$\mathcal{C}$ is proper. Clearly, $\mathcal{C} \cap \mathcal{C}'$ is also the intersection of $\mathcal{C}$ with any other conic of the pencil generated by $\mathcal{C}$ and $\mathcal{C}'$:

$$Q(x, y, z) = Q'(x, y, z) = 0$$

if and only if

$$Q(x, y, z) = \lambda Q(x, y, z) + \lambda' Q'(x, y, z) = 0 \text{ for } \lambda' \neq 0.$$

Hence $\mathcal{C} \cap \mathcal{C}'$ is the intersection of $\mathcal{C}$ with a degenerate conic $\mathcal{C}''$ in the pencil. But $\mathcal{C}''$ consists of two secant lines or of a double line. Figure 15, in which the "heavy" lines represent double lines, shows the "four" expected intersection points (notice, however, that the pairs of conics shown on this figure do not define distinct pencils (see Exercise VI.43).

4. The cross-ratio of four points on a conic and Pascal's theorem

A conic $\mathcal{C}$ and a point m are given in a plane P. We consider the pencil of lines through m. Recall that this is the line $m^\star$ of $P^\star$.

Any point m of $\mathcal{C}$ defines a mapping

$$\pi_m : \mathcal{C} \longrightarrow m^\star$$

which, with any point n of $\mathcal{C}$, associates the line mn, with the convention that $\pi_m(m)$ is the tangent to $\mathcal{C}$ at m. Let us prove that π_m, or more exactly π_m^{-1} is a *parametrization* of the conic $\mathcal{C}$ by the projective line $m^\star$ (this is the first assertion in the next proposition).

Proposition 4.1. *For any point m of $\mathcal{C}$, the mapping π_m is a bijection. Moreover, if m and n are two points of $\mathcal{C}$, the composed mapping*

$$\pi_n \circ \pi_m^{-1} : m^\star \longrightarrow n^\star$$

is a homography.

Remark 4.2. There is a converse to this proposition: if m and n are two points in the plane and if $f : m^\star \to n^\star$ is a homography, there exists a conic $\mathcal{C}$ through m and n such that

$$\mathrm{Im}(\mathcal{C}) = \{D \cap f(D) \mid D \in m^\star\}$$

(see [**Ber77**] or [**Sid93**]). The set consisting of the proposition and its converse is often called the Chasles–Steiner theorem. The transformation $m^\star \to n^\star$ is shown in Figure 16.

Corollary 4.3. *Let $\mathcal{C}$ be a proper conic with nonempty image and let $m_1, \ldots, m_4$ be four points of $\mathcal{C}$. The cross-ratio $[mm_1, mm_2, mm_3, mm_4]$ does not depend on the choice of the point m on $\mathcal{C}$.*

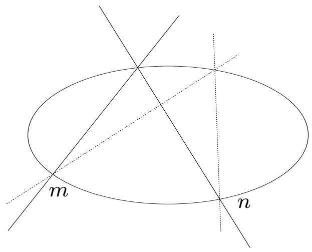

Fig. 16

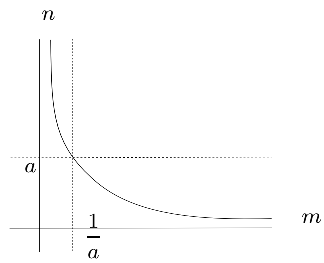

Fig. 17

This cross-ratio is called the *cross-ratio* of the four points $m_1, \ldots, m_4$ on the conic $\mathcal{C}$.

Proof of the proposition. To avoid using technical results on homographies, let us use coordinates. All we have to do is to choose these cleverly. We choose the line mn as the line at infinity (assuming $m \neq n$, otherwise there is nothing to prove). In the remaining affine plane, the conic $\mathcal{C}$ is a hyperbola. We choose the origin at its center and the two basis vectors as directing vectors of the asymptotes, scaling them so that the equation of the affine conic is $xy = 1$. The points m and n are the points at infinity of the x- and y-axes respectively, so that the pencil $m^\star$ consists of the lines $y = a$ parallel to the x-axis and the pencil $n^\star$ of the lines $x = b$.

Thus the mapping $\pi_n \circ \pi_m^{-1}$ is (see Figure 17)

$$\begin{array}{rcl} \mathbf{K} \cup \{\infty\} = m^\star & \longrightarrow & n^\star = \mathbf{K} \cup \{\infty\} \\ a & \longmapsto & \dfrac{1}{a} \end{array}$$

a projective transformation indeed... $\square$

We are now going to prove Pascal's theorem, special cases of which we have already encountered (see Exercise III.51). Pascal has proved his theorem in a circle (the general case of conics being a consequence of the special case of circles) probably using Menelaüs' theorem (Exercise I.37).

Theorem 4.4 (Pascal's theorem). *Let $\mathcal{C}$ be (the image of) a proper conic and let a, b, c, d, e and f be six points of $\mathcal{C}$. Then the intersection points of ab and de, bc and ef, cd and fa are collinear.*

Proof. Let us give names to some of the intersection points:

$$\begin{cases} x = bc \cap ed \\ y = cd \cap ef \\ z = ab \cap de \\ t = af \cap cd \end{cases}$$

