Preface

The author believes that the theory of operator algebras should be viewed as a number theory in analysis. Number theory has been attracting the interest of humans ever since civilization began. Every culture in the world throughout history has given special meanings to certain numbers.

For example, a number may represent a position, quantity and/or quality. Today's civilization would be just impossible without numbers. People have been attracted to the mysteries of numbers throughout history. Accordingly, number theory is the oldest and most developed area of mathematics. Throughout the mathematical path to the present day, people have gradually learned properties of numbers. It is surprising to find that the number zero was not recognized until Hindus found it about one thousand years ago (although it is recognized that Mayans found it as well). Compared to this old field of mathematics, the theory of operator algebras is very new; its foundation was given by the pioneering work of J. von Neumann and his collaborator F. J. Murray in the early part of the twentieth century, i.e. in the thirties. Subsequent major development occurred only a decade later in the late forties and the early fifties. But since then it has marked steady progress reaching new heights today. The theory handles self-adjoint algebras of bounded operators on a Hilbert space. The advent of quantum physics at the turn of century forced one to consider non-commutative variables. One needed to broaden the concept of numbers. Integers, rational numbers, real numbers and complex numbers are all commutative. Among the few noncommutative mathematical systems available at the beginning of quantum mechanics were matrix algebras, which did not accommodate the needs of quantum physics because the Heisenberg uncertainty principle and/or Heisenberg commutation relation do not allow one to stay in the realm of finite matrices. One needs to consider algebras of operators on a Hilbert space of infinite dimension. Some of these operators correspond to important physical quantities. One has to include operators in the list of "numbers". Number theory tells us to put numbers in a field to study them more efficiently. Similarly, the theory of operator algebras puts operators of interest in an algebra and we study the algebra and its structure first. The infinite dimensionality of the underlying Hilbert space poses big challenges and also presents interesting new phenomenon which do not occur in the classical frame work. We have already seen some of them in the first volume. For example, the continuity of dimensions in a factor of type II_1 is one of them. The infinite dimensionality of our objects forces us to create sophisticated methods to handle approximations. Simple minded counting does not lead to the heart of the matter. For example, it is impossible to introduce a simple minded coordinate system in an infinite dimensional operator algebra, thus mathematical induction based on a basis does not fly. The early part of the theory, in the period of the forties through the early sixties were spent on this issue. Luckily there is a remarkable similarity between the theory of measures on a locally compact space and the theory of operator algebras. The first volume was devoted to the pursuit of this similarity.

The second volume of "Theory of Operator Algebras" is devoted to the study of the structure of von Neumann algebras of type III and their automorphism groups, cf. Chapter VI through Chapter XII; and the third volume is devoted to the study of the fine structure analysis of approximately finite dimensional factors and their automorphism groups, cf. Chapter XIII through Chapter XVIII. The last chapter, Chapter XIX, is an introduction to the theory of subfactors and their symmetries. One should note that the class of von Neumann algebras of type III is given by exclusion, i.e., by the absence of a non-trivial trace or a non-zero finite projection. This situation presented the major obstruction for the study of von Neumann algebras from the beginning of the subject until the advent of Tomita-Takesaki theory in the late sixties whilst many examples had been found to be of type III: the infinity of non-isomorphic factors were first established for factors of type III by Powers in 1967, [670], before the discovery of infinitely many non-isomorphic factors of type II_1 or II_∞, [635, 686], and most examples from quantum physics were shown to be of type III, [430]. It was the Tomita-Takesaki theory which broke the ice. It is still amazing that the subject defined by exclusion admits such a fine structural analysis since usually exclusion does not allow one to find any alternative and is viewed as pathological. Of course, a von Neumann algebra of type III had been pathological until we discover their fine structure. We will explore this in full detail through the second volume.

Each chapter has its own introduction which describes the content of that chapter and the basic strategy so that the reader can get a quick overview of the chapter.

In the second and third volume, we present two major items in the theory of von Neumann algebras: one is the analogy with integration theory on an abstract measure space and the other is the emphatic importance of automorphisms of algebras, i.e. we emphasize the symmetries of our objects following the modern point of view of E. Galois.

In general, the theory of von Neumann algebras is considered to be non-commutative integration. In Volume I, the similarity between von Neumann algebras and measure spaces are examined from the point of view of Banach space duality. In the second and third volume, non-commutative integration goes far beyond the analogy with ordinary integration. Since it is not our main interest to examine how ordinary integration should be formulated based on commutative von Neumann algebras, it is not discussed here in detail beyond a few comments. Still it is possible to develop a theory which covers the ordinary integration theory based on the operator algebra approach. In fact, such a theory has been explored by G. K. Pedersen, [653, Chapter 6], and it does eliminate pathological uninteresting measure spaces easily. The main difference between the operator algebra approach and the conventional approach to integration theory relies on the fact that in operator algebras one considers functions first, or equivalently variables, and then one views the underlying points as the spectrum of the variables; whilst in the ordinary approach one considers points first and views variables as functions on the set of points. We would like to point out here, however, that in practice we never observe points directly only approximately by successive evaluations of coordinates. Besides this philosophical difference, there is another major difference between the ordinary integration theory and the non-commutative integration theory which rests on the fact that a weight, a non-commutative counterpart of a σ-finite measure, gives rise to a one-parameter automorphism group, called the modular automorphism group, of the von Neumann algebra in