# Preface

The Geometrization Program of Thurston has been the driving force for research in 3-manifold topology in the last 25 years. This has inspired a surge of activity investigating hyperbolic 3-manifolds (and Kleinian groups), as these manifolds form the largest and least well-understood class of compact 3-manifolds. Familiar and new tools from diverse areas of mathematics have been utilised in these investigations – from topology, geometry, analysis, group theory and, from the point of view of this book, algebra and number theory. The important observation in this context is that Mostow Rigidity implies that the matrix entries of the elements of $\mathrm{SL}(2, \mathbb{C})$, representing a finite-covolume Kleinian group, can be taken to lie in a field which is a finite extension of $\mathbb{Q}$. This has led to the use of tools from algebraic number theory in the study of Kleinian groups of finite covolume and thus of hyperbolic 3-manifolds of finite volume. A particular subclass of finite-covolume Kleinian groups for which the number-theoretic connections are strongest is the class of arithmetic Kleinian groups. These groups are particularly amenable to exhibiting the interplay between the geometry, on the one hand and the number theory, on the other.

This book is designed to introduce the reader, who has begun the study of hyperbolic 3-manifolds or Kleinian groups, to these interesting connections with number theory and the tools that will be required to pursue them. There are a number of texts which cover the topological, geometric and analytical aspects of hyperbolic 3-manifolds. This book is constructed to cover arithmetic aspects which have not been discussed in other texts. A central theme is the study and determination of the invariant number field and the invariant quaternion algebra associated to a Kleinian group of finite covolume, these arithmetic objects being invariant with respect to the commensurability class of the group. We should point out that this book does not investigate some classical arithmetic objects associated to Kleinian groups via the Selberg Trace Formula. Indeed, we would suggest that, if prospective readers are unsure whether they wish to follow the road down which this book leads, they should dip into Chapters 4 and 5 to see what is revealed about examples and problems with which they are already familiar. Thus this book is written for an audience already familiar with the basic aspects of hyperbolic 3-manifolds and Kleinian groups, to expand their repertoire to arithmetic applications in this field. By suitable selection, it can also be used as an introduction to arithmetic Kleinian groups, even, indeed, to arithmetic Fuchsian groups.

We now provide a guide to the content and intent of the chapters and their interconnection, for the reader, teacher or student who may wish to be selective in choosing a route through this book. As the numbering is intended to indicate, Chapter 0 is a reference chapter containing terminology and background information on algebraic number theory. Many readers can bypass this chapter on first reading, especially if they are familiar with the basic concepts of algebraic number theory. Chapter 1, in essence, defines the target audience as those who have, at least, a passing familiarity with some of the topics in this chapter. In Chapters 2 to 5, the structure, construction and applications of the invariant number field and invariant quaternion algebra associated to any finite-covolume Kleinian group are developed. The algebraic structure of quaternion algebras is given in Chapter 2 and this is further expanded in Chapters 6 and 7, where, in particular, the arithmetic structure of quaternion algebras is set out. Chapter 3 gives the tools and formulas to determine, from a given Kleinian group, its associated invariant number field and quaternion algebra. This is then put to effect in Chapter 4 in many examples and utilised in Chapter 5 to investigate the geometric ramifications of determining these invariants.

From Chapter 6 onward, the emphasis is on developing the theory of arithmetic Kleinian groups, concentrating on those aspects which have geometric applications to hyperbolic 3-manifolds and 3-orbifolds. Our definition of arithmetic Kleinian groups, and arithmetic Fuchsian groups, given in Chapter 8, proceeds via quaternion algebras and so naturally progresses from the earlier chapters. The geometric applications follow in Chapters 9, 11 and 12. In particular, important aspects such as the development of the volume formula and the determination of maximal groups in a commensurability class form the focus of Chapter 11 building on the ground work in Chapters 6 and 7.

Using quaternion algebras to define arithmetic Kleinian groups facilitates the flow of ideas between the number theory, on the one hand and the geometry, on the other. This interplay is one of the special beauties of the subject which we have taken every opportunity to emphasise. There are other, equally meritorious approaches to arithmetic Kleinian groups, particulary via quadratic forms. These are discussed in Chapter 10, where we also show how these arithmetic Kleinian groups fit into the wider realm of general discrete arithmetic subgroups of Lie groups.