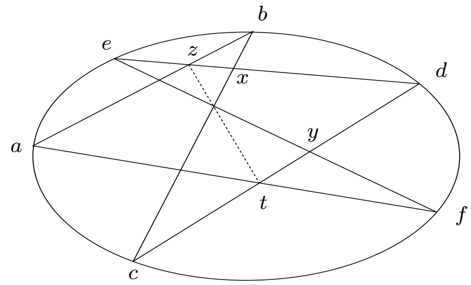

Fig. 18. The "mystic hexagram" of Pascal

(see Figure 18). We want to prove that the intersection point of the lines bc and ef lies on the line zt. To do this, let us calculate the cross-ratio $[z, x, d, e]$ (these four points are collinear on the line ed):

$$\begin{aligned}
[z, x, d, e] &= [bz, bx, bd, be] \\
&= [ba, bc, bd, be] \quad \text{(these are other names for the same lines)} \\
&= [fa, fc, fd, fe] \quad \text{(because } f \text{ is another point of } \mathcal{C}) \\
&= [t, c, d, y] \quad \text{(using the secant } cd).
\end{aligned}$$

We thus have the equality

$$[z, x, d, e] = [t, c, d, y].$$

Let m be the intersection point of zt and bc. The perspectivity[13] of center m (from the line ed to the line cd) maps z to t, x to c, d to d and thus, as it preserves the cross-ratio, it must map e to y, hence e, m and y are collinear. Therefore m, which belongs to bc and zt, also belongs to ef... and this is what we wanted to prove. □

5. Affine quadrics, *via* projective geometry

Let us consider now the real projective plane as the completion of some affine plane. We have said that any proper projective conic defines an affine conic. We have also seen that there is only one type of (nonempty) real proper conic over $\mathbf{R}$ and over $\mathbf{C}$. The question of the affine classification is thus reduced to the question of the relative positions of the conic and the line at infinity. We have already noticed that the intersection of the conic and the line at infinity is the set of isotropic vectors of the dehomogenized quadratic form q.

Figure 19 shows the affine conics, from top to bottom:

---

(13) See if necessary Exercise V.12.

– in the projective completion of the affine plane $\mathcal{P}$, together with the line at infinity,
– in the affine plane $\mathcal{P}$ itself,
– in the 3-dimensional vector space $\mathcal{P} \times \mathbf{R}$, where the affine plane is the $z = 1$-plane.

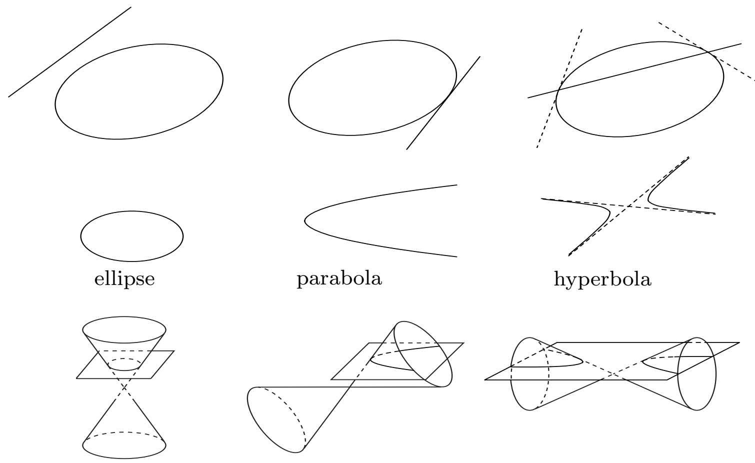

Fig. 19

For those of our readers who prefer equations to very clear pictures as those shown here, here are some equations. Let us choose the line of equation $z = 0$ as line at infinity and describe the conic by a homogeneous equation $Q(x, y, z) = 0$ for some quadratic form Q of signature $(2, 1)$.

There are three possibilities:

– The conic does not intersect the line at infinity, it is thus contained in the affine plane, this is an ellipse in this plane. In equations: the conic has an equation $x^2 + y^2 - z^2 = 0$, its affine part is the ellipse of equation $x^2 + y^2 = 1$.
– The conic is tangent to the line at infinity, its affine part is a parabola. In equations: take $x^2 - yz = 0$ for the conic; an equation of the affine part is $y = x^2$.
– The conic intersects the line at infinity at two points; its affine part is a hyperbola, the asymptotes are the tangents at the two points at infinity. In equations: $x^2 - y^2 - z^2 = 0$ for the projective conic, $x^2 - y^2 = 1$ for its affine part.

What we have just described can be rephrased, in a more pedantic way, as "two proper affine conics are (affinely) equivalent if and only if their points at infinity are two projectively equivalent 0-dimensional quadrics". We have used both the fact that there is only one type of projective conics and the affine classification: we knew, *e.g.,* that the affine conics with no point at infinity were all affinely equivalent... but the affine classification can also be deduced from the projective classification, as I show now, in any dimension.

Proposition 5.1. *To give the affine type of a proper affine quadric is equivalent to giving the types of the projectivized quadric and of its intersection with the hyperplane at infinity.*

Proof. Let $\mathcal{C}$ and $\mathcal{C}'$ be two proper quadrics in an affine space $\mathcal{E}$. We choose an origin O, on $\mathcal{C}$ to make life easier. We will use, as ever, the vector space $E \oplus \mathbf{K}$ in which $\mathcal{E}$ is the affine hyperplane $z = 1$ and O is the point $(0, 1)$. The projective completion of $\mathcal{E}$ is $P(E \oplus \mathbf{K})$, the hyperplane at infinity is $P(E)$.

An equation for $\mathcal{C}$ is $f(m) = q(\overrightarrow{OM}) + L(\overrightarrow{OM})$ or $f(u, 1) = q(u) + L(u)$. An equation for $\mathcal{C}'$ is $f'(u, 1) = q'(u) + L'(u) + c'$. The projective quadrics under consideration are the completions

$$Q(u, z) = q(u) + zL(u), \quad Q'(u, z) = q'(u) + zL'(u) + z^2c'$$

in $P(E \oplus \mathbf{K})$ and the quadrics at infinity $q(u)$, $q'(u)$ in $P(E)$.

