolio given in (1.2)), i.e., $\Delta = \dfrac{C_u - C_d}{S_u - S_d}$. This represents the sensitivity of C to the price S of the underlying asset.

The *elasticity* Ω of the option is equal to $\dfrac{C_u - C_d}{C} \Big/ \dfrac{S_u - S_d}{S}$, i.e., $\Omega = \dfrac{S}{C}\Delta$ where C is the price of the option. We denote by m_C the expectation of the rate of return on the option. The risk of the option is measured by the variance of the rate of return on the option:

$$\begin{aligned} m_C &= \frac{pC_u + (1-p)C_d}{C} - 1 \\ v_C &= \{p(1-p)\}^{1/2} \, \frac{C_u - C_d}{C} \, . \end{aligned}$$

We have that $v_C = \Omega \, v_S$: the risk of the call is equal to the product of the elasticity of the option by the volatility of the underlying asset. The greater the volatility of the underlying asset, the greater is the risk attached to the call.

Proposition 1.1.4. *The volatility of an option is greater than the volatility of the underlying asset:*

$$v_C \geq v_S \, .$$

The excess rate of return of the call is greater than the excess rate return of the asset:

$$m_C - r \geq m_S - r \, .$$

Notice that this last property makes it worthwhile to purchase a call.

Proof. First, we show that $\Omega \geq 1$.

We have seen how $C = \dfrac{\pi C_u + (1-\pi) \, C_d}{1+r}$ where $\pi = \dfrac{(1+r) \, S - S_d}{S_u - S_d}$. Thus

$$(1+r) \, (S(C_u - C_d) - C(S_u - S_d)) + (S_u C_d - S_d C_u) = 0 \, .$$

Using the relation $C_u = (S_u - K)^+$ and the equivalent formula for C_d, we check that $S_u C_d - S_d C_u \leq 0$, and hence that $\Omega \geq 1$.

We would like to establish a relationship between m_C and m_S. To do this, we use the hedging portfolio (α, β), which satisfies

$$\begin{cases} S_u \beta + (1+r)\alpha = C_u \\ S_d \beta + (1+r)\alpha = C_d \, , \end{cases}$$

as well as the equality $C = \alpha + S\beta$. We then obtain (using $\beta = \Delta$)

$$S_u\Delta - C_u = (1+r)(S\Delta - C)$$
$$S_d\Delta - C_d = (1+r)(S\Delta - C) ,$$

and hence

$$p(S_u\Delta - C_u) + (1-p)(S_d\Delta - C_d) = (1+r)(S\Delta - C) .$$

Rearranging terms,

$$m_S S\Delta - m_C C = r(S\Delta - C)$$

where

$$m_C - r = \Omega(m_S - r) .$$

The excess rate of return on the call is equal to Ω, the elasticity of the option, multiplied by the excess rate of return on the asset (with $\Omega \geq 1$). □

In Chap. 3, we will study these concepts in continuous time.

1.1.6 Incomplete Markets

A Finite Number of States of the World When the asset takes the value s_j at time 1 in state of the world j with $j = 1, \dots, k$, for $k > 2$, it is no longer possible to replicate the option, as we obtain k equations ($k > 2$) with 2 unknowns. We consider contingent claims that are of the form $H = (h_1, h_2, \dots, h_k)$, where h_j corresponds to the payoff in state of the world j. This contingent claim is replicable if there exists a pair (α, θ) such that $\alpha(1+r) + \theta S_1 = H$, that is such that

$$\alpha(1+r) + \theta s_j = h_j ; \forall j .$$

In this case, the price of the contingent claim H is the initial value $h = \alpha + \theta S$ of the replicating portfolio.

The set $\mathcal{P}$ of risk-neutral probability measures is by definition the set of probability measures Q that assign strictly positive probability to each state of the world, and satisfy

$$E_Q(S_1) = S(1+r) .$$

The set of risk-neutral probabilities $(q_1, q_2, \dots, q_k)$ is determined by

$$q_j > 0 \quad \text{for } j = 1, 2, \dots, k$$
$$\sum_{j=1}^{k} q_j = 1$$
$$\sum_{j=1}^{k} q_j s_j = (1+r)S .$$

The price range associated with the contingent claim H is defined by

$$\left] \inf_{Q \in \mathcal{Q}} E_Q(\tilde{H}), \sup_{Q \in \mathcal{Q}} E_Q(\tilde{H}) \right[\ ,$$

where $\tilde{H}$ is the discounted value of H, i.e., $\tilde{H} = H/(1+r)$ in our model. We will come back to the price range later. In the meantime, we note that if the market is incomplete, and if H is replicable, then the value of $E_Q(H/(1+r))$ does not depend on the choice of risk-neutral measure Q. Indeed, if there exists (α, θ) such that

$$\alpha(1+r) + \theta s_j = h_j, \ \forall j$$

then for any choice of risk-neutral probability measure $(q_j, 1 \le j \le k)$, we have

$$E_Q(H) = \sum_{j=1}^{k} q_j h_j = \sum_{j=1}^{k} q_j(\alpha(1+r) + \theta s_j) = \alpha(1+r) + \theta(1+r)S \ .$$

A Continuum of States of the World Let $(\Omega, \mathcal{A}, Q)$ be a given probability space. Let S be the price of the asset at time 0. Suppose that there exist two numbers S_d and S_u such that the price at time 1 is a random variable S_1 taking values in $[S_d, S_u]$, and with a density f that is strictly positive on $[S_d, S_u]$. Suppose moreover that $S_d < (1+r)\,S < S_u$. Let $\mathcal{P}$ be the set of risk-neutral probability measures, that is, the set of probability measures P such that $E_P\left(\dfrac{S_1}{1+r}\right) = S$ (condition (1.4)). We need these probability measures to be equivalent to Q. In other words, we need S_1 to admit under P (or Q) a density function that is strictly positive on $[S_d, S_u]$.

Proposition 1.1.5. *For any convex function g (for example $g(x) = (x - K)^+$), we have*

$$\sup_{P \in \mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right) = \frac{g(S_u)}{1+r}\,\frac{S(1+r) - S_d}{S_u - S_d} + \frac{g(S_d)}{1+r}\,\frac{S_u - S(1+r)}{S_u - S_d} \ .$$

If g is of class C^1, we have

$$\inf_{P \in \mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right) = \frac{g\big((1+r)S\big)}{1+r} \ .$$

Proof. Let g be a convex function. Let μ and ν be the slope and y-intersect of the line that goes through the points with coordinates $(S_d, g(S_d))$ and $(S_u, g(S_u))$. We then have:

$$\begin{cases} \forall x \in [S_d, S_u],\ g(x) & \le \ \mu x + \nu \\ \qquad\qquad\quad g(S_d) & = \ \mu S_d + \nu \\ \qquad\qquad\quad g(S_u) & = \ \mu S_u + \nu \ , \end{cases}$$

and hence, for all $P \in \mathcal{P}$,

$$E_P(g(S_1)) \leq \mu E_P(S_1) + \nu = \mu S(1+r) + \nu \ .$$

As $\mu = \frac{g(S_u)-g(S_d)}{S_u-S_d}$ and $\nu = g(S_d) - S_d \frac{g(S_u)-g(S_d)}{S_u-S_d}$, we obtain an upper bound.

Let P^* be the probability measure such that

$$\begin{cases} P^*(S_1 = S_u) &= p \\ P^*(S_1 = S_d) &= 1-p \\ E_{P^*}(S_1) &= S(1+r) \ . \end{cases}$$

The last condition above determines p (equal to the π appearing in formula (1.3)):

$$p = \frac{S(1+r) - S_d}{S_u - S_d}, \qquad 1 - p = \frac{S_u - S(1+r)}{S_u - S_d} \ .$$

We have $E_{P^*}(g(S_1)) = \mu S(1+r) + \nu$. The supremum is attained under P^*. We notice that this probability measure does not belong to $\mathcal{P}$, as it does not correspond to the case where S_1 has a strictly positive density function. However we can approach P^* with a sequence of probability measures P_n belonging to $\mathcal{P}$, in the sense that $E_{P^*}(g(S_1)) = \lim E_{P_n}(g(S_1))$.

Similarly, we can obtain a lower bound

$$\inf_{P \in \mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right) = \frac{g(S(1+r))}{1+r} \ .$$

Indeed, if γ and δ are the slope and the y-intersect of the tangent to the curve $y = g(x)$ at the point with coordinates $(S(1+r), g(S(1+r)))$, then

$$g(x) \geq \gamma x + \delta, \quad \gamma S(1+r) + \delta = g(S(1+r)) \ .$$

Hence $E_P(g(S_1)) \geq E_P(\gamma S_1 + \delta) = g(S(1+r))$, and the minimum is attained by the Dirac measure at $S(1+r)$. □

This result can be interpreted in terms of volatility. If S_1 takes values in $[S_d, S_u]$ and has expectation $S(1+r)$, then its variance is bounded below by 0 (this value is attained when $S_1 = S(1+r)$), and achieves a maximum when S_1 takes only the extreme values S_d and S_u.

As we remarked earlier, if there does not exist a portfolio that replicates the option, we cannot assign the option a unique price. We define the selling price of the option as the minimal expenditure enabling the seller to hedge himself: it is the smallest amount of money to be invested in a portfolio (α, β) with final value greater than the value of the option $g(S_1)$. Hence the selling price is

$$\inf_{(\alpha,\beta)\in\mathcal{A}} (\alpha + \beta S)$$

with $\mathcal{A} = \{(\alpha,\beta)|\ \alpha(1+r) + \beta x \geq g(x),\ \forall x \in [S_d, S_u]\}$. We have

$$\inf_{(\alpha,\beta)\in\mathcal{A}} (\alpha + \beta S) = \sup_{P\in\mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right).$$

Indeed, by definition of $\mathcal{A}$, we have $\alpha(1+r) + \beta S_1 \geq g(S_1)$, and hence

$$\inf_{(\alpha,\beta)\in\mathcal{A}} (\alpha + \beta S) \geq \sup_{P\in\mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right).$$

Moreover, using the pair $(\mu,\ \nu)$ from the previous section, we can check that $\left(\frac{\nu}{1+r}, \mu\right)$ is in $\mathcal{A}$:

$$\inf_{(\alpha,\beta)\in\mathcal{A}} (\alpha + \beta S) \leq \ \mu S + \frac{\nu}{1+r} \ = \sup_{P\in\mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right).$$

The two problems,

$$\sup_{P\in\mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right) \quad \text{and} \quad \inf_{(\alpha,\beta)\in\mathcal{A}} (\alpha + \beta S)$$

are called “dual problems”.

We define the buying price of an option as the maximum amount that can be borrowed against the option. The buying price of a call is then defined by:

$$\sup_{(\alpha,\beta)\in\mathcal{C}} (\alpha + \beta S)$$

with $\mathcal{C} = \{(\alpha,\beta)|\ \alpha(1+r) + \beta x \leq g(x),\ \forall x \in [S_d, S_u]\}$.

