Preface

In the present monograph, we consider the extremal length method in its form of *the method of moduli of families of curves* in applications to the problems of conformal, quasiconformal mapping, and Teichmüller spaces. This method going back to H. Grötzsch, A. Beurling, L. V. Ahlfors, J. Jenkins is now one of the basic methods in various parts of Analysis. Several surveys and monographs, e.g., [30], [64], [78], [107], [139] are devoted to the development of this method and applications. However, we want to give here a useful guide: how one can start to solve extremal problems of conformal mapping beginning with simple but famous classical theorems and ending at difficult new results. Some more non-traditional applications we consider in the quasiconformal case. The modulus method permits us to consider the problems in question from a single point of view.

At the mid-century it was established that the classical methods of the geometric function theory could be extended to complex hyperbolic manifolds. The Teichmüller spaces turned out to be the most important of them. Recently, it has become clear that some forms of the extremal length method could be applied to examine different properties of Teichmüller spaces (see e.g. [43], [44]).

Thus, we are concerned with *the modulus method and its applications to extremal problems for conformal, quasiconformal mappings, the extension of moduli onto Teichmüller spaces.* The book is intended for different groups of readers:

(1) Non-experts who want to know about how one can use the modulus technique to solve extremal problems of Complex Analysis. One can find proofs of classical theorems of conformal and quasiconformal mapping by means of the modulus method as well as many examples of symmetrization and polarization. Graduate students will find here some useful exercises to check their understanding.

(2) Experts who will find new results about solution of difficult extremal problems for conformal and quasiconformal mappings and about the extension of the modulus onto Teichmüller spaces.

For the most part of this book we assume the background provided by the usual graduate courses in complex analysis, in particular, the theory of conformal mappings.

This book is not an exhaustive survey of quasiconformal or conformal mapping. Here we mostly consider applications of the modulus method to extremal problems and to the Teichmüller theory. Some of results are known but we present them from the modern point of view and some of them appear here for the first time. One can find either the results in the proper development of the modulus method or its applications.

We omit some difficult proofs of theorems which one can find in already existing monographs, however, we prove various introductory theorems in Chapter 2 to give the reader the flavour of the modulus method. To facilitate matters along this line, we present some exercises (marked by $\star$) which are either simple examples or else theorems that we suggest to prove independently.

First of all, this book reflects the scientific interests and results of the author and does not pretend to be an exhaustive treatment of the field. In particular, we have deliberately omitted a number of results by R. Kühnau and his alumni. Most of them are covered in depth in [75]. We only mention here the work by F. Gardiner and H. Masur [44] on the relationship between a special embedding of a Teichmüller space by extremal lengths and the Thurston embedding that turns out to be the starting point of a new interesting direction. A thorough treatment of the proper development of the modulus method in connection with the extremal partitions one can find in the already mentioned books by G. Kuz'mina [78], M. Ohtsuka [107], K. Strebel [141], and of symmetrization and polarization in a series of articles by A. Solynin [133]–[136] and V. Dubinin [30]. So this work is neither a complete exposition of the classical theory nor a complete survey of the latest results. But we hope one can find here a step to new ways of investigation to make progress in modern and classical problems.

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