Let us assume first that the two affine quadrics are equivalent. That is, there is an affine transformation

$$\psi : \mathcal{E} \longrightarrow \mathcal{E} \text{ such that } f' \circ \psi(M) = \lambda f(M)$$

(for some nonzero scalar λ). In our notation,

$$\psi(u, 1) = (v + \overrightarrow{\psi}(u), 1) \text{ where } v = \overrightarrow{O\psi(O)}$$

so that the condition is equivalent to

$$q' \circ \overrightarrow{\psi}(u) = \lambda q(u), \quad L' \circ \overrightarrow{\psi}(u) + 2\varphi'(v, \overrightarrow{\psi}(u)) = \lambda L(u), \quad q'(v) + L'(v) + c' = 0.$$

The first equation gives the equivalence of the quadrics at infinity, the last one just says that $\psi(O) \in \mathcal{C}'$.

Recall (from Proposition V-5.7) that ψ extends to a linear isomorphism

$$\begin{array}{rcl} \Psi : E \oplus \mathbf{K} & \longrightarrow & E \oplus \mathbf{K} \\ (u, z) & \longmapsto & (zv + \overrightarrow{\psi}(u), z). \end{array}$$

Hence

$$\begin{aligned} Q'(\Psi(u, z)) &= q'(zv + \overrightarrow{\psi}(u)) + zL'(zv + \overrightarrow{\psi}(u)) + z^2c' \\ &= q'(\overrightarrow{\psi}(u)) + z(L'(\overrightarrow{\psi}(u)) + 2\varphi'(v, \overrightarrow{\psi}(u))) \\ &\qquad + z^2(q'(v) + L'(v) + c'). \end{aligned}$$

Eventually, $Q' \circ \Psi(u, z) = \lambda Q(u, z)$, so that the two projective quadrics are indeed equivalent.

Conversely, assume that the projective quadrics are equivalent and the quadrics at infinity too. That is, there are two linear isomorphisms

$$\Phi : E \oplus \mathbf{K} \longrightarrow E \oplus \mathbf{K} \text{ and } \psi : E \longrightarrow E$$

such that

$$Q' \circ \Phi = \lambda Q \text{ and } q' \circ \psi = \mu q$$

for some nonzero scalars λ, μ that we can (and will) take equal to 1. Our task is to deduce that there exists a linear isomorphism

$$\Theta : E \oplus \mathbf{K} \to E \oplus \mathbf{K}$$

such that $\Theta \mid_E = \psi$ and $Q' \circ \Theta = Q$. Then Θ will define a projective transformation of $P(E \oplus \mathbf{K})$ preserving the hyperplane at infinity and hence an affine transformation of $\mathcal{E}$. As it extends ψ, it will be of the form

$$\Theta(u, 1) = (v + \psi(u), 1)$$

so that the same computation as above gives

$$f' \circ \Theta(u, 1) = f(u, 1)$$

and the affine quadrics will be equivalent.

Now the existence of Θ is a consequence of Witt's theorem (Theorem 8.10 below): $Q = Q' \circ \Phi$ is a quadratic form on $E \oplus \mathbf{K}$ and $\psi : E \to E$ satisfies $q' \circ \psi = q$. Hence

$$F = \Phi^{-1} \circ \psi : E \longrightarrow E \longrightarrow \Phi^{-1}(E) = E'$$

satisfies $Q \circ F = Q \mid_E$, in other words, F is an isometry from $(E, Q \mid_E)$ to $(E', Q \mid_{E'})$. Witt's theorem asserts[14] that it can be extended to a linear isomorphism

$$\widetilde{F} : E \oplus \mathbf{K} \longrightarrow E \oplus \mathbf{K}$$

that is an isometry for Q. Now we have

$$Q' \circ (\Phi \circ \widetilde{F}) = Q \circ \widetilde{F} = Q.$$

Therefore $\Theta = \Phi \circ \widetilde{F}$ is the expected isomorphism. $\square$

The classification of proper projective real quadrics with nonempty images in $\mathbf{P}_3(\mathbf{R})$ is very simple, as we have seen (Proposition 3.6): there are only two types,

$$\mathrm{I}: \quad x^2 + y^2 + z^2 - t^2 = 0 \quad \text{ and } \quad \mathrm{II}: \quad x^2 + y^2 - z^2 - t^2 = 0.$$

Notice that, for the homogenized form $q + t(L + ct)$ to have rank 4, the initial form q must have rank at least 2, hence the conic at infinity can be:

(14) This is the place where we need the quadrics to be proper (namely Q to be nondegenerate).

- empty (a),
- proper and nonempty (b),
- or degenerate with image a point (c) or two intersecting lines (d).

We get the following five affine types:

- ellipsoid (Ia), $x^2 + y^2 + z^2 = 1$,
- hyperboloids of one (IIb) or two (Ib) sheets,
- hyperbolic (IId) or elliptic (Ic) paraboloids

that are depicted in Figure 20.