Similarly, we get:

$$\sup_{(\alpha,\beta)\in\mathcal{C}} (\alpha + \beta S) \ = \inf_{P\in\mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right).$$

1.2 A One-Period Model with $(d+1)$ Assets and k States of the World

We now construct a model that is slightly more complex than the previous one. We consider the case of a one-period market with $(d+1)$ assets and k states of the world. Here again, we do not claim to describe the real world (and nor will we at any point of the book). Instead, we aim to draw out concepts with which we can develop acceptable forms of model.

If S^i is the price at time 0 of the i-th asset ($i = 0, \dots, d$), then let its value at time 1 in state j be denoted by v_j^i.

A portfolio $(\theta^0, \theta^1, \dots, \theta^d)$ is made up of θ^i stocks of type i, and therefore its value at time 0 is $\sum_{i=0}^d \theta^i S^i$, and its value at time 1 is $\sum_{i=0}^d \theta^i v_j^i$ if we are in the j-th state of the world.

Notation 1.2.1. The column vector S has components S^i, and the column vector θ has components θ^i.

Let V be the matrix of prices at time 1: that is the $(k \times (d+1))$-matrix whose i-th column is made up of the prices of the i-th asset at time 1, that is $(v_j^i, 1 \le j \le k)$.

We use matrix notation: $\theta \cdot S = \sum_{i=0}^d \theta^i S^i$ is the scalar product of θ and S, and $V\theta$ denotes the $\mathbb{R}^k$-vector with components $(V\theta)_j = \sum_{i=0}^d \theta^i v_j^i$.

We write V^T for the matrix transpose of V, and S^T for the vector transpose of S.

A riskless asset is an asset worth $(1+r)$ at time 1 whatever the state of the world, and worth 1 at time 0. The rate of interest r is used as both a lending rate and as a borrowing rate for the sake of simplicity. Thus, $\dfrac{1}{1+r}$ is the price that must be paid at time 0 in order to hold one euro at time 1 in all states of the world.

Notation 1.2.2. $\mathbb{R}_+^k$ denotes the set of vectors of $\mathbb{R}^k$ that have non-negative components. $\mathbb{R}_{++}^k$ denotes the set of vectors of $\mathbb{R}^k$ that have strictly positive components. Δ^{k-1} refers to the unit simplex in $\mathbb{R}^k$:

$$\Delta^{k-1} = \left\{ \lambda \in \mathbb{R}_+^k \mid \sum_{i=1}^k \lambda_i = 1 \right\}.$$

Let z and z' be two vectors in $\mathbb{R}^k$. We write $z \ge z'$ to express $z_i \ge z'_i$ for all i.

Exercise 1.2.3. Show that if V is a $k \times (d+1)$-matrix, then there is an equivalence between the statements:

(i) The rank of the mapping associated with V is k.
(ii) The linear mapping associated with V is surjective, and the one associated with V^T is injective.

1.2.1 No Arbitrage Opportunities

We now introduce the concept of an arbitrage opportunity, which was touched upon earlier.

The Assumption of No Arbitrage Opportunities

Definition 1.2.4. *An arbitrage is a portfolio* $\theta = (\theta^0, \theta^1, \dots, \theta^d)$ *with a non-positive initial value* $S \cdot \theta = \sum_{i=0}^{d} \theta^i S^i$ *and a non-negative value* $V\theta$ *at time 1, with at least one strict inequality. In other words, either* $S \cdot \theta < 0$ *and* $V\theta \geq 0$*, or* $S \cdot \theta = 0$ *and* $V\theta \geq 0$ *with a strict inequality in at least one state of the world.*

We say that there are ***no arbitrage opportunities*** *when there is no arbitrage. That is to say, the following conditions must hold:*

(i) $V\theta = 0$ *implies* $S \cdot \theta = 0$,

(ii) $V\theta \geq 0$, $V\theta \neq 0$ *implies* $S \cdot \theta > 0$.

Indeed, in the first case, if we had $V\theta = 0$ and $S \cdot \theta < 0$ (or $S \cdot \theta > 0$), then the portfolio θ (or $-\theta$) would be an arbitrage. In the second case, if $V\theta \geq 0$, $V\theta \neq 0$ and $S \cdot \theta \leq 0$, then θ would be an arbitrage.

An arbitrage opportunity is a means of obtaining wealth without any initial capital. Obviously an arbitrage opportunity could not exist without being very quickly exploited. We therefore make the following assumption, referred to as the assumption of *no arbitrage opportunity* (NAO).

The NAO Assumption: there exists no arbitrage opportunity.

Using the same notation as before, we recall a result from linear programming:

Lemma 1.2.5 (Farkas' Lemma). *The implication* $V\theta \geq 0 \Rightarrow S \cdot \theta \geq 0$ *holds if and only if there exists a sequence* $(\beta_j)_{j=1}^{k}$ *of non-negative numbers such that* $S^i = \sum_{j=1}^{k} v_j^i \, \beta_j, \quad i \in \{0, \dots, d\}$.

We remark that the assumption of NAO is a little bit stronger than the assumptions of Farkas' Lemma, as according to the former, if the portfolio's payments are non-negative, and strictly positive in at least one state of the world, then the price of the portfolio is strictly positive. From this we will deduce (with a proof that is in fact simpler than that of Farkas' Lemma) that the β_j are strictly positive.

We recall the Minkowski separation theorem.

Theorem 1.2.6 (The Minkowski Separation Theorem). *Let* C_1 *and* C_2 *be two non-empty disjoint convex sets in* $\mathbb{R}^k$*, where* C_1 *is closed and* C_2

is compact. Then there exists a family $(a_1, \ldots, a_k)$ of non-zero coefficients, and two distinct numbers b_1 and b_2 such that

$$\forall\, x \in C_1,\ \forall\, y \in C_2, \qquad \sum_{j=1}^{k} a_j x_j \ \le\ b_1 \ <\ b_2 \ \le\ \sum_{j=1}^{k} a_j y_j \ .$$

Theorem 1.2.7. *The NAO assumption is equivalent to the existence of a sequence $(\beta_j)_{j=1}^{k}$ of strictly positive numbers, called state prices, such that*

$$S^i \ =\ \sum_{j=1}^{k} v_j^i \,\beta_j; \quad i \in \{0, \ldots, d\}\ . \tag{1.6}$$

Proof. (of Theorem 1.2.7)
Let S^T be the row vector $(S^0, S^1, \ldots, S^d)$ and let U be the vector subspace of $\mathbb{R}^{k+1}$

$$U := \left\{ z \in \mathbb{R}^{k+1} \mid z = \begin{pmatrix} -S^T \\ V \end{pmatrix} x;\ x \in \mathbb{R}^{d+1} \right\} .$$

The assumption of NAO implies that $U \cap \mathbb{R}_+^{k+1} = \{0\}$, so that in particular, $U \cap \Delta^k = \emptyset$. According to Minkowski's theorem, there exists a set of non-zero coefficients $\{\beta_j\,;\ j = 0, \ldots, k\}$ and two numbers b_1 and b_2, such that

$$\sum_{j=0}^{k} \beta_j z_j \ \le\ b_1 \ <\ b_2 \ \le\ \sum_{j=0}^{k} \beta_j w_j; \qquad z \in U,\ w \in \Delta^k\ .$$

As $0 \in U$, $b_1 \ge 0$, and hence, by choosing a vector w whose components are all zero except for the j-th, which is equal to 1, we deduce that $\beta_j > 0$, $\forall\, j \in \{0, \ldots, k\}$. Without loss of generality, we can take $\beta_0 = 1$.

Then let β be the vector $(\beta_1, \ldots, \beta_k)^T$. Taking into account the form of the elements of U, we write the inequality $z_0 + \sum_{j=1}^{k} \beta_j z_j \ \le\ 0$ as $(-S + V^T\beta) \cdot x \le 0$. Hence $S = V^T\beta$, i.e., $S^i = \sum_{j=1}^{k} \beta_j v_j^i$ with $\beta_j > 0$, $j \in \{1, \ldots, k\}$.

The proof of the converse is trivial. □

The vector β is called a state price vector: β_j corresponds to the price at time 0 of a product that is worth 1 at time 1 in state j, and 0 in all the other states. We will come back to this interpretation later.

Probabilistic Interpretation of the State Prices Until now in this section, we have not used probabilities. We will now give a probabilistic interpretation of the NAO assumption and of Theorem 1.2.7. Introducing probabilities will enable us to study more general models, and to exploit the concept of NAO.

If asset 0 is riskless, then we have

$$v_j^0 \;=\; 1+r, \quad j \in \{1,\ldots,k\}\ ,$$

and hence, using (1.6), for $i=0$:

$$\frac{1}{1+r} \;=\; \sum_{j=1}^{k} \beta_j\ .$$

Let us set $\pi_j = (1+r)\beta_j$. The π_j are positive numbers such that $\sum_{j=1}^{k} \pi_j = 1$. Therefore, they can be interpreted as probabilities on the different states of the world. We have

$$S^i \;=\; \frac{1}{1+r} \sum_{j=1}^{k} \pi_j v_j^i \quad i \in \{1,\ldots,d\}\ .$$

We have thus constructed a probability measure under which the price S^i of the i-th asset is the expectation of its price at time 1, discounted using the riskless rate.

If we construct a portfolio θ, we get:

$$(1+r)\sum_{i=0}^{d} \theta^i S^i \;=\; \sum_{j=1}^{k} \pi_j \sum_{i=0}^{d} \theta^i v_j^i\ ,$$

where π is (as in Sect. 1.1) a probability measure that is neutral with respect to risk: a riskless investment with initial value $\sum_{i=0}^{d} \theta^i S^i$ yields $(1+r)\sum_{i=0}^{d} \theta^i S^i$, which is equal to the expectation (under π) of the value of the portfolio at time 1.

