Preface

One of the key tools in classical representation theory is the fact that a representation of a group can also be viewed as an action of the group algebra on a vector space. This has been (one of) the motivations to introduce algebras, and modules over algebras. During the passed century, it has become clear that several different notions of module can be introduced, with a variety of applications in different mathematical disciplines. For example, actions by group algebras can also be used to develop Galois descent theory, with its applications in number theory. Graded modules originated from projective algebraic geometry. In fact a group grading can be considered as a coaction by the group algebra, i.e. the dual of an action. One may then consider various types of modules over bialgebras and Hopf algebras: Hopf modules (in integral theory), relative Hopf modules (in Hopf-Galois theory), dimodules (when studying the Brauer group). Perhaps the most important ones are the Yetter-Drinfeld modules, that have been studied in connection with the theory of quantum groups, the quantum Yang-Baxter equation, braided monoidal categories, and knot theory.

Frobenius fuctors generalize the classical concept of Frobenius algebra that appeared first 100 years ago in the work of Frobenius on representation theory. The study of Frobenius algebras has seen a revival during the passed five years, serving as an important tool in problems arising from different fields: Jones theory of subfactors of von Neumann algebras ([98], [100]), topological quantum field theory ([3], [8]), geometry of manifolds and quantum cohomology ([79], [129] and the references indicated there), the quantum Yang-Baxter equation ([15], [42]), and Yetter-Drinfeld modules ([49], [88]).

Separable functors are a generalization of the theory of separable field extensions, and of separable algebras. Separability plays a crucial role in several topics in algebra, number theory and algebraic geometry, for example in classical Galois theory, ramification theory, Azumaya algebras and the Brauer group theory, Hochschild cohomology and étale cohomology. A more recent application can be found in the Jones theory of subfactors of von Neumann algebras, already mentioned above with respect to Frobenius algebras.

In this monograph, we present - from a purely algebraic point of view - a unification schedule for actions and coactions and their properties, where we are mainly interested in generalizations of Frobenius and separability prop-

erties. The unification theory takes place at four different levels.
First, we have a unification on the level of categories of modules: *Doi-Koppinen modules* were introduced first, and all modules mentioned above can be viewed as special cases. *Entwined modules* arose from noncommutative geometry; they are at the same time more general and easier to deal with, and provide new fields of applications. Secondly, there is a unification at the level of functors between module categories: one can introduce morphisms of entwining structures, and then associate such a morphism a pair of adjoint functors. Many "classical" pairs of adjoint functors (the induction functor, forgetful functors, restriction of (co)scalars, functors forgetting a grading, and their adjoints) are in fact special cases of this construction. A third unification takes place at the level of the properties of these pairs of adjoint functors. Here the inspiration comes from two at first sight completely different algebraic notions, having their roots in representation theory: separable algebras and Frobenius. We give a categorical approach, leading to the introduction of *separable functors* and *Frobenius functors*. Not only this explains the at first sight mysterious fact that both separable and Frobenius algebras can be characterized using Casimir elements, it also enables us to prove Frobenius and separability type properties in a unified framework, with several new versions of Maschke's Theorem as a consequence.
The fourth unification is based on the theory of Yetter-Drinfeld modules, their relation with the quantum Yang-Baxter equation, and the FRT Theorem. The pentagon equation has appeared in the theory of duality for von Neumann algebras, in connection with C^*-algebras. Here we explain how they are related to Hopf modules. In a similar way, another nonlinear equation which we called the Long equation is related to the category of Long dimodules, that finds its origin in generalizations of the Brauer-Wall group. Finally, the FS equation can be used to characterize Frobenius algebras, as well as separable algebras, providing yet another explanation of the relationship between the two notions. For all these equations, we have a version of the FRT Theorem.
In Chapter 1, some preliminary results are given. We have included a Section about coalgebras and bialgebras, and one about adjoint functors. Section 1.2 deals with a classical treatment of Frobenius and separable algebras over fields, and we explain how they are connected to classical representation theory.
Chapter 2 provides a discussion of entwining structures and their representations, entwined modules, and we discuss how they generalize other types of modules and how they are related to the smash (co)product and the factorization problem of an algebra through two subalgebras. We also give the general pair of adjoint functors mentioned earlier. First properties of the category of entwined modules are discussed, for example we discuss when the category of entwined modules is a monoidal category. We use entwining structures mainly as a tool to unify all kinds of modules, but we want to point