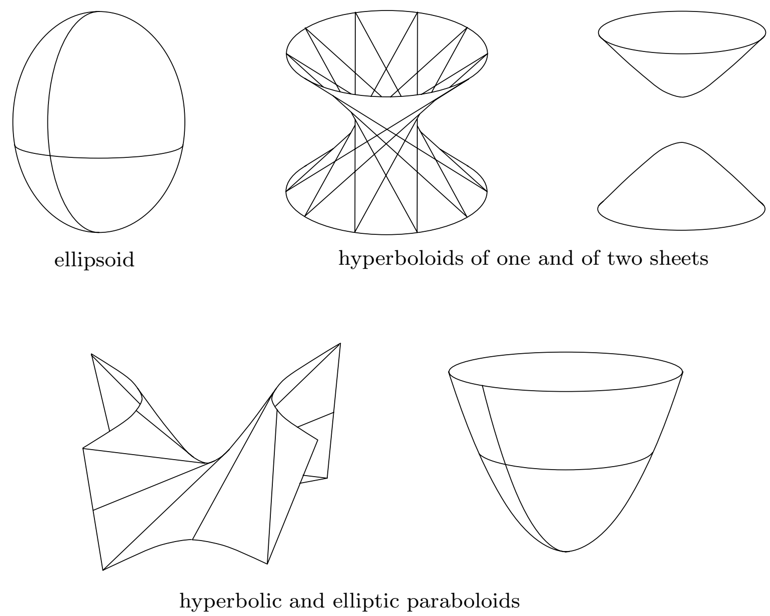

Fig. 20. Affine quadrics in $\mathbf{R}^3$

From the affine viewpoint, the only thing which is left to explain is the question of the center. We have:

Proposition 5.2. *The center of an affine quadric is the pole of the hyperplane at infinity.*

Remark 5.3. And this is why we were not able to find the center of a parabola: it was at infinity!

Proof. Let O be the center of the affine central quadric $\mathcal{C}$. This is a center of symmetry for $\mathcal{C}$. Let D be a line contained in the hyperplane at infinity. The point O and the line D span a projective plane that intersects $\mathcal{C}$ along a conic

$\mathcal{C}_D$. Let d be a line through O in this plane. It intersects $\mathcal{C}_D$ at two points m and m'. By symmetry, the tangent lines to $\mathcal{C}_D$ at m and m' are parallel, hence (see Remarks 3.5), the pole of d is on the line at infinity D. Thus the polar line of O with respect to $\mathcal{C}_D$ is D. Therefore the polar hyperplane of O with respect to $\mathcal{C}$ is indeed the hyperplane at infinity. □

6. Euclidean conics, *via* projective geometry

We are now going to find again the metric properties of affine Euclidean conics using projective geometry[15]. Therefore, our plane is now an affine Euclidean plane $\mathcal{P}$. As usual (see § V-3), we consider $\mathcal{P}$ as the affine plane of equation $z = 1$ in a vector space of dimension 3 (namely $\mathcal{P} \times \mathbf{R}$). Now, the Euclidean structure of $\mathcal{P}$ is important, we thus also endow E with a Euclidean structure that restricts to the given structure on $\mathcal{P}$:

– Concretely, an orthonormal affine frame of $\mathcal{P}$ being chosen, it defines a basis of the vector space E that we decide to be an orthonormal basis.
– Even more concretely,
  - the vector space E (with its basis) is $\mathbf{R}^3$ endowed, *e.g.,* with the standard Euclidean form $x^2 + y^2 + z^2$;
  - the affine plane $\mathcal{P}$, of equation $z = 1$, is directed by the (vector) plane P of equation $z = 0$, endowed with the standard Euclidean form $x^2 + y^2$;
  - we are now in the projective plane $P(E)$, the completion of $\mathcal{P}$.

In $P(E)$, the Euclidean form defines a conic, that of equation

$$x^2 + y^2 + z^2 = 0.$$

I have already mentioned that its image is empty... but that it will nevertheless play an important role. To begin with, let us give points to it: in order to do this, it suffices to complexify the space E. As we have already identified E with $\mathbf{R}^3$ by the choice of a basis, complexification is a benign process, we simply consider the coordinates (x, y, z) as being complex numbers.

The image of the projective projective conic of equation $x^2 + y^2 + z^2 = 0$ has become nonempty. Since we are interested in *affine* conics in $\mathcal{P}$, it is natural to wonder where our Euclidean conic intersects the line at infinity (that of equation $z = 0$). This is at the two points of homogeneous coordinates $(1, \pm i, 0)$.

[15] This exposition of the metric properties of conics *via* projective geometry is due to Plücker.

Definition 6.1. The two points I and J of $P(E)$ where the projective conic defined by the Euclidean structure intersects the line at infinity are called the *circular points*.

Remarks 6.2

- There is no use in making precise which is I and which is J.
- The coordinates of the points I and J are conjugated. In particular, if I satisfies an equation with real coefficients (for instance if I lies on a real conic), then J satisfies the same equation (J lies on the same conic).
- These points are imaginary *and* at infinity, two good reasons why some of our (young) readers have never met them before.

Before coming to conics, let us notice that the orthogonality (in the usual Euclidean sense) of the affine lines of $\mathcal{P}$, a relation between two lines $\mathcal{D}$ and $\mathcal{D}'$, can be expressed in terms of the circular points.

Proposition 6.3. *The affine lines $\mathcal{D}$ and $\mathcal{D}'$ are orthogonal if and only if, on the line at infinity, one has*

$$[I, J, \infty_{\mathcal{D}}, \infty_{\mathcal{D}'}] = -1.$$

Proof. We have already proved above (Remark 2 about polarity and duality) that, if a and b (here I and J) are the intersection points of a conic (here that defined by the Euclidean form) with a line (here the line at infinity), two points m and n (here $\infty_{\mathcal{D}}$ and $\infty_{\mathcal{D}'}$) represent orthogonal lines in the 3-dimensional vector space (here, thus, for the Euclidean scalar product) if and only if the equality

$$[a, b, m, n] = -1$$

holds. We just need now to remember that the lines in the vector plane of equation $z = 0$ that represent the points at infinity of the affine lines $\mathcal{D}$ and $\mathcal{D}'$ in the affine plane of equation $z = 1$ are the directions of $\mathcal{D}$ and $\mathcal{D}'$. They are thus orthogonal if and only if $\mathcal{D}$ and $\mathcal{D}'$ are. $\square$

Let us come now to conics. To begin with, we characterize circles among the conics.