The rate of return on asset i in state j is by definition equal to $(v_j^i - S^i)/S^i$. The expectation of the rate of return on i is, under probability measure π, equal to the rate of return on the riskless asset:

$$\sum_{j=1}^{k} \pi_j\, \frac{v_j^i - S^i}{S^i} \;=\; r\ .$$

Proposition 1.2.8. *Under the assumption of NAO, if asset 0 is riskless, then there exists a probability measure π on the states of the world, under which the price at time* 0 *of asset i is equal to the expectation of its price at time* 1*, discounted by the riskless rate:*

$$S^i \;=\; \frac{1}{1+r} \sum_{j=1}^{k} \pi_j v_j^i\ . \tag{1.7}$$

Exercise 1.2.9. Let $V\theta$ be the vector with components $(V\theta)_j = \sum_{i=0}^{d} v_j^i \theta^i$, and let $S \cdot \theta$ denote the scalar product $S \cdot \theta = \sum_{i=0}^{d} S^i \theta^i$.

a. Let $z \in \operatorname{Im} V$. Let θ be any vector satisfying $z = V\theta$. Show that, under the assumption of NAO, the mapping $\pi : z \to S \cdot \theta$ does not depend on the choice of θ, and defines a positive linear functional on $\operatorname{Im} V$.

b. Show that π can be extended to a positive linear functional $\overline{\pi}$ on $\mathbb{R}^k$. To do this, show that for all $\hat{z} \notin \operatorname{Im} V$, there exists $\phi(\hat{z}) \in \mathbb{R}$ such that

$$\max\{\pi(z'),\ z' \le \hat{z},\ z' \in \operatorname{Im} V\} < \phi(\hat{z}) < \min\{\pi(z'),\ z' \ge \hat{z},\ z' \in \operatorname{Im} V\} \ .$$

Next show that the mapping $z + \lambda\hat{z} \to \pi(z) + \lambda\phi(\hat{z})$ is linear and positive, and extends π to the space generated by $\operatorname{Im} V$ and $\hat{z}$.

c. Show, using the Riesz representation theorem, that $\overline{\pi}(z) = \beta \cdot z$ with $\beta \in \mathbb{R}^k_{++}$.

d. Thence deduce Theorem 1.2.7

Exercise 1.2.10. Suppose that there are constraints on portfolios, modeled by a closed convex cone $\mathcal{C}$: $\theta \in \mathcal{C}$. For example:

$$\begin{cases} \theta^i \text{ unconstrained for } 0 \le i \le r \\ \theta^i \ge 0 \qquad\qquad \text{for } r+1 \le i \le r+p \\ \theta^i \le 0 \qquad\qquad \text{for } r+p+1 \le i \le d. \end{cases}$$

Adapt the definition of NAO to the restriction to $\mathcal{C}$.

1. Suppose that there are $k \ge 4$ states of the world, and 4 assets. Asset 0 is riskless, and the rate of interest is r. The other assets are risky, and their returns are given by a matrix V. Suppose that the constraints are $\theta^2 \ge 0$ and $\theta^3 \le 0$. Let $\tilde{V} = \begin{bmatrix} \ldots & \ldots & V & \ldots \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

   Show that NAO with restrictions on portfolios can be expressed as
   (i) $\tilde{V}\theta = 0 \ \Rightarrow\ S \cdot \theta = 0$.
   (ii) $\tilde{V}\theta \in \mathbb{R}^k_+,\ \tilde{V}\theta \ne 0 \ \Rightarrow\ S \cdot \theta > 0$.
   Hence deduce that there exists a probability measure π such that

$$S^1 = \frac{1}{1+r}\sum_{j=1}^{k} v_j^1 \pi_j,\ S^2 \ge \frac{1}{1+r}\sum_{j=1}^{k} v_j^2 \pi_j \text{ and } S^3 \le \frac{1}{1+r}\sum_{j=1}^{k} v_j^3 \pi_j \ .$$

2. For $\overline{\theta} \in \mathcal{C}$, write $N_{\mathcal{C}}(\overline{\theta}) = \{p \in \mathbb{R}^{d+1} | p^T(\theta - \overline{\theta}) \leq 0, \forall \theta \in \mathcal{C}\}$. Show, by generalizing the proof of Theorem 1.2.7, that NAO restricted to $\mathcal{C}$, is equivalent to the existence of $\beta \in \mathbb{R}^k_{++}$ such that $-S + V^T\beta \in N_{\mathcal{C}}(0)$.
3. Recover the results of 1.

Exercise 1.2.11.

1. Suppose that there are 2 states of the world, and 2 assets, one riskless (the rate of interest is taken to be r) and the other a stock worth either S_u or S_d at time 1. Suppose that the risky asset has purchase price S_0 and selling price $S_0' \leq S_0$. We use the notation $\theta = \theta^+ - \theta^-$ for the amount of stock held, and θ^0 for the amount of riskless asset held. The cost of this portfolio is then $\theta^0 + \theta^+ S_0 - \theta^- S_0'$, and it pays
$$\begin{cases} (1+r)\theta^0 + (\theta^+ - \theta^-)S_u \text{ in the high state, after an up-move} \\ (1+r)\theta^0 + (\theta^+ - \theta^-)S_d \text{ in the low state, after a down-move} \end{cases}$$
Show, using Farkas' Lemma, that there is NAO if and only if there exists at least one probability measure π such that
$$S_0' \leq \frac{S_u \pi}{1+r} + \frac{S_d(1-\pi)}{1+r} \leq S_0 \ .$$
The reader can introduce the matrix:
$$\begin{bmatrix} (1+r) & S_u & -S_u \\ (1+r) & S_d & -S_d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
2. Suppose that there are d assets, with an injective gains matrix V. Suppose that the cost $\phi(\theta)$, $\phi : \mathbb{R}^d \to \mathbb{R}$ of a portfolio $\theta \in \mathbb{R}^d$ is a sublinear function, that is, one satisfying
$$\begin{cases} \phi(\theta_1 + \theta_2) \leq \phi(\theta_1) + \phi(\theta_1) & \forall (\theta_1, \theta_2) \in \mathbb{R}^{2d} \\ \phi(t\theta) = t\phi(\theta) & \forall\, t \geq 0 \ . \end{cases}$$
Notice that in particular, we have $\phi(0) = 0$ and $-\phi(-\theta) \leq \phi(\theta)$. Let $U := \{(z_1, z_2) \in \mathbb{R}^k \times \mathbb{R} \mid \exists \theta \text{ such that } z_1 \leq -\phi(\theta) \text{ and } z_2 = V\theta\}$. Show that U is a convex cone.
We say that there is NAO if $V\theta = 0 \Rightarrow \phi(\theta) = 0$, and $V\theta \in \mathbb{R}^k_+$, $V\theta \neq 0 \Rightarrow \phi(\theta) > 0$. Show that under the assumption of NAO, $U \cap \mathbb{R}^{k+1}_+ = \{0\}$. Show, by adapting the proof of Theorem 1.2.7, that NAO is equivalent to the existence of a strictly positive β such that
$$-\phi(-\theta) \leq \beta^T V\theta \leq \phi(\theta).$$
Hence recover the results of the first question.

1.2.2 Complete Markets

Definition and Characterization

Definition 1.2.12. *A market is complete if, for any vector w of $\mathbb{R}^k$, we can find a portfolio θ such that $V\theta = w$; that is to say, there exists θ such that*

$$\sum_{i=0}^{d} \theta^i \, v_j^i \;=\; w_j, \quad j \in \{1, \ldots, k\} \, .$$

A market is complete if we can choose a portfolio at time 0 in such a way as to attain any given vector of wealth at time 1.

Proposition 1.2.13. *A market is complete if and only if the matrix V is of rank k.*

Proof. Matrix V has rank k if and only if the mapping associated with V is surjective; the equation $V\theta = w$ then has at least one solution. □

Economic Interpretation of State Prices In a complete market, for any $j \in \{1, \ldots, k\}$, there exists a portfolio θ_j such that the payoff of θ_j satisfies $V\theta_j = (\delta_{1,j}, \ldots, \delta_{k,j})^T$, with $\delta_{i,j} = 0$ when $i \neq j$, and $\delta_{j,j} = 1$ (the asset is then called an Arrow–Debreu asset). In an arbitrage-free market, the initial value of θ_j is $S \cdot \theta_j \;=\; \beta^T V \theta_j \;=\; \beta_j$. Therefore we can interpret β_j as the price to be paid at time 0 in order to have one euro at time 1 in state j and nothing in the other states of the world. Hence the terminology "state price".

Moreover, we note that if there exists β such that $V^T\beta = S$, then, as the mapping associated with matrix V^T is injective, the vector β is unique.

The Risk-Neutral Probability Measure In a complete market, there necessarily exists a riskless portfolio, that is a portfolio θ such that $(V\theta)_j = a$ for all $j \in \{1, \ldots, k\}$. The initial value of this portfolio is taken to be V_0. The rate of return on the portfolio is $(a - V_0)/V_0$, and will be denoted by r. Without loss of generality, we can assume asset 0 to be riskless, and we can normalize its price so that it is 1 at time 0, its value at time 1 being $1 + r$. If there exists a probability measure π satisfying $V^T\pi = (1 + r)S$, then it is unique. We then call it the "risk-neutral measure".

1.2.3 Valuation by Arbitrage in the Case of a Complete Market

Let z be a vector of $\mathbb{R}^k$. Under the assumption of NAO, if there exists a portfolio $\theta = (\theta^0, \theta^1, \ldots, \theta^d)$ taking the value z at time 1, i.e., such that

$$\sum_{i=0}^{d} \theta^i \, v_j^i \;=\; z_j \; ,$$

then we say that z is replicable. The value of the portfolio at time 0 is $z_0 = \sum_{i=0}^{d} \theta^i S^i$, and this value does not depend on the hedging portfolio chosen. Indeed, suppose that there exist two portfolios θ and $\widetilde{\theta}$ such that $V\theta = V\widetilde{\theta}$ and $S \cdot \theta > S \cdot \widetilde{\theta}$. The portfolio $\widetilde{\theta} - \theta$ is an arbitrage opportunity. In the complete market framework, there always exists a hedging portfolio.

Proposition 1.2.14. *In a complete and arbitrage-free market, the initial value of the payoff $z \in \mathbb{R}^k$, delivered at time 1, is given by*

$$\frac{1}{1+r} \sum_{j=1}^{k} \pi_j \, z_j \; = \; \sum_{j=1}^{k} \beta_j \, z_j \; .$$

Remark 1.2.15. The initial value of z is a linear function of z.

Proof. (of Proposition 1.2.14)

The value of any hedging portfolio is $z_0 = \sum_{i=0}^{d} \theta^i \, S^i$. It is enough to use (1.6) or (1.7) and write

$$z_0 \; = \; \sum_{i=0}^{d} \theta^i \, \sum_{j=1}^{k} \beta_j \, v_j^i \; = \; \beta \cdot z = \frac{1}{1+r} \sum_{j=1}^{k} \pi_j z_j \; .$$

□

Remark 1.2.16. The expression above has a two-fold advantage. It does not depend on the portfolio, and it can be interpreted, as follows: the price at time 0 of the replicating portfolio $(z_j;\; j = 1, \dots, k)$ is the discounted expectation under π of its value at time 1.

In the case of an option on the i-th asset, we have $z_j = \sup\,(v_j^i - K,\, 0)$, and hence we get the arbitrage price

$$\frac{1}{1+r} \sum_{j=1}^{k} \pi_j \; \sup\,(v_j^i - K,\, 0) \; .$$

1.2.4 Incomplete Markets: the Arbitrage Interval

Generally speaking, it is not possible to valuate a product by arbitrage in an incomplete market. If z is not replicable, then we can define an *arbitrage interval*. We associate with any portfolio θ, its corresponding initial value $\theta \cdot S$.