out that they were originally introduced with a completely different motivation, coming from noncommutative geometry: one can generalize the notion of principal bundles to a very general setting in which the role of coordinate functions on the base is played by a general noncommutative algebra A, and the fibre of the principal bundle by a coalgebra C, where A and C are related by a map $\psi : A \otimes C \to C \otimes A$, called the entwining map, that has to satisfy certain compatibility conditions (see [32] and [33]). Entwined modules, as representations of an entwining structure, were introduced by Brzeziński [23], and he proved that Doi-Koppinen Hopf modules and, a fortiori, graded modules, Hopf modules and Yetter-Drinfeld modules are special cases. Entwined modules can also be applied to introduce coalgebra Galois theory, we come back to this in Section 4.8, where we also explain the link to descent theory.

The starting points of Chapter 3 are *Maschke's Theorem* from Representation Theory (a group algebra is semisimple if and only if the order of the group does not divide the characteristic of the base field), and the classical result that a finite group algebra is *Frobenius*. Larson and Sweedler have given Hopf algebraic generalizations of these two results, using integrals.

Both the Maschke and Frobenius Theorem can be restated in categorical terms. Let us first look at Maschke's Theorem. If we replace the base field k by a commutative ring, then we obtain the following result: if the order of the group G is invertible in k, then every exact sequence of kG-modules that splits as a sequence of k-modules is split as a sequence of kG-modules. If k is field, this implies immediately that kG is semisimple; in fact it turns out that all variations of Maschke's Theorem that exist in the literature admit such a formulation. In fact we have more: the kG-splitting maps are constructed deforming the k-splitting maps in a *functorial* way. A proper definition of functors that have this functorial Maschke property was given by Năstăsescu, Van den Bergh, and Van Oystaeyen [145]. They called these functors *separable functors* because for a given ring extension $R \to S$, the restriction of scalars functor is separable if and only if S/R is separable in the classical sense. A Theorem of Rafael [158] gives necessary and sufficient conditions for a functor with an adjoint to be separable: the unit (or counit) of the adjunction has to be split (or cosplit). We will see that the separable functor philosophy can be applied successfully to any adjoint pair of functors between categories of entwined modules. We will focus mainly on the functors forgetting the action and the coaction, as this is more transparent and leads to several interesting results.

A similar functorial approach can be used for the Frobenius property. It is well-known that a k-algebra S is Frobenius if and only if the restriction of scalars functors is at the same time a left and right adjoint of the induction functor. This has lead to the introduction of *Frobenius functors*: this is a functor which has a left adjoint that is also a right adjoint. An adjoint pair of Frobenius functors is called a Frobenius pair.