Proposition 6.4. *Circles are the proper conics that pass through the circular points.*

Proof. Circles are the conics that have an equation of the form[16]

$$x^2 + y^2 + L(x, y) + c = 0$$

in an orthonormal frame, that is, conics such that the quadratic part of one of their equations is the Euclidean form itself. The points at infinity of a circle are given by the equations

$$z = 0 \text{ and } x^2 + y^2 = 0.$$

These are indeed the circular points.

Conversely, if a conic has an affine equation

$$q(x, y) + L(x, y) + c = 0,$$

to say that it passes through the circular points is to say that the quadratic form q vanishes at I and J and thus that it is proportional to the Euclidean form $x^2 + y^2$: the form

$$ax^2 + bxy + cy^2$$

vanishes at $(1, \pm i)$ if and only if

$$a \pm bi - c = 0,$$

that is, if and only if $b = 0$ and $a = c$, that is, if and only if $q(x, y) = a(x^2 + y^2)$. □

Remarks 6.5

- Compare Corollary 3.9 "five points of the plane always lie on a conic" with the fact that three points of the plane always lie on a circle.
- If a (real) conic contains the point I, it must contain the conjugate point J as well. This is thus a circle. Circles are actually the conics containing one of the circular points.
- A pencil of circles (in the sense of § III-4) is a pencil of conics containing the circular points. See also Exercise VI.52.
- While two conics can intersect at four points, two circles intersect at at most two points. This can be understood by a calculation: since the two equations have the same highest degree terms $(x^2 + y^2)$, to looking for the intersection of two circles amounts to look for the intersection of a circle and a line... the radical axis of the two circles. This can also be understood geometrically: the two circles intersect at the two circular points[17] and at their intersection points at finite distance.

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[16] There are circles consisting of a single point, or even empty circles, in this family, but this does not matter.

[17] The circular points lie on all circles; they can thus be considered as "universally cyclic".

Tangents through the circular points. Consider now a real affine conic $\mathcal{C}$ and its tangents through the circular points. They will give us all the expected metric properties.

Let us assume first that I (and thus J as well) lies on $\mathcal{C}$.

Proposition 6.6. *The tangents to a circle at the circular points intersect at the center of this circle.*

Proof. The tangents to $\mathcal{C}$ at I and J intersect at a point F which is a real point: their equations are conjugated to each other, thus their intersection point is real. As it lies at the intersection of the tangents at I and J, this point is the pole of the line IJ, alias the line at infinity (Figure 21). Hence, thanks to Proposition 5.2, F is the center of the circle! □

Let us assume now that neither I, nor J lies on $\mathcal{C}$, but that the line IJ, namely the line at infinity, is tangent to $\mathcal{C}$. We know that $\mathcal{C}$ is then a parabola.

Proposition 6.7. *Let $\mathcal{C}$ be a parabola. The line IJ is tangent to $\mathcal{C}$. The second tangent to $\mathcal{C}$ through I and the second tangent to $\mathcal{C}$ through J intersect at the focus F of $\mathcal{C}$.*

Let us finally assume that I and J do not lie on $\mathcal{C}$ and that the line IJ is not tangent to $\mathcal{C}$. Then $\mathcal{C}$ is an ellipse or a hyperbola.

Proposition 6.8. *Let $\mathcal{C}$ be a central conic that is not a circle. Let D_1 and D_2 be the tangents to $\mathcal{C}$ through I, D'_1 and D'_2 the respective conjugated complex lines. The lines D'_1 and D'_2 are the tangents to $\mathcal{C}$ through J. The intersection points F_1 and F_2 of D_1 and D'_1, D_2 and D'_2 (resp.) are the foci of $\mathcal{C}$. Their polars with respect to $\mathcal{C}$ are the corresponding directrices.*

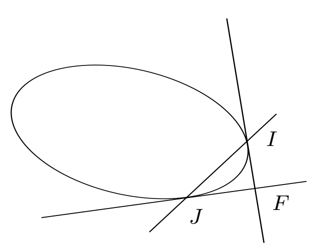

Fig. 21

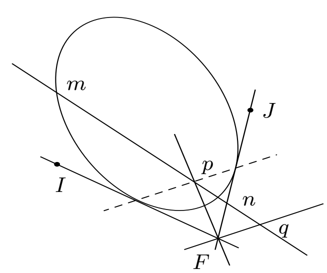

Fig. 22

Proof of Propositions 6.7 and 6.8. In the two cases, consider one of the tangents D through I and the conjugate line D', which is a tangent through J. Let F be the intersection of D and D' and let Δ be the polar of F with respect to $\mathcal{C}$. Choose any two points m and n on $\mathcal{C}$; let p be the intersection point of mn with Δ and q the point of mn defined by the equality

$$[m, n, p, q] = -1.$$

Thus p and q are orthogonal with respect to $\mathcal{C}$. Hence both F and q are on the polar of p with respect to $\mathcal{C}$, so that this polar is the line Fq (Figure 22). In particular, if $q' = Fq \cap \Delta$ and if a and b are the points of contact of D and D' with $\mathcal{C}$, we have

$$[a, b, p, q'] = -1, \text{ that is, } [FI, FJ, Fp, Fq] = -1.$$

Viewed on the line at infinity, this equality simply expresses the fact that the lines Fp and Fq are orthogonal. As the pencil (Fm, Fn, Fp, Fq) is harmonic, this means that Fp and Fq are the bisectors of (Fm, Fn) (see if necessary Exercise V.21). We then have:

$$\begin{aligned}\frac{Fm}{Fn} &= \frac{pm}{pn} \text{ (see Exercise III.26)}\\ &= \frac{d(m,\Delta)}{d(n,\Delta)} \text{ by similarity.}\end{aligned}$$

We fix the point n on $\mathcal{C}$ and we deduce $Fm = ed(m, \Delta)$, and this is what we wanted to prove. □

Remark 6.9. This is also why we have not been able to find a directrix for the circle: this was the line at infinity!