We define the selling price of z as the smallest amount of wealth that can be invested in a portfolio θ in such a way that the final value of this portfolio is greater than z. In the following, we suppose that there is a riskless asset. The super-replication price is then

$$\overline{S}(z) := \inf\{\theta \cdot S \mid (V\theta)_j \geq z_j;\, \forall j\}\ .$$

We define the purchase price of z as the maximal amount of money that can be borrowed against z, i.e.,

$$\underline{S}(z) := \sup\{\theta \cdot S \mid (V\theta)_j \leq z_j;\, \forall j\}\ .$$

First we note that $\overline{S}(z)$ is well-defined. Indeed, let us take $\tilde{\theta}$ to be an element of the non-empty set $\{\theta \mid V\theta \geq z\}$. The set $\{\theta \mid S\cdot\theta \leq S\cdot\tilde{\theta} \text{ and } V\theta \geq z\}$ is a compact set (from the NAO condition), on which the function $S\cdot\theta$ attains its minimum.

Moreover, we can easily show that if $\bar{S}(z) \neq \underline{S}(z)$ and if the price $S(z)$ of the contingent asset z satisfies $S(z) \geq \bar{S}(z)$ or $S(z) \leq \underline{S}(z)$, then an arbitrage occurs if we use strategies that include this new asset. If $\bar{S}(z) \neq \underline{S}(z)$ and if the price $S(z)$ of the contingent asset z satisfies $\underline{S}(z) < S(z) < \bar{S}(z)$, then there is NAO when we use strategies that include this new asset. Let us show that, indeed, if a portfolio (θ_z, θ) satisfies $\theta_z z + V\theta \geq 0$ and $\theta_z z + V\theta \neq 0$, then $\theta_z S(z) + S\cdot\theta > 0$.

- If $\theta_z = 0$, it follows from NAO.
- If $\theta_z < 0$, we have $V\frac{\theta}{-\theta_z} \geq z$, so that $S\cdot\frac{\theta}{-\theta_z} \geq \bar{S}(z) > S(z)$, and hence $\theta_z S(z) + S\cdot\theta > 0$.
- If $\theta_z > 0$, we have $z \geq V\frac{\theta}{-\theta_z}$, so that $S\cdot\frac{\theta}{-\theta_z} \leq \underline{S}(z) < S(z)$, and hence $\theta_z S(z) + S\cdot\theta > 0$.

In addition, we check that $\theta_z z + V\theta = 0$ implies $\theta_z S(z) + S\cdot\theta = 0$. Indeed, as $V\frac{\theta}{-\theta_z} = z$, so $\underline{S}(z) = \bar{S}(z) = S(z) = S\cdot\frac{\theta}{-\theta_z}$, and hence $\theta_z S(z) + S\cdot\theta = 0$.

Therefore, there is NAO when we use strategies that include the new asset.

Finally, $\overline{S}(z)$ is sublinear: it satisfies

$$\overline{S}(z+z') \leq \overline{S}(z) + \overline{S}(z') \quad \text{and} \quad \overline{S}(az) = a\overline{S}(z) \quad \forall a \in \mathbb{R}_+\ .$$

Moreover, $-\overline{S}(-z) = \underline{S}(z)$.

Let us now show that

$$\overline{S}(z) = \max\{\beta^T z \mid \beta \geq 0,\ V^T\beta = S\}\ .$$

Indeed, for any θ such that $V\theta \geq z$, and any $\beta \geq 0$ such that $V^T\beta = S$, we have $S \cdot \theta = \beta^T V\theta \geq \beta^T z$. Hence

$$\min\{S \cdot \theta \mid V\theta \geq z\} \;\geq\; \max\{\beta^T z \mid \beta \geq 0,\; V^T\beta = S\} \,.$$

In addition, if $\overline{\theta}$ is a solution to $\min_{\{V\theta \geq z\}} S\cdot\theta$, then there exists a Lagrange multiplier[3] $\overline{\beta} \geq 0$ such that $S = V^T\overline{\beta}$ and $\overline{\beta}^T(V\overline{\theta} - z) = 0$. Hence

$$\overline{S}(z) = S \cdot \overline{\theta} = \overline{\beta}^T z \;\leq\; \max\{\beta^T z \mid \beta \geq 0,\; V^T\beta = S\} \,.$$

The required equality follows.

If there is a riskless asset, we can normalize $\overline{\beta}$, and hence

$$\overline{S}(z) = \max\left\{\frac{E_\pi(z)}{1+r}\,\middle|\quad V^T\pi = (1+r)S\right\} \,.$$

The expression above represents the maximum of the expectation across all the probability measures under which discounted prices are martingales. In this way, we have generalized the results of Sect. 1.1.6.

Exercise 1.2.17. Arbitrage bounds in the presence of portfolio constraints.

We use the notation of Exercise 1.2.10, and restrict ourselves to portfolios belonging to $\mathcal{C}$. Let

$$\overline{S}(z) := \inf\{\theta \cdot S \,|\, \theta \in \mathcal{C}, V\theta \geq z\} \,.$$

(If there exists no $\theta \in \mathcal{C}$ such that $V\theta \geq z$, we set $\overline{S}(z) := \infty$).

1. Show that $\overline{S}(z)$ is well-defined and sublinear.
2. Show that $\overline{S}(z) = \max\{\beta^T z \mid \beta \geq 0,\, -S + V^T\beta \in N_{\mathcal{C}}(0)\}$. (Recall that if $\overline{\theta}$ minimizes $\theta \cdot S$ under the constraints $\theta \in \mathcal{C}$ and $V\theta \leq z$, then there exists $\beta \geq 0$ and $v \in N_{\mathcal{C}}(\overline{\theta})$ such that $S = \beta^T V - v$ and $\beta^T(V\overline{\theta} - z) = 0$).

Exercise 1.2.18. Arbitrage bounds in the case of transaction costs.

We use the assumptions and notation of Exercise 1.2.11. Suppose that there is a riskless asset. In addition, for any z there exists θ such that $V\theta \geq z$, and we define

$$\overline{S}(z) := \inf\{\phi(\theta) \,|\, V\theta \geq z\} \,.$$

1. Show that $\overline{S}(z)$ is well-defined, and sublinear.
2. Show that $\overline{S}(z) = \max\{\beta^T z \mid \beta \geq 0,\, \phi(\theta) \geq \beta^T V\theta,\, \forall\, \theta\}$.
3. Calculate the purchase price for a call with strike K, where the rest of the data is as in as in question 1 of Exercise 1.2.11. (First consider the case $S_0 \geq \frac{S_u}{1+r}$, and next the case $S_0 < \frac{S_u}{1+r}$).

[3] See annex.

1.3 Optimal Consumption and Portfolio Choice in a One-Agent Model

The two models introduced previously were purely financial. We now consider a very simple economy, which has a single good for consumption, taken as the numéraire, and a single economic agent. This agent has known resources $R_0 > 0$ a time 0, and his resources at time 1 given by $R_j > 0$ in state of the world j.

In order to modify his future revenue, the agent can buy a portfolio of assets at time 0, on condition that he does not run into debt. We assume that the $(d + 1)$ assets have the same characteristics as in the previous section.

The agent consumes: c_0 is the amount of his consumption at time 0; c_j that of his consumption at time 1 in state of the world j.

The agent constructs a portfolio θ. The set of consumption–portfolio pairs that are compatible with the agent's revenue, is defined by the following inequalities:

$$\begin{aligned} &\text{(i)} \quad R_0 \;\geq\; c_0 + \sum_{i=0}^{d} \theta^i \, S^i \\ &\text{(ii)} \quad R_j \;\geq\; c_j - \sum_{i=0}^{d} \theta^i \, v_j^i, \quad j \in \{1, \ldots, k\} \,. \end{aligned} \tag{1.8}$$

The first constraint states that money invested in the portfolio comes from the portion of revenue that has not been consumed, and the second, that consumption at time 1 is covered by his resources and by the portfolio.

The set of consumption strategies that are compatible with the agent's revenue is then:

$$B(S) := \left\{ c \in \mathbb{R}_+^{k+1};\ \exists\, \theta \in \mathbb{R}^{d+1},\ \text{satisfying (1.8)} \right\} .$$

The agent has "preferences" on $\mathbb{R}_+^{k+1}$, that is to say, a preorder (a reflexive and transitive binary relation), written $\succeq$, which is complete (any two elements of $\mathbb{R}_+^{k+1}$ can be compared). We say that $u :\ \mathbb{R}_+^{k+1} \to \mathbb{R}$ is a utility function that represents the preorder of preferences if $u(c) \geq u(c')$ is equivalent to $c \succeq c'$. Historically, the concept of a utility function came before that of a preorder of preferences. Utility functions have long been part of the basis of economic theory ("marginalist" theory). Later, much work sought to give foundations to utility theory, by taking the preorders as a starting point.

We assume here that the investor's preferences are represented by a function u from $\mathbb{R}_+^{k+1}$ into $\mathbb{R}$, which is strictly increasing with respect to each of its variables, strictly concave and differentiable. We suppose that the agent maximizes his utility under budgetary constraints (1.8). The derivative u' is called the marginal utility. We assume moreover that $\dfrac{\partial u}{\partial c_i}(c_0, \dots, c_i, \dots, c_k) \to \infty$ when $c_i \to 0$. This condition means that the agent has a strong aversion to consuming nothing at time 0 or at time 1 in one of the states of the world.

1.3.1 The Maximization Problem

Let u be a utility function. We say that $c^* \in B(S)$ is an optimal consumption if

$$u(c^*) = \max \{u(c);\ c \in B(S)\} .$$

Existence of an Optimal Consumption

Proposition 1.3.1. *There is an optimal solution if and only is S satisfies the NAO assumption. The optimal solution is strictly positive.*

Proof. Suppose that there exists an optimal solution (c_0^*, c_1^*) financed by θ^*, and an arbitrage θ^a. We then have $S \cdot \theta^a \le 0$ and $V\theta^a \ge 0$ where at least one of the inequalities is strict. It is true that the consumption $(c_0^* - S \cdot \theta^a, c_1^* + V\theta^a) \in B(S)$ (an associated portfolio is $\theta^* + \theta^a$). Using the property of an arbitrage strategy, $c_0^* - S \cdot \theta^a \ge c_0^*$, $c_1^* + V\theta^a \ge c_1^*$ with at least one strict inequality. As $u(c)$ is strictly increasing, this contradicts the optimality of (c_0^*, c_1^*).