Some readers may wish to use this book as an introduction to arithmetic Kleinian groups. A short course covering the general theory of quaternion algebras over number fields, suitable for such an introduction to either arithmetic Kleinian groups or arithmetic Fuchsian groups, is essentially self-contained in Chapters 2, 6 and 7. The construction of arithmetic Kleinian groups from quaternion algebras is given in the first part of Chapter 8 and the main consequences of this construction appear in Chapter 11. However, if the reader wishes to investigate the role played by arithmetic Kleinian groups in the general framework of all Kleinian groups, then he or she must further assimiliate the material in Chapter 3, such examples in Chapter 4 as interest them, the remainder of Chapter 8, Chapter 9 and as much of Chapter 12 as they wish.

For those in the field of hyperbolic 3-manifolds and 3-orbifolds, we have endeavoured to make the exposition here as self-contained as possible, given the constraints on some familiarity with basic aspects of algebraic number theory, as mentioned earlier. There are, however, certain specific exceptions to this, which, we believe, were unavoidable in the interests of keeping the size of this treatise within reasonable bounds. Two of these are involved in steps which are critical to the general development of ideas. First, we state without proof in Chapter 0, the Hasse-Minkowski Theorem on quadratic forms and use that in Chapter 2 to prove part of the classification theorem for quaternion algebras over a number field. Second, we do not give the full proof in Chapter 7 that the Tamagawa number of the quotient $\mathcal{A}_A^1/A_k^1$ is 1, although we do develop all of the surrounding theory. This Tamagawa number is used in Chapter 11 to obtain volume formulas for arithmetic Kleinian groups and arithmetic Fuchsian groups. We should also mention that the important theorem of Margulis, whereby the arithmeticity and non-arithmeticity in Kleinian groups can be detected by the denseness or discreteness of the commensurator, is discussed, but not proved, in Chapter 10. However, this result is not used critically in the sequel. Also, on a small number of occasions in later chapters, specialised results on algebraic number theory are employed to obtain specific applications.

Many of the arithmetic methods discussed in this book are now available in the computer program Snap. Once readers have come to terms with some of these methods, we strongly encourage them to experiment with this wonderful program to develop a feel for the interaction between hyperbolic 3-manifolds and number theory.

Finally, we should comment on our method of referencing. We have avoided "on the spot" references and have placed all references in a given chapter in the Further Reading section appearing at the end of each chapter. We should also remark that these Further Reading sections are intended to be just that, and are, by no means, designed to give a historical account of the evolution of ideas in the chapter. Thus regrettably, some papers critical to the development of certain topics may have been omitted while, perhaps, later refinements and expository articles or books, are included. No offence or prejudice is intended by any such omissions, which are surely the result of shortcomings on the authors' part possibly due to the somewhat unsystematic way by which they themselves became acquainted with the material contained here.

We owe a great deal to many colleagues and friends who have contributed to our understanding of the subject matter contained in these pages. These contributions have ranged through inspiring lectures, enlightening conversations, helpful collaborations, ongoing encouragement and critical feedback to a number of lecture courses which the authors have separately given on parts of this material. We especially wish to thank Ted Chinburg, Eduardo Friedman, Kerry Jones, Darren Long, Murray Macbeath, Gaven Martin, Walter Neumann and Gerhard Rosenberger. We also wish to thank Fred Gehring, who additionally encouraged us to write this text, and Oliver Goodman for supplying Snap Data which is included in the appendix. Finally, we owe a particular debt of gratitude to two people: Dorothy Maclachlan and Edmara Cavalcanti Reid. Dorothy has been an essential member of the backroom staff, with endless patience and support over the years. More recently, Edmara's patience and support has been important in the completion of the book.

In addition to collaborating, and working individually, at our home institutions of Aberdeen University and the University of Texas at Austin, work on the text has benefited from periods spent at the University of Auckland and the Instituto de Matemática Pura e Aplicada, Rio de Janiero. Furthermore, we are grateful to a number of sources for financial support over the years (and this book has been several years in preparation) – Engineering and Physical Sciences Research Council (UK), Marsden Fund (NZ), National Science Foundation (US), Royal Society (UK), Sloan Foundation (US) and the Texas Advanced Research Program. The patient support provided by the staff at Springer-Verlag has also been much appreciated.

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