7. Circles, inversions, pencils of circles

In § III-4, we considered equations of circles to describe the pencils of circles in the affine Euclidean plane. In this chapter, we have considered more general equations of conics. I explain what was happening in § III-4 using (at last!) the "good" space of circles, the one that contains circles, lines... together with the points of the plane, a space in which a pencil of circles is, simply, a line.

More or less all what is explained here may be generalized to the space of spheres of an affine Euclidean space of dimension n (see [**Ber77**, Chapter 20]).

The space of circles. We are now in an affine Euclidean plane $\mathcal{P}$. Circles are the conics that have an equation whose quadratic part is a nonzero multiple of the Euclidean norm:

$$f(M) = \lambda \left\|\overrightarrow{OM}\right\|^2 + L(\overrightarrow{OM}) + c$$

or

$$f(M) = \lambda \left\| \overrightarrow{OM} \right\|^2 + 2\overrightarrow{OM} \cdot u + c$$

for some scalars λ (nonzero) and c and some vector u of the vector plane P directing $\mathcal{P}$.

Let us denote by $\mathcal{C}(\mathcal{P})$ and call *space of circles* of $\mathcal{P}$ the projective space deduced from the vector space $P \times \mathbf{R}^2$ of these equations. The $\mathbf{R}^2$ summand is the one where the ordered pair (λ, c) lives, the P summand the one where the vector u leaves. The space of circles is a real projective space of dimension 3, a projective subspace of the space of all affine conics in $\mathcal{P}$.

If such an equation indeed describes a circle, it must have the form

$$f(M) = \lambda \left(\left\| \overrightarrow{AM} \right\|^2 - R^2 \right)$$

for some point A, with λ nonzero and $R^2 > 0$. As

$$R^2 = \|u\|^2 - \lambda c,$$

we must have

$$\lambda \neq 0 \text{ and } \|u\|^2 - \lambda c > 0.$$

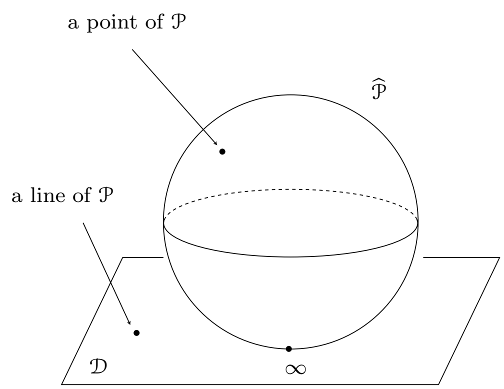

Fig. 23. The fundamental quadric in the space of circles

The space $\mathcal{C}(\mathcal{P})$ thus consists of equations of:

- genuine circles (corresponding to $\lambda \neq 0$ and $\|u\|^2 - \lambda c > 0$),
- "point-circles" (corresponding to $\lambda \neq 0$ and $\|u\|^2 - \lambda c = 0$),
- circles of imaginary radius[(18)], with empty image (corresponding to $\lambda \neq 0$ and $\|u\|^2 - \lambda c < 0$),
- affine lines of $\mathcal{P}$ (corresponding to $\lambda = 0$ and $u \neq 0$)

[(18)] Sometimes called "pseudo-real circles" (see [**DC51**]). These are real circles, since they have a real equation, without any real point, since their image is empty.

– and of the equation corresponding to $\lambda = 0$ and $u = 0$ (the image of the corresponding conic is empty).

The fundamental quadric. An important role is played by the projective quadric of equation

$$r(u, \lambda, C) = \|u\|^2 - \lambda c.$$

The image of this quadric consists of point-circles, to which the point $(0, 0, 1)$ has to be added. In other words, it is identified with the union of the set of points of $\mathcal{P}$ and an additional point, that we shall not fail to call a point at infinity.

This quadric is indeed our old $\widehat{\mathcal{P}} = \mathcal{P} \cup \{\infty\}$, even as a topological space: the signature of the quadratic form r is $(3, 1)$ since

$$r(u, \lambda, c) = \|u\|^2 + \frac{1}{4}\left[(\lambda - c)^2 - (\lambda + c)^2\right],$$

its image is contained in the affine subspace[19] $\lambda + c \neq 0$ of $\mathcal{P}$... where it (the image quadric) is clearly homeomorphic to a sphere of dimension 2.

Remark 7.1. This way, we have *embedded* the sphere $\widehat{\mathcal{P}} = \mathcal{P} \cup \{\infty\}$, which has already proved to be useful for the study of circles in § V-7, in the space of circles.

Let $\mathcal{D}$ be the projective hyperplane of equation $\lambda = 0$. We have seen that it consists of the affine lines of $\mathcal{P}$ ($\lambda = 0$ and $u \neq 0$), a set to which the point ∞ should be added.

One should notice that $\widehat{\mathcal{P}}$ and $\mathcal{D}$ are tangent at ∞ (Figure 23).

A summary. The space $\mathcal{C}(\mathcal{P})$ contains both the set $\mathcal{D}$ of the affine lines[20] of $\mathcal{P}$ (a projective plane) and the set of points of $\mathcal{P}$ (a quadric $\widehat{\mathcal{P}}$) tangent at a point (denoted ∞). This is another way to explain why the plane $\mathcal{P}$ has been completed, in § V-7, by the addition of a point rather than a line.