Conversely, let us show that under the assumption of NAO, if V is injective, then there exists an optimal solution. A more general result will be proved in Chap. 6. Let us show that the set

$$\left\{\theta \in \mathbb{R}^{d+1};\ \exists\, c \in \mathbb{R}_+^{k+1},\ \text{satisfying (1.8)}\right\}$$

is bounded. Suppose that, on the contrary, there exists a sequence (c_n, θ_n) satisfying (1.8), and such that $\|\theta_n\| \to \infty$, and let $\hat{\theta}$ be a limit point of the sequence $\frac{\theta_n}{\|\theta_n\|}$. We have

$$\frac{S \cdot \theta_n}{\|\theta_n\|} + \frac{c_{0n}}{\|\theta_n\|} \le \frac{R_0}{\|\theta_n\|} ,$$

$$\frac{c_{jn}}{\|\theta_n\|} \le \frac{R_j}{\|\theta_n\|} + \frac{V\theta_n}{\|\theta_n\|} ,$$

for all n, and hence $S \cdot \hat{\theta} \le 0$ and $V\hat{\theta} \ge 0$. By the NAO assumption, $V\hat{\theta} = 0$, and $\hat{\theta} = 0$, so contradicting the fact that $\|\hat{\theta}\| = 1$. We deduce that $B(S)$ is closed and bounded, and thus compact, and hence that an optimal solution c^* does exist. Let us now show that c^* is strictly positive.

As the function u is strictly increasing, the budget constraints (1.8) are binding. Hence there exists θ^* such that

$$\begin{cases} c_0^* + \sum_{i=0}^{d} \theta^{i*} S^i - R_0 \ = \ 0 \\ c_j^* - \sum_{i=0}^{d} \theta^{i*} v_j^i - R_j \ = \ 0, \quad j \in \{1, \ldots, k\} \ . \end{cases}$$

Let ε satisfy $c_0^* + \varepsilon S \cdot \theta^* > 0$ and $c_j^* - \varepsilon (V\theta^*)_j > 0$ for any $j \in \{1, \ldots, k\}$. The consumption $(c_0, c_1, \ldots, c_k)$ where $c_0 = \varepsilon S \cdot \theta^* + c_0^*$ and $c_j = c_j^* + \varepsilon (V\theta^*)_j$ for any $j \in \{1, \ldots, k\}$ is in $B(S)$ (an associated portfolio is $(1 - \varepsilon)\theta^*$). As u is concave,

$$u(c) - u(c^*) \geq \varepsilon \left(S \cdot \theta^* \frac{\partial u}{\partial c_0}(c) - \sum_{j=1}^{k} (V\theta^*)_j \frac{\partial u}{\partial c_j}(c) \right) .$$

For ε small enough, if $c_0^* = 0$ or if $c_j^* = 0$ for $j \in \{1, \ldots, k\}$, the last expression above is strictly positive: since if $c_0^* = 0$ (respectively $c_j^* = 0$), $S \cdot \theta^* = R_0 > 0$ (respectively $(V\theta^*)_j < 0$), and when $\varepsilon \to 0$, $\frac{\partial u}{\partial c_0}(c) \to \infty$ (respectively $\frac{\partial u}{\partial c_j}(c) \to \infty$). This contradicts the optimality of c^*. □

Remark 1.3.2. It is important to take note of the conditions under which this proposition holds. In the first part of the proof, we used the fact that u is **strictly** increasing with respect to all of its variables. In the second part, we used the non-negativity of consumption. The following exercises provide very simple counterexamples to the statement of the proposition when these conditions are no longer satisfied.

Exercise 1.3.3.

1. Consider an economy in which there are two dates, 0 and 1. At time 1, there are two possible states of the world. At time 0, an agent holding one euro, can buy a portfolio made up of two assets whose payoffs are represented by the payment vectors $[1, 0]$ and $[0, 1]$ respectively, and whose prices are $S^1 = 1,\ S^2 = 0$. Further assume that the agent consumes c_0. At time 1, in addition to the payment vector of his portfolio, the agent receives $[1, 2]$ and consumes (c_1, c_2). Suppose that the agent has utility function
$$u(c_0, c_1, c_2) = c_0 + \min\{c_1, c_2\} \ .$$
Show that the agent's consumption–portfolio problem admits a solution (notice that the maximum utility that the agent can achieve, is 2). Is the solution unique? Show that the financial market admits an arbitrage. Comment on these results.

2. The data here is the same as that of the previous question, except that the agent's utility function is given by

$$u(c_0, c_1, c_2) = -(c_0 - 1)^2 - (c_1 - 1)^2 - (c_2 - 2)^2 \ .$$

Show that the agent's consumption–portfolio problem admits a solution. Comment on the result.

3. We no longer assume the consumption to be positive. At time 0, an agent holding one euro, can buy an asset, whose payment vector is $[1, 1]$, and whose price is $S^1 = 1$. At time 1, in addition to the payment vector of his portfolio, the agent receives $[1, 2]$. We assume that his utility function is

$$u(c_0, c_1, c_2) = c_0 + c_1 + c_2 \ .$$

Show that the financial market does not admit arbitrage, and that nevertheless, the agent's consumption–portfolio problem does not admit a solution.

Asset Valuation Formula As c^* is strictly positive, it follows from the method of Lagrange multipliers, that a necessary and sufficient condition for c^* to be optimal, is for there to exist $\theta^* \in \mathbb{R}^{d+1}$ and $\lambda^* \in \mathbb{R}_+^{k+1}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial c_0}(c^*) - \lambda_0^* \ = \ 0 \\ \dfrac{\partial u}{\partial c_j}(c^*) - \lambda_j^* \ = \ 0 \ , \qquad j \in \{1, \ldots, k\} \ , \end{cases} \tag{1.9.i}$$

$$\lambda_0^* S^i - \sum_{j=1}^{k} \lambda_j^* v_j^i \ = \ 0 \ , \qquad i \in \{0, \ldots, d\} \ , \tag{1.9.ii}$$

$$\begin{cases} \lambda_0^* \left(c_0^* + \displaystyle\sum_{i=0}^{d} \theta^{i*} S^i - R_0 \right) \ = \ 0 \\ \lambda_j^* \left(c_j{}^* - \displaystyle\sum_{i=0}^{d} \theta^{i*} v_j^i - R_j \right) \ = \ 0 \ , \qquad j \in \{1, \ldots, k\} \ . \end{cases} \tag{1.9.iii}$$

The assumption that u is strictly increasing, implies that its derivatives are strictly positive. Hence, from (1.9.i), we have $\lambda^* \in \mathbb{R}_{++}^{k+1}$ and we can write expression (1.9.iii) as

$$\begin{cases} c_0^* + \displaystyle\sum_{i=0}^{d} \theta^{i*} S^i - R_0 \ = \ 0 \\ c_j^* - \displaystyle\sum_{i=0}^{d} \theta^{i*} v_j^i - R_j \ = \ 0, \quad j \in \{1, \ldots, k\} \ . \end{cases} \tag{1.9.iv}$$

Defining β_j as

$$\beta_j = \frac{\lambda_j^*}{\lambda_0^*} = \frac{\partial u/\partial c_j}{\partial u/\partial c_0}(c^*) , \tag{1.10}$$

the β_j are strictly positive, and, using (1.9.ii), we obtain a formula for evaluating the price of the assets:

$$S^i = \sum_{j=1}^{k} \beta_j v_j^i . \tag{1.11}$$

The interest rate is given by the expression:

$$1 + r = \frac{\partial u/\partial c_0(c^*)}{\sum_{j=1}^{k} \partial u/\partial c_j(c^*)} . \tag{1.12}$$

Finally, eliminating θ^{i*} from the equations in (1.9.iv), we get

$$c_0^* + \sum_{j=1}^{k} \beta_j c_j^* = R_0 + \sum_{j=1}^{k} \beta_j R_j .$$

The Complete Market Case In the case of a complete market with no arbitrage, the optimization problem under constraints, defined by (1.8), takes a simpler form. As the market is complete, there exists a unique β such that $S = V^T\beta$. Let us define the inequality

$$c_0 + \sum_{j=1}^{k} \beta_j c_j \le R_0 + \sum_{j=1}^{k} \beta_j R_j . \tag{1.13}$$

This is the budgetary constraint placed on an agent who buys a consumption of c_j at a contingent price of β_j.

If the market is complete

$$B(S) = \left\{c \in \mathbb{R}_+^{k+1} \text{ satisfying } (1.13)\right\} .$$

Indeed, if $c \in B(S)$, using (1.9.iv) to eliminate θ, we can show that c satisfies (1.13).

Conversely, let c satisfy (1.13). If the market is complete, there exists θ such that

$$c_j - \sum_{i=0}^{d} \theta^i v_j^i - R_j = 0 \text{ for all } j \in \{1, \ldots, k\} .$$

Using (1.9.iv) and (1.13), we can show that (1.8) is satisfied, and hence that $c \in B(S)$.

Thus we are brought back to a maximization problem under a single budgetary constraint $c_0 + \sum_{j=1}^{k} c_j \beta_j \leq \sum_{j=1}^{k} R_j \beta_j + R_0$. Formula (1.10) then follows trivially. We observe that the price in state j is proportional to the marginal utility of consumption in state j.

As we showed previously, if there is a riskless asset, the β_j can be interpreted in terms of risk-neutral probabilities $\beta_j = \frac{\pi_j}{1+r}$. The risk-neutral probability of state j is therefore proportional to the marginal utility of consumption in state j. We note that by using risk-neutral probabilities, we can write constraint (1.12), if asset 0 is riskless, as

$$c_0 + \sum_{j=1}^{k} \frac{c_j}{1+r} \pi_j \leq R_0 + \sum_{j=1}^{k} \frac{R_j}{1+r} \pi_j .$$

The consumption at time 0, plus the value, discounted by the risk-free return, of the expectation with respect to π of consumption at time 1, is less than or equal to the revenue at time 0, plus the discounted expectation of the revenue at time 1. This formulation of the constraint will be used in continuous time, in Chaps. 4 and 8, as it allows us to transform a path-wise constraint into a constraint on an expected value.

The Incomplete Market Case We suppose that there is a riskless asset. We write $\mathcal{P}$ for the set of probability measures π satisfying $V^T \pi = (1+r)S$. If $c \in B(S)$ and $\pi \in \mathcal{P}$, we have

$$c_0 + \sum_{j=1}^{k} \frac{c_j}{1+r} \pi_j \leq R_0 + \sum_{j=1}^{k} \frac{R_j}{1+r} \pi_j .$$

We use the notation

$$V(\pi) = \max u(c)$$

where the maximum is taken over the c that satisfy the constraint

$$c_0 + \sum_{j=1}^{k} \frac{c_j}{1+r} \pi_j \leq R_0 + \sum_{j=1}^{k} \frac{R_j}{1+r} \pi_j .$$

Thus we obtain

$$u(c^*) \leq \min_{\pi \in \mathcal{P}} V(\pi) .$$

As the corresponding necessary and sufficient first order conditions are satisfied, it follows from (1.10) and (1.11) that $u(c^*) = V(\beta(1+r))$ where β is defined as in (1.10). Therefore we have:

$$u(c^*) = \min_{\pi \in \mathcal{P}} V(\pi) .$$

We refer to such a π as a “minimax” probability measure.