Let $\eta : 1 \to GF$ be the unit of an adjunction; as we have seen, to conclude that F is separable, we need a natural transformation $\nu : GF \to 1$. Our strategy will be to describe *all* natural transformations $GF \to 1$; we will see that they form a k-algebra, and that the natural transformations that split the unit are idempotents (*separability idempotents*) in this algebra.
A look at the definition of adjoint pairs of functors tells us that we have to investigate natural transformations $GF \to 1$ and $1 \to FG$; the difference is that the normalizing properties for the separability property and the Frobenius property are not the same. But still we can handle both problems in a unified framework, and this is what we will do in Chapter 3. In Chapter 4, we will apply the results from Chapter 3 in some important subcases. We have devoted Sections to relative Hopf modules and Hopf-Galois theory, graded modules, Yetter-Drinfeld modules and the Drinfeld double, and Long dimodules. For example, we prove that, for a finitely generated projective Hopf algebra H, the Drinfeld double $D(H)$ is a Frobenius extension of H if and only if H is unimodular.
Part I tells us that Hopf modules, Yetter-Drinfeld modules and Long dimodules over a Hopf algebra H can be regarded as special cases as Doi-Koppinen Hopf modules and entwined modules, and that a unified theory can be developed. In Part II, we look at these three types of modules from a different point of view: we will see how they are connected to three different nonlinear equations. The celebrated FRT Theorem shows us the close relationship between Yetter-Drinfeld modules and the *quantum Yang-Baxter* equation (QYBE) (see e.g. [115], [108], [128]). We will discuss how the two other types of modules, Hopf modules and Long dimodules, are related to other nonlinear equations. It comes as a surprise that the nonlinear equation related to the category of Hopf modules ${}_H\mathcal{M}^H$ is the *pentagon* (or *fusion*) equation, which is even older, and somehow more basic then the quantum Yang-Baxter equation. Using Hopf modules, we will present two different approaches to solving this equation: a first approach is to prove an FRT type Theorem for the pentagon equation; a second, completely different, approach was developed by Baaj and Skandalis for unitary operators on Hilbert spaces ([10]) and, more recently, by Davydov ([65]) for arbitrary vector spaces. We will conclude Chapter 6 with a few open problems that may have important consequences: from a philosophical point of view the theory presented herein views a finite dimensional Hopf algebra H simply as an invertible matrix $R \in \mathcal{M}_{n^2}(k) \cong \mathcal{M}_n(k) \otimes \mathcal{M}_n(k)$ that is a solution for the pentagon equation $R^{12}R^{13}R^{23} = R^{23}R^{12}$. Furthermore, in this case $\dim(H)|n$. This point of view could be crucial in reducing the problem of classifying finite dimensional Hopf algebras (currently in full development and using different and complex techniques) to the elementary theory of matrices from linear algebra. At this point a new Jordan theory (we called it *restricted Jordan theory*) has to be developed.
In Chapter 8, we will focus on the Frobenius-separability equation, all solu-

tions of which are also solutions of the braid equation. An FRT type theorem will enable us to clarify the structure of two fundamental classes of algebras, namely separable algebras and Frobenius algebras. The fact that separable algebras and Frobenius algebras are related to the same nonlinear equation is related to the fact that separability and Frobenius properties studied in Chapters 3 and 4 are based on the same techniques.
As we already indicated, the quantum Yang-Baxter equation has been intensively studied in the literature. For completeness sake, and to illustrate the similarity with our other nonlinear equations, we decided that to devote a special Chapter to it. This will also allow us to present some recent results, see Section 5.5.
The three authors started their common research on Doi-Koppinen Hopf modules in 1995, with a three month visit by the second and third author to Brussels. The research was continued afterwards within the framework of the bilateral projects “Hopf algebras and (co)Galois theory” and “Hopf algebras in Algebra, Topology, Geometry and Physics” of the Flemish and Romanian governments, and “New computational, geometric and algebraic methods applied to quantum groups and differential operators” of the Flemish and Chinese governments.
We benefitted greatly from direct and indirect contributions from - in alphabetical order - X-TO-Status: 00000003 Margaret Beattie, Tomasz Brzeziński, Sorin Dăscălescu, Jose (Pepe) Gómez Torrecillas, Bogdan Ichim, Bogdan Ion, Lars Kadison, Claudia Menini, Constantin Năstăsescu, Şerban Raianu, Peter Schauenburg, Mona Stanciulescu, Dragos Ştefan, Lucien Van hamme, Fred Van Oystaeyen, Yinhuo Zhang, and Yonghua Xu. Chapters 2 and 3 are based on an seminar given by the first author in Brussels during the spring of 1999. The first author wishes to thank Sebastian Burciu, Corina Calinescu and Erwin De Groot for their useful comments. Finally we wish to thank Paul Taylor for his kind permission to use his “diagrams” software.
A few words about notation: in Part I, we work over a commutative ring k; unadorned Hom, $\otimes$, $\mathcal{M}$ etc. are assumed to be taken over k. In Part II, we are always assuming that we work over a field k. For k-modules M and N, I_M will be the identity map on M, and $\tau:\ N\otimes M\to M\otimes N$ will be the switch map mapping $m\otimes n$ to $n\otimes m$. Also it is possible to read part II without reading part I first: one needs the generalities of Chapter 1, and the definitions in the first Sections of Chapter 2.

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