Circles and their centers. The quadric $\widehat{\mathcal{P}}$ divides the projective space $\mathcal{C}(\mathcal{P})$ into two components. I shall let the *interior* of $\widehat{\mathcal{P}}$ denote the set of points satisfying $r(u, \lambda, c) < 0$, namely, the set of circles of imaginary radii, and the *exterior* of $\widehat{\mathcal{P}}$ the set of genuine circles.

Any circle has a center... in other words, it is possible to associate, with any point C of $\mathcal{C}(\mathcal{P})$, the other point A where the line $C\infty$ intersects $\widehat{\mathcal{P}}$ (we are going to use systematically and intensively the intersection of the lines

(19) This is the affine space of dimension 3 in which all the picture of this section are drawn.
(20) The set of the affine lines $\mathcal{P}$ is here $\mathcal{D} - \{\infty\}$. This is the complement of a point in a projective plane (see Exercise V.41).

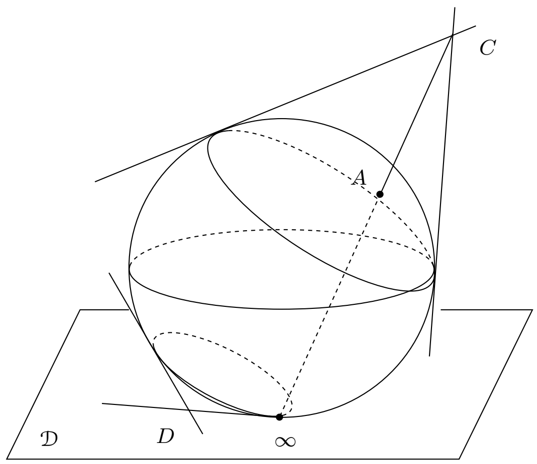

Fig. 24. Circles, their centers, lines

with the quadric). It should be noticed that the circles of imaginary radius have a center and that the lines also have a center... at ∞ (Figure 24).

However, the point C of $\mathcal{C}(\mathcal{P})$ is a circle of $\mathcal{P}$, hence a subset of $\widehat{\mathcal{P}}$ (something which is also shown by Figure 24): the points of C are indeed the points M of $\widehat{\mathcal{P}}$ such that the line CM is tangent to $\widehat{\mathcal{P}}$. The circle C of $\widehat{\mathcal{P}}$ is thus the intersection of $\widehat{\mathcal{P}}$ with the polar plane of the point C with respect to the quadric $\widehat{\mathcal{P}}$, a plane that I will denote by $C^{\perp}$.

One should also see in this description a confirmation of the fact that the imaginary circles have no points and that all the lines pass through ∞.

Inversions. Any circle C which is not a point defines an inversion, namely a transformation of $\widehat{\mathcal{P}}$. This is the involution that associates, with the point M of $\widehat{\mathcal{P}}$, the second intersection point M' of CM with $\widehat{\mathcal{P}}$ (Figure 25). It should be noticed that:

- his transformation indeed exchanges the center A of C, pole of the inversion, and the point ∞;
- the lines also define inversions, with pole at infinity (these are, of course, the reflections).

An inversion transforms any circle into a circle; we should also see it act on $\mathcal{C}(\mathcal{P})$. With Γ, the inversion of circle C associates the point Γ' such that

$$[C, N, \Gamma, \Gamma'] = -1,$$

where N denotes the intersection point of $C\Gamma$ with the plane $C^{\perp}$ defined by C (Figure 25). The inversion defined this way is a *projective transformation* of $\mathcal{C}(\mathcal{P})$.

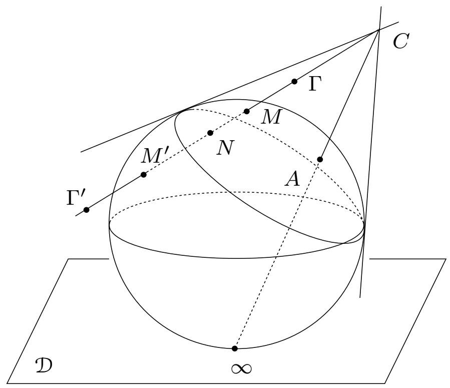

Fig. 25. Images of circles and points by an inversion

Orthogonal circles. The polarity with respect to $\widehat{\mathcal{P}}$ describes the orthogonality of circles: the circles C and C' are orthogonal if C' is in the polar plane $C^{\perp}$ of C (and conversely). For example, the points of C (if it has points) are point-circles orthogonal to C'.

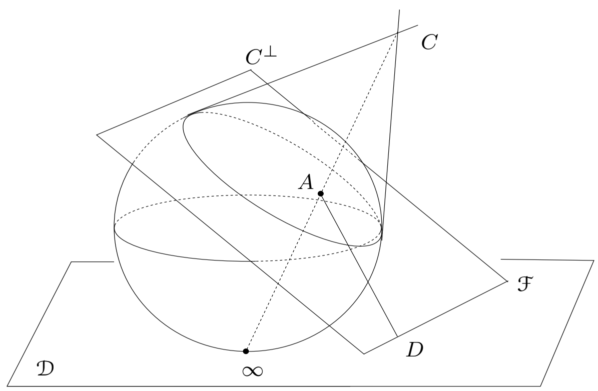

Fig. 26. Diameters of a circle

The plane $C^{\perp}$ intersects the space $\mathcal{D}$ of lines along a projective line $\mathcal{F}$ that is the set of diameters of C (Figure 26).

It should also be noticed that:

- the circles orthogonal to C are the fixed points of the inversion of circle C acting on the space of circles;
- the two notions of orthogonality we have for lines coincide: on $\mathcal{D}$, we have $\lambda = 0$ and the two orthogonalities are defined by the Euclidean scalar product in the plane.