1.3.2 An Equilibrium Model with a Representative Agent

We take as given the endowments $(R_0, \ldots, R_k)$ of the agent, and the asset prices S. A pair $((R_0, \ldots, R_k), S)$ is an equilibrium if the optimal solution to the agent's consumption–portfolio problem is $((R_0, \ldots, R_k), 0_{d+1})$. In other words, at price S, the agent does not carry out any transactions. Let z be a contingent claim, and let $S(z)$ be its price. We say that the claim is valued at equilibrium if, when it is introduced into the financial markets in equilibrium, the optimal demand θ_z for the claim is zero. In other words, writing R for the agent's random endowments at time 1 and C for his consumption vector at time 1, the optimal solution to the problem

$$\begin{aligned}&\max u(c_0, c) \quad \text{under the constraints}\\ &c_0 + \theta \cdot S + \theta_z S(z) \le R_0\\ &C \le R + V\theta + \theta_z z\end{aligned}$$

is given by $(R_0, \ldots, R_k)$ and by the associated portfolio $(0_{d+1}, 0_z)$.

Proposition 1.3.4. *If $((R_0, \ldots, R_k), S)$ is an equilibrium, then the interest rate and the asset prices are given by:*

$$1 + r = \frac{\partial u/\partial c_0(R_0, \ldots, R_k)}{\sum_{j=1}^{k} \partial u/\partial c_j(R_0, \ldots, R_k)}$$

and

$$S^i = \sum_{j=1}^{k} \frac{\partial u/\partial c_j(R_0, \ldots, R_k)}{\partial u/\partial c_0(R_0, \ldots, R_k)} v_j^i \quad \text{for all } i = \{1, \ldots, d\}\,.$$

The equilibrium price of a contingent claim $z \in \mathbb{R}^k$ is:

$$S(z) = \sum_{j=1}^{k} \frac{\partial u/\partial c_j(R_0, \ldots, R_k)}{\partial u/\partial c_0(R_0, \ldots, R_k)}\, z_j\,.$$

Proof. The first part of the proposition follows from (1.10) and (1.11) with $(c^*) = (R_0, \ldots, R_k)$. To prove the second part, we suppose that the agent's consumption–portfolio problem

$$\begin{aligned}&\max u(c_0, c) \text{ under the constraints}\\ &c_0 + \theta \cdot S + \theta_z S(z) \le R_0\\ &C \le R + V\theta + \theta_z z\end{aligned}$$

has for optimal solution $((R_0, \ldots, R_k), 0_{d+1}, 0_z)$. According to the Kuhn–Tucker theorem, there exists $\tilde{\lambda} \in \mathbb{R}_+^{k+1}$ such that

i) $\frac{\partial u}{\partial c_0}(R_0, \ldots, R_k) - \tilde{\lambda_0} \ = \ 0$

ii) $\frac{\partial u}{\partial c_j}(R_0, \ldots, R_k) - \tilde{\lambda_j} \ = \ 0 \ , \qquad j \in \{1, \ldots, k\}$

iii) $\tilde{\lambda_0} S^i = \sum_{j=1}^k \tilde{\lambda_j} v_j^i \ , \qquad i \in \{0, \ldots, d\}$

iv) $\tilde{\lambda_0} S(z) = \sum_{j=1}^k \tilde{\lambda_j} z_j .$

The equilibrium price of a contingent asset follows trivially from these formulae. □

Exercise 1.3.5. Consider an economy with two dates 0 and 1. At time 1, there are two states of the world. At time 0, an agent holding one euro can buy a portfolio made up of two assets with respective payment vectors $[1, 1]$ and $[2, 0]$. Assume moreover that he consumes c_0. At time 1, in addition to the payment vector of his portfolio, the agent receives $[1, 2]$ and consumes (c_1, c_2). Suppose the agent has utility function $u(c_0, c_1, c_2) = \log(c_0) + \frac{1}{2}(\log(c_1) + \log(c_2))$. Suppose that the agent's optimal strategy is to buy nothing. What are the assets' equilibrium prices? What is the risk-neutral probability measure?

1.3.3 The Von Neumann–Morgenstern Model, Risk Aversion

First of all, we present the theory for decisions taken over one period. In the interests of simplicity, we assume here that there is only a single consumption good.

Let $\mathcal{P}$ be the set of probability measures on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. In particular, if there are only a finite number of states, if state j occurs with probability μ_j, and if consumption C at time 1 is a random variable taking values c_j, then the probability law μ_C of C with $\mu_C = \sum_{j=1}^k \mu_j \delta_{c_j}$ is an element of $\mathcal{P}$.

We make the assumption that only the consequences of random events (that is possible cash flows and their probabilities) are taken into account. This assumption comes down to supposing that the agent's preferences are not expressed on the positive or zero random variables, but directly on $\mathcal{P}$. We use the notation $\succeq$ for the preorder of the agent's preferences, which is assumed to be complete. We say that $u : \mathcal{P} \to \mathbb{R}$ is a utility function that represents the preorder of preferences if $u(\mu) \geq u(\mu')$ is equivalent to $\mu \succeq \mu'$.

We say that the utility is Von Neumann–Morgenstern if there exists $v : \mathbb{R}_+ \to \mathbb{R}$ such that

$$u(\mu) = \int_0^\infty v(x) \mathrm{d}\mu(x) \ .$$

In the particular case $\mu_C = \sum_{j=1}^k \mu_j \delta_{c_j}$, the VNM utility is written

$$u(\mu_C) = \sum_{j=1}^{k} \mu_j v(c_j) \ ,$$

so that our criterion is to maximize the expectation of the consumption's utility.

We do not discuss here the abundant literature that establishes axioms on the preorder of preferences on $\mathcal{P}$ in such a way that it admits a VNM representation.

We say that the agent is *risk averse* if

$$v(E(C)) \geq E(v(C)), \text{ for all } C \ .$$

Thus an investor prefers a future consumption $E(C)$ with certainty, to a consumption c_1 with probability μ_1, a consumption c_2 with probability $\mu_2, \ldots$, a consumption c_k with probability μ_k.

If the agent has a preorder on all finite probabilities, the presence of risk aversion is equivalent to the concavity of v. Indeed, if v is concave, then according to Jensen's inequality[4], we have $v(E(C)) \geq E(v(C)),$ for all C. Conversely, we suppose that $v(E(C)) \geq E(v(C))$ for all C. Let $(x, y) \in \mathbb{R}^2_+$. We consider the random variable C worth x with probability α and y with probability $1 - \alpha$. As

$$v(E(C)) = v(\alpha x + (1-\alpha)y) \geq E(v(C)) = \alpha v(x) + (1-\alpha)v(y) \ ,$$

by letting α, x and y vary, we obtain the concavity of v.

We say that an investor is *risk-neutral*, if v is an affine function. Then

$$v(E(C)) = E(v(C)), \text{ for all } C \ .$$

When an agent is risk averse, we define the *risk premium* $\rho(C)$ linked to the random consumption C: it is the amount the investor is prepared to give up in order to obtain, with certainty, a consumption level equal to $E(C)$. As v is a continuous, strictly increasing and strictly concave function, for all C there exists $\rho(C) \geq 0$ such that

$$v(E(C) - \rho(C)) \ = \ E\left(v(C)\right) \ . \tag{1.14}$$

The amount $E(C) - \rho(C)$ is called the *certainty equivalent* of C, and ρ is called the risk premium. When the investor is risk-neutral, $E[v(C)] = v[E(C)]$, so that $\rho(C) \ = \ 0$. We now assume that there are a finite number of states, that consumption C at time 1 is a random variable taking values c_j with probability μ_j, and that v is of class C^2. Using Taylor's expansion, on the

[4] See for example Chung [58].

condition that the values c_j taken by the consumption C are close enough to $E(C)$, we get

$$v(c_j) \simeq v[E(C)] + [c_j - E(C)]v'[E(C)] + \frac{[c_j - E(C)]^2}{2} v''[E(C)] .$$

Taking expectations on both sides,

$$E[v(C)] = \sum_{j=1}^{k} \mu_j v(c_j) \simeq v[E(C)] + v''[E(C)] \frac{\mathrm{Var\,C}}{2} .$$

Expanding the first term of (1.13), using Taylor's expansion once again, we get

$$v[E(C) - \rho(C)] \simeq v[E(C)] - \rho(C)v'[E(C)]$$

and hence we can evaluate $\rho(C)$:

$$\rho(C) \simeq -\frac{v''[E(C)]}{2v'[E(C)]} \operatorname{Var} C .$$

The coefficient $I_a(v,x) = -\dfrac{v''(x)}{v'(x}$ is called the *absolute risk aversion coefficient of* v. Thus the certainty equivalent of C is approximately equal to $E(C) - \dfrac{I_a(v, E(C))}{2} \operatorname{Var} C$, which as a first approximation justifies the choice of a mean–variance criterion.

Exercise 1.3.6. We denote by $\mathcal{N}(\mu, \sigma^2)$ the normal distribution with mean μ and variance σ^2. Let $C \overset{\text{law}}{=} \mathcal{N}(\mu, \sigma^2)$ and $v(c) = -\mathrm{e}^{-\beta c}$, $\beta > 0$. Show that $I_a(v,x) = \beta$ for any x and that $E(C) - \rho(C) = \mu - \frac{\beta}{2}\sigma^2$. (Here we can use that fact that $C \overset{\text{law}}{=} \mu + \sigma Y$, where $Y \overset{\text{law}}{=} \mathcal{N}(0,1)$). In this particular case, the certainty equivalent of C is exactly equal to $E(C) - \dfrac{\alpha}{2} \operatorname{Var} C$.

Exercise 1.3.7. Calculate the absolute risk aversion index in the following cases: $v(c) = \frac{c^\gamma}{\gamma}$ with $0 < \gamma < 1$, $v(c) = \ln c$.

