# Foreword

In recent decades, the theory of inverse and ill-posed problems has impressively developed into a highly respectable branch of Applied Mathematics and has had stimulating effects on Numerical Analysis, Functional Analysis, Complexity Theory, and other fields. The basic problem is to draw useful information from noise contaminated physical measurements, where in the case of ill-posedness, naive methods of evaluation lead to intolerable amplification of the noise. Usually, one is looking for a function (defined on a suitable domain) that is close to the true function assumed to exist as underlying the situation or process the measurements are taken from, and the above mentioned gross amplification of noise (mathematically often caused by the attempt to invert an operator whose inverse is unbounded) makes the numerical results so obtained useless, these "results" hiding the true solution under large amplitude high frequency oscillations.

There is an ever growing literature on ways out of this dilemma. The way out is to suppress unwanted noise, thereby avoiding excessive suppression of relevant information. Various methods of "regularization" have been developed for this purpose, all, in principle, using extra information on the unknown function. This can be in the form of general assumptions on "smoothness", an idea underlying, e.g., the method developed by Tikhonov and Phillips (minimization of a quadratic functional containing higher derivatives in an attempt to reproduce the measured data) and various modifications of this method. Another efficient method is the so-called "regularization by discretization" method where one has to find a kind of balance between the fineness of discretization and its tendency to amplify noise. Yet another method, the so-called "descriptive regularization" method, consists in exploiting a priori known characteristics of the unknown function, such as regions of nonegativity, or monotonicity, or convexity that can be used in a scheme of linear or nonlinear fitting to the measured data, fitting optimal with respect to appropriate constraints. Many ramifications and combinations of these and other methods have been analyzed theoretically and used in numerical calculations. Our monograph deals with the method called the "moment method". The moments considered here are of the form

$$\mu_n = \int_\Omega u(x) d\sigma_n, \qquad n = 1, 2, 3, ...,$$

where $\Omega$ is a domain in $\mathbf{R}^k$, $d\sigma_n$ is, either a Dirac measure, $n \in \mathbf{N}$, or a measure absolutely continuous with respect to the Lebesgue measure, i.e.,

$$d\sigma_n = g_n(x) dx, \qquad n \in \mathbf{N},$$

$g_n(x)$ being Lebesgue integrable on $\Omega$. The idea of the moment method is to reconstruct an unknown function $u(x)$ from a given set $(\mu_n)_{n \in I}$, $I \subset \mathbf{N}$, of the moments of $u(x)$. Then the problem arises as to whether a knowledge of moments of $u(x)$ uniquely determines this function. For the moment problems considered in this monograph, unless stated otherwise, the knowledge of the complete sequence of moments of $u(x)$ uniquely determines the function. In practice, one has available only a finite set $\mu_1, ..., \mu_m$ of moments, and furthermore these are usually contaminated with noise, the reason being that they are results of experimental measurements. The question then is: To what extent, can the true function $u(x)$ be recovered from the finite set $(\mu_i)_{1 \le i \le m}$ of moments? Note that in the latter situation, the question of existence of a solution $u$ plays a minor role. The moments being only approximately known, the problem is reduced to one of ”regularization”, namely, to the problem of fitting the function $u(x)$ as closely as possible to the available data, that is, to the given approximate values of the moments, $u(x)$ being assumed to lie in a nice function space and to obey a known or stipulated restriction to the size of an appropriate functional. In our theory of regularization, the index $m$, i.e., the number of the given moment values mentioned above, will play the role of the regularization parameter. In illustration of the theory, we shall study several concrete cases, discussing inverse problems of function theory, potential theory, heat conduction and gravimetry. We will make essential use of analyticity or harmonicity of the functions involved, and so the theory of analytic functions and harmonic functions will play a decisive role in our investigations. We hope that this monograph, which is a fruit of several years of joint efforts, will stimulate further research in theoretical as well as in practical applications.

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