Pencils of circles. We have already used the lines of $\mathcal{C}(\mathcal{P})$ quite a lot. A line is a set of circles, and this is what is called a *pencil* of circles. The orthogonal line $\mathcal{F}^{\perp}$ with respect to the quadric is the orthogonal pencil. The various types of pencils listed in § III-4 are simply the various ways in which a line $\mathcal{F}$ can intersect a quadric in a real projective space (see § 1).

- If the two intersection points are imaginary, all the circles of the pencil are genuine circles. This is a pencil of intersecting circles. The two base points are the points of $\mathcal{F}^{\perp} \cap \widehat{\mathcal{P}}$. The line $\mathcal{F}$ intersects $\mathcal{D}$ at a point D which is the radical axis of the pencil.
- Conversely, a line intersecting $\widehat{\mathcal{P}}$ at two points (other that ∞) is a nonintersecting pencil: these pencils contain point-circles and imaginary radius circles. One would have noticed that $\mathcal{F}$ intersects $\widehat{\mathcal{P}}$ at two points if and only if $\mathcal{F}^{\perp}$ does not intersect $\widehat{\mathcal{P}}$.
- A line $\mathcal{F}$ tangent to $\widehat{\mathcal{P}}$ at a point not equal to ∞ is a pencil of tangent circles.
- If $\mathcal{F}$ intersects $\widehat{\mathcal{P}}$ at ∞ and at another point A, all the circles of $\mathcal{F}$ have their center at A; the pencil $\mathcal{F}$ is thus a pencil of concentric circles. There is no radical axis, it has gone to infinity (and more precisely to ∞).
- If $\mathcal{F}$ is contained in $\mathcal{D}$ but does not pass through ∞, the pencil consists of lines. As was noticed above, these lines are the diameters of a circle (Figure 26); this is thus a pencil of concurrent lines. The line $\mathcal{F}^{\perp}$ contains ∞ and is as in the previous case.
- The last case is that of a line $\mathcal{F}$ contained in $\mathcal{D}$ and passing through ∞. This time, the lines of the pencil $\mathcal{F}$ intersect only at ∞; this is a pencil of parallel lines. The orthogonal pencil has the same nature.

Inverse of a pencil. An inversion transforms a pencil into a pencil (a projective transformation transforms a line into a line).

The circular group. Consider now the group $\mathcal{G}$ of projective transformations of $\mathcal{C}(\mathcal{P})$ that preserve the quadric $\mathcal{C}(\mathcal{P})$. They act on $\widehat{\mathcal{P}}$ and the restriction defines a homomorphism Φ from $\mathcal{G}$ to the group of all transformations of $\widehat{\mathcal{P}}$.

Theorem 7.2. *The restriction to $\widehat{\mathcal{P}}$ defines an isomorphism of $\mathcal{G}$ onto the circular group of $\mathcal{P}$.*

Proof. Notice first that $\widehat{\mathcal{P}}$ contains projective frames of $\mathcal{C}(\mathcal{P})$ and thus that any element of $\mathcal{G}$, which is a projective transformation, is well determined by its restriction to $\widehat{\mathcal{P}}$. Hence Φ is injective.

On the other hand, the circular group is generated by the inversions, of which we know that they come from $\mathcal{G}$. The image of Φ thus contains all the circular group.

We still need to check that this image is contained in the circular group. By definition, the elements of $\Phi(\mathcal{G})$ preserve the set of circles of $\widehat{\mathcal{P}}$. Theorem V-7.12 thus allows us to conclude. $\square$

8. Appendix: a summary of quadratic forms

We have already used examples of quadratic forms, the scalar products, in the Euclidean chapters of this book. In this appendix, $\mathbf{K}$ denotes one of the fields $\mathbf{R}$ or $\mathbf{C}$, E is a vector space of finite dimension over $\mathbf{K}$, $\varphi : E \times E \to \mathbf{K}$ is a symmetric bilinear form and eventually $q : E \to \mathbf{K}$ is the quadratic form associated with φ. It is said that φ is the *polar form* of q.

Recall that q is defined from φ by

$$q(x) = \varphi(x, x)$$

or φ from q by

$$\varphi(x, y) = \frac{1}{2}[q(x + y) - q(x) - q(y)]$$

or even by

$$\varphi(x, y) = \frac{1}{4}[q(x + y) - q(x - y)].$$

A nonzero vector x is *isotropic* if $q(x) = 0$. The set of isotropic vectors is a *cone* (if x is isotropic, λx is isotropic for any scalar λ), the *isotropy cone*.

Calculus. The polar form is essentially the differential of the quadratic form: fix a norm $\|\cdot\|$ on E and write

$$q(x + h) = q(x) + 2\varphi(x, h) + q(h).$$

The term $q(h)$ is quadratic (!) in h and hence is an $o(\|h\|)$. Thus q is differentiable at x (for all x) and its differential is the linear form

$$dq_x : h \longmapsto 2\varphi(x, h).$$

Calculation in a basis. Let $(e_1, \dots, e_n)$ be a basis of E. The form φ is determined by the values of the $\varphi(e_i, e_j)$, a square table of numbers that can be put in a matrix $A = (\varphi(e_i, e_j)_{1 \leqslant i,j \leqslant n})$. The matrix A is symmetric, which means that it satisfies the equality ${}^tA = A$. It is possible to write $\varphi(x, y)$ as a product of matrices (writing x for the column matrix that gives the coordinates of the vector x in the basis $(e_1, \dots, e_n)$):

$$\varphi(x, y) = {}^txAy.$$