1.3.4 Optimal Choice in the VNM Model

We return to the two date model considered in Sect. 1.3.1, and assume that at time 1, state j occurs with probability $\mu = (\mu_j)_{j=1}^k$. We suppose here that there is a riskless asset, and that the market is complete. The investor has preferences on $\mathbb{R} \times \mathcal{P}$. Let us look at the special case in which the preferences can be represented by a utility function that is "additively separable" with respect to time:

$$u(c_0, C) \;=\; v_0(c_0) + \alpha\, E(v(C)) \;=\; v_0(c_0) + \alpha \sum_{j=1}^{k} \mu_j\, v(c_j)$$

$$0 < \alpha < 1$$

where α is a discount factor, and where v^0 and v are strictly concave, strictly increasing C^2 functions satisfying

$$\lim_{x \to 0} v_0'(x) = \infty \,, \qquad \lim_{x \to 0} v'(x) = \infty \,,$$

$$\lim_{x \to \infty} v_0'(x) = 0 \,, \qquad \lim_{x \to \infty} v'(x) = 0 \,.$$

Let $I_0 :]0,\infty[\,\to\,]0,\infty[$ (respectively $I :]0,\infty[\,\to\,]0,\infty[$) be the inverse function of $v^{0\prime}$ (respectively of v'). The functions I_0 and I are continuous and strictly decreasing. Denote the riskless rate by r.

In this special case, formulae (1.11) and (1.12) become:

$$1 + r = \alpha \frac{\sum_{j=1}^{k} \mu_j v'(c_j^*)}{v_0'(c_0^*)} = \alpha \frac{E(v'(C^*))}{v_0'(c_0^*)} \tag{1.15}$$

$$S^i \;=\; \alpha \frac{\sum_{j=1}^{k} \mu_j v'(c_j^*) v_j^i}{v_0'(c_0^*)} \;=\; \frac{1}{1+r} E(V^i v'(C^*)) \tag{1.16}$$

where V^i and C^* are random variables taking the values v_j^i and c_j^* respectively.

Let us show how to obtain the optimal solution in explicit form. Indeed, the investor solves the following problem $\mathcal{P}$:

$$\max v_0(c_0) + \alpha \sum_{j=1}^{k} \mu_j\, v(c_j) \;\text{ under the constraint}$$

$$c_0 + \sum_{j=1}^{k} \frac{c_j}{1+r}\, \pi_j \;\leq\; R_0 + \sum_{j=1}^{k} \frac{R_j}{1+r}\, \pi_j \;.$$

Let λ be the Lagrange multiplier associated with the constraint. We have

$$\begin{aligned} & v_0'(c_0^*) = \lambda \\ & \alpha \mu_j v'(c_j^*) = \lambda \tfrac{\pi_j}{1+r}, \quad \forall\, j = 1, \ldots, k \,, \end{aligned} \tag{1.17}$$

and hence

$$\begin{aligned} & c_0^* = I_0(\lambda) \\ & c_j^* = I\left(\frac{\pi_j \lambda}{\mu_j (1+r) \alpha} \right), \quad \forall\, j = 1, \ldots, k \,. \end{aligned} \tag{1.18}$$

The Lagrange multiplier λ is determined by the budget constraint, and satisfies the following equation:

$$I_0(\lambda) + \frac{1}{1+r}\sum_{j=1}^{k}\pi_j I\left(\frac{\lambda\pi_j}{\mu_j(1+r)\alpha}\right) = R_0 + \frac{1}{1+r}\sum_{j=1}^{k}\pi_j R_j \ . \qquad (1.19)$$

As the function $x \to I_0(x) + \frac{1}{1+r}\sum_{j=1}^{k}\pi_j I\left(\frac{\pi_j x}{\mu_j(1+r)\alpha}\right)$ is a decreasing function from $]0,\infty[$ into itself, equation (1.16) has a unique solution. Once the Lagrange multiplier has been determined, we can deduce the optimal consumption from (1.15), and then finally obtain the optimal portfolio using the relation $V\theta^* = C^* - R$.

Let us show that under the assumption that the agent is in equilibrium, we can give an estimate of the risk-neutral probability.

Replacing c_0^* by R_0 and C^* by R, it follows from (1.14) that

$$\lambda = v_0^{'}(R_0) = \alpha(1+r)E(v'(R))$$

and that

$$\pi_j \ = \mu_j \ \frac{v'(R_j)}{E(v'(R))} \ . \qquad (1.20)$$

This expression for the risk-neutral probability does not depend on future consumption. Using Taylor's expansion, we get:

$$v'(R_j) \ \simeq \ v'[E(R)] + [R_j - E(R)]v''[E(R)] \ .$$

Taking expectations,

$$E[v'(R)] \ \simeq \ v'[E(R)] \ .$$

Hence

$$\frac{\pi_j}{\mu_j} \simeq \ 1 + \frac{[R_j - E(R)]v''[E(R)]}{v'[E(R)]} = 1 + I_a(v, E(R))(E(R) - R_j) \ .$$

The greater the agent's index of absolute aversion to risk for $E(R)$ and the greater the difference between the average value of his resources and his resources in a given state j, the greater is the risk-neutral probability of state j occurring.

If the investor had a neutral attitude to risk ($v' =$ cst), he would be prepared to pay $\frac{\alpha\mu_j}{v_0'(R_0)}$ at time 0 in order to receive 1 euro tomorrow in the state of the world j. If he is risk averse, he is prepared to pay $\frac{\alpha\mu_j v'(R_j)}{v_0'(R_0)}$ today so as to receive 1 euro tomorrow in state of the world j.

To summarize, we have used two different approaches:

- the assumption of no arbitrage opportunities enabled us to construct a probability measure under which we are neutral with respect to risk,
- the introduction of a utility function and of exogenous (or subjective) probabilities led us firstly to define the concept of risk aversion, secondly to obtain valuation formula (1.16), and finally to exhibit a risk-neutral probability measure.

We remark on the fact that these two risk-neutral probability measures are equivalent.

Exercise 1.3.8.

1. We assume that the market is complete and that $v_0(c) = v(c) = \log(c)$. Calculate I_0, I and the optimal solution. Carry out the corresponding calculations when $v_0(c) = 0$ and $v(c) = \log(c)$, and similarly obtain I_0, I and the optimal solution for $v_0(c) = v(c) = c^\alpha,\ 0 < \alpha < 1$.

2. We consider an economy with two dates, 0 and 1. At time 1, there are two states of the world. At time 0, an agent does not own anything and can buy for a price $[1, 1]$, a portfolio of two assets whose respective payment vectors are $[1, -1]$ and $[2, -2]$. At time 1, in addition to the payment vector of his portfolio, the agent receives $[1, 1]$ and consumes (c_1, c_2). Suppose that the agent has a VNM utility function, that he attributes the probabilities $(\frac{1}{2}, \frac{1}{2})$ to the two states of the world, and that his utility index is $u(c) = \log c$. Show that the agent's consumption–portfolio problem has a solution, and that nevertheless, the market admits an arbitrage. Comment on these results.

Exercise 1.3.9. We consider an economy with two dates 0 and 1. There are three states of the world at time 1. At time 0, an agent does not own anything, and he can buy a portfolio of three assets that have respective payment vectors $[1, 1, 1]$, $[3, 2, 1]$ and $[1, 2, 6]$, and respective prices $S^1 = 1$, $S^2 = 2$ and $S^3 = 3$. He must not run into debt. The agent does not consume at time 0. At time 1, in addition to the payment vector of his portfolio, the agents receives $[1, 2, 1]$ in the different states, and consumes (c_1, c_2, c_3).

1. Calculate the state prices and the interest rate. Show that the market is complete. Calculate the risk-neutral probability. Show that the set of consumptions that the agent can achieve at time 1 is given by
$$\{c \in \mathbb{R}^3 \ \mid \ c_1 + c_2 + c_3 \leq 4\} \,.$$
2. We assume that the agent attributes the probabilities $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to the different states of the world, and that he has a VNM utility function of index $u(x) = \log(x)$. Calculate his optimal consumption and portfolio.

How would the results change if the agent attributed the probabilities $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ to the different states of the world?

3. Suppose that the agent can only buy a portfolio that contains the two first assets. Calculate the interest rate, and characterize the set of risk-neutral probabilities. Find the set of its extrema. Show that the set of consumptions that can be attributed to the agent at time 1 is

$$\{c \in \mathbb{R}^3 \ | \ c_2 \leq 2, \ \frac{1}{2}(c_1 + c_3) \leq 1\} \ .$$

Calculate the optimal consumption and portfolio when the agent attributes probabilities $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to the states of the world, and has a VNM utility function with index $u(x) = \log(x)$.

4. We suppose that the agent can, without running into debt, purchase a portfolio made up of the three assets, and that in addition he can buy a positive amount of asset 2.
Characterize the set of risk-neutral probabilities. Find the set of its extrema. Calculate the agent's optimal consumption and portfolio when he affects the probabilities $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to the states of the world, and has a VNM utility function with index $u(x) = \log(x)$.

1.3.5 Equilibrium Models with Complete Financial Markets

The Representative Agent We now study a simple model in order to illustrate the effect of introducing financial markets into an economy. This model will be further developed in Chap. 6.

Consider an exchange economy with a single consumption good and m economic agents. We suppose that there are $(d+1)$ assets, with the same characteristics as in the previous sections. We assume the market to be complete.

Agent h has an initial endowment of e_{h0} units of the good at time 0, and knows that he will receive e_{hj} units of the good at time 1 in state of the world j. To modify his future resources, he can, at time 0, buy a portfolio of securities $\theta_h = (\theta_h^0, \dots, \theta_h^d)$ on condition that he does not run into debt.

Given a price S for the assets, we define the agent's budget set as the set of consumption plans which he can carry out with his initial wealth and future income:

$$B_h(S) := \{c \in \mathbb{R}_+^{k+1} \mid \exists \theta \in \mathbb{R}^{d+1},$$
$$e_{h0} \geq c_0 + \theta \cdot S; \ e_{hj} \geq c_j - (V\theta)_j, j \in \{1, \dots, k\}\}.$$

As in Sect. 1.3.3, we suppose that agent h has preferences that are represented by a VNM utility function of the form

$$u_h(c_0, C) = v_{h0}(c_0) + \alpha \sum_{j=1}^{k} \mu_j v_{h1}(c_j) \ .$$

We suppose here that all the agents have the same discount factor α.

Definition 1.3.10. *The collection* $\{\overline{S}, (\overline{c}_h, \overline{\theta}_h); h = 1, \ldots, m\}$ *is an equilibrium of the economy with financial markets if, given* $\overline{S}$

1. $\overline{c}_h$ *maximizes* $u_h(c_{h0}, C_h)$ *under the constraint* $c_h = (c_{h0}, C_h) \in B_h(\overline{S})$,
2. $\sum_{h=1}^{m} \overline{c}_{hj} = \sum_{h=1}^{m} e_{hj} \quad := e_j, \ \ j \in \{1, \ldots, k\}$,
3. $\sum_{h=1}^{m} \overline{\theta}_h = 0$.

In other words, the market in the good clears (equality 2) and the security market also clears (equality 3).

Remark 1.3.11. If v_h^0 and v_h are strictly increasing, and if V is injective, then equality 3 is implied by 1 and 2. Indeed, as the utility functions are strictly increasing, at equilibrium the constraints are binding. Therefore we have $e_{hj} = \overline{c}_{hj} - (V\overline{\theta}_h)_j$ for all h and $j \in \{1, \ldots, k\}$. As $\sum_{h=1}^{m} e_{hj} = \sum_{h=1}^{m} \overline{c}_{hj}$ for $j \in \{1, \ldots, k\}$, we have $V\left(\sum_{h=1}^{m} \overline{\theta}_h\right) = 0$, which implies $\sum_{h=1}^{m} \overline{\theta}_h = 0$.

We suppose now that an equilibrium exists. We can use the first order necessary and sufficient conditions of the precious section. Hence for all h,

$$S^i = \alpha \sum_{j=1}^{k} \mu_j \frac{v_h'(\overline{c}_{hj})}{v_{h0}'(\overline{c}_{h0})} v_j^i = \sum_{j=1}^{k} \frac{1}{1+r} \ \mu_j \frac{v_h'(\overline{c}_{hj})}{E(v_h'(\overline{C}_h))} v_j^i \ , \tag{1.19}$$

$$\frac{1}{1+r} = \alpha \sum_{j=1}^{k} \mu_j \frac{v_h'(\overline{c}_{hj})}{v_{h0}'(\overline{c}_{h0})} \ . \tag{1.20}$$

Under the assumption of complete markets, the equation $S = V^T \beta$ has a unique solution. Under this assumption, the ratios

$$\frac{v_h'(\overline{c}_{hj})}{v_{h0}'(\overline{c}_{h0})} \quad \text{are therefore independent of } h.$$

Let us then consider a fictitious agent, the "representative agent", whose utility is

$$u(c_0, C) := v_0(c_0) + \alpha \sum_{j=1}^{k} \mu_j v(c_j) \ ,$$

where

$$v_0(c):=\max\left\{\sum_{h=1}^{m}\frac{v_{h0}(c_h)}{v'_{h0}(\bar{c}_{h0})};\ \sum_{h=1}^{m}c_h=c\right\}$$

$$v(c):=\max\left\{\sum_{h=1}^{m}\frac{v_h(c_h)}{v'_{h0}(\bar{c}_{h0})};\ \sum_{h=1}^{m}c_h=c\right\}\ .$$

Using the first order necessary and sufficient conditions of these new optimization problems, we check that

$$u(e_0,e)=\sum_{h=1}^{m}\frac{v_{0h}(\bar{c}_{h0})}{v'_{h0}(\bar{c}_{h0})}+\alpha\sum_{j=1}^{k}\sum_{h=1}^{m}\frac{v_h(\bar{c}_{hj})}{v'_{h0}(\bar{c}_{h0})}\mu_j\ ,$$

where e is a random variable taking the value e_j with probability μ_j. Using the first order conditions and the implicit function theorem, we show that v_0 and v are differentiable, that $v'_0(e_0)=1$ and that

$$v'(e_j)=\frac{v'_h(\bar{c}_{hj})}{v'_{h0}(\bar{c}_{h0})}\quad j=1,\ldots,k\ .$$

Hence, (1.20) can be rewritten as

$$\frac{1}{1+r}=\alpha E(v'(e)) \tag{1.21}$$

$$S^i=\frac{1}{1+r}\sum_{j=1}^{k}\mu_j\frac{v'(e_j)}{E(v'(e))}v_j^i=\frac{1}{1+r}\left[E(V^i)+\mathrm{Cov}\left(\frac{v'(e)}{E(v'(e))},V^i\right)\right] \tag{1.22}$$

where V^i is a random variable taking value v_j^i.

The formula above plays a very important role in the financial literature, as it shows that when there is a equilibrium, the price of an asset is only a function of aggregate endowment (and not of each individual's endowment). In the next section, we will look at the relationship more closely.

Exercise 1.3.12. Consider an economy with two dates, two agents, two states of the world, and one good in each state. Suppose that the agents have utility functions $v_{0h}(c)=v_h(c)=\log(c),\ h=1,2$ and assign probabilities $\frac{1}{2}$ to the states of the world. Assume that the agents have endowments $e_{01}=1$, $e_1=(1,3)$ and $e_{02}=2$, $e_2=(3,1)$. Assume that the two assets are traded at time 0: the riskless asset, and an asset that pays 1 in state 1 and 0 in state 2. Find the equilibrium of this economy.

The Capital Asset Pricing Model (CAPM) Formula This model will be developed in greater generality in Chap. 6.

We suppose that the agents have quadratic utility functions (i.e., $v'_h(c) = -a_h c + b_h$ with $a_h > 0$ for all h), and that at equilibrium the agents have strictly positive consumption. In this particular case, we can easily check that v' is linear and decreasing, that is

$$v'(c) = -ac + b \quad (\text{with } a > 0) .$$

Equation (1.22) then becomes

$$S^i = \frac{1}{1+r}\left[E(V^i) - \frac{a}{E(v'(e))}\,\mathrm{Cov}(e, V^i)\right] . \qquad (1.23)$$

Formula (1.23) is called the CAPM (Capital Asset Pricing Model) formula. As long as $E(v'(e)) > 0$, the price of asset i is therefore greater than the discounted expectation of returns, if it is negatively correlated with e (i.e., $\mathrm{Cov}(e, V^i) \le 0$): the asset provides a form of insurance.

If we introduce $\rho^i = \frac{V^i}{S^i}$, the return on asset i, and M, such that $e = VM$ (M is called the market portfolio), we can express (1.23) in the form

$$E(\rho^i) - (1+r) = \frac{a}{E(v'(e))}\ \mathrm{Cov}\ (\rho^i, e) = \frac{aS\cdot M}{E(v'(e))}\,\mathrm{Cov}(\rho_i, \rho_M)\ ,$$

setting $\rho_M = \frac{e}{S\cdot M}$ (ρ_M is the return on the market portfolio). We then get, in particular:

$$E(\rho_M) - (1+r) = \frac{aS\cdot M}{E(v'(e))}\,\mathrm{Var}\ \rho_M\ .$$

From this we deduce

$$E(\rho^i) - (1+r) = \frac{\mathrm{Cov}\ (\rho^i, \rho_M)}{\mathrm{Var}\,\rho_M}\{E(\rho_M) - (1+r)\}\ . \qquad (1.24)$$

This formula, which links the excess return on an asset to the return on the market portfolio, is called the beta formula, where the β^i coefficient is given by $\frac{\mathrm{Cov}(\rho^i, \rho_M)}{\mathrm{Var}\,\rho_M}$. We note that β^i is the coefficient of the regression of ρ^i on ρ_M. In valuation models for financial assets (or in the CAPM: Capital Asset Pricing Model), it is interpreted as a sensitivity factor to the risk of asset i. We find that the risk premium for asset i, that is $E(\rho^i) - (1+r)$, is a linear function of its β.

An Approximate CAPM Formula More generally, taking any utility function for the representative agent, let us suppose that the e_j are close to $E(e)$. We then obtain an approximation of the CAPM formula. Indeed,

$$\frac{v'(e_j)}{E(v'(e))} \simeq 1 + \alpha[E(e) - e_j] ,$$

where α is the representative agent's index of absolute aversion to the risk in $E(e)$. Formula (1.22) then becomes

$$S^i \simeq \frac{E(V^i)}{1+r} - \frac{\alpha}{1+r} \operatorname{Cov}(e, V^i) . \qquad (1.25)$$

Notes

The financial literature in discrete time is extensive, and it would be quite impossible to give a detailed bibliography here. We restrict ourselves to a few books, which provide the basics: Huang and Litzenberger [195], (1988), Pliska [297], (1997), Mel'nikov [267], (1999), the first part of Shiryaev [330], (1999), Le Roy and Werner [248], (2001), Cvitanic and Zapatero [78], (2002) and the first part of Föllmer and Schied [160], (2002).

The initial formulation and use of the NAO assumption are due to Ross [312, 313], (1976, 1978). Varian [351], (1988) gives a summary of this approach and Cochrane [61] develops its applications to asset pricing. For the probabilistic aspects, see Bingham and Kiesel [33] (1998) and Björk [34] (1998).

For the axiomatic approach to the Von Neumann–Morgenstern utility, the reader can consult Huang and Litzenberger [195], (1988), Kreps [240], (1990), Le Roy and Werner [248], (2001), and Föllmer and Schied [160], (2002).

The problem of choosing an optimal consumption and portfolio in incomplete markets, or in the presence of portfolio constraints, was originally studied by He and Pearson [183], (1991). Different solution methods are presented in Pliska [297], (1997) and Mel'nikov [267], (1999).

The options literature goes back to Merton [269, 270], (1973). That too is vast. We have only given the basic definitions here. The reader is referred to Cox–Rubinstein [71], (1985). Wilmott's books [361, 362], (1998, 2001) provide a good introduction to the problem of valuation and hedging. A more detailed study of our own and other references will be given in the chapter on exotic options.

We will study the equilibria of financial markets more thoroughly in Chap. 6.

For optimization in finite dimensions and duality properties, the reader can refer to Rockafellar [308], (1970), Luenberger [256],(1969), Hiriart-Urruty [190], (1996), and Florenzano and Le Van [157], (2000).

ANNEX 1

Optimization under Constraints, the Kuhn–Tucker Theorem with Linear Constraints

Let C be an open convex set in $\mathbb{R}^n$.

We consider the following problem denoted $P_{\alpha\beta}$ and formulated for $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{R}^p$ and $\beta = (\beta_1, \dots, \beta_q) \in \mathbb{R}^q$, by:

$$\begin{array}{ll} \max f(x), & \text{under the constraints} \\ f_i(x) \leq \alpha_i, & \forall i = 1, \dots, p, \\ g_j(x) = \beta_j, & \forall j = 1, \dots, q, \\ x \in C & \end{array}$$

where the function $f : C \to \mathbb{R}$ is concave and differentiable, and where the functions $f_i : C \to \mathbb{R}, \quad i = 1, \dots, p$ and $g_j \quad j = 1, \dots, q$ are affine. We call f the **objective function**. We write K for the **admissible set**

$$K = \{x \in C \mid f_i(x) \leq \alpha_i, \quad \forall i = 1, \dots, p, \quad g_j(x) = \beta_j, \quad \forall j = 1 \dots q\} .$$

Theorem *Let $\bar{x} \in K$. Then $\bar{x}$ is a solution to $P_{\alpha\beta}$ if and only if there exists $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^p_+ \times \mathbb{R}^q$ such that*

1. $\nabla f(\bar{x}) = \sum_{i=1}^p \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j=1}^q \bar{\mu}_j \nabla g_j(\bar{x})$,
2. $\bar{\lambda}_i (f_i(\bar{x}) - \alpha_i) = 0, \qquad \forall i = 1, \dots, p.$

We call $(\bar{\lambda}, \bar{\mu})$ **the Lagrange multipliers** or the **Kuhn–Tucker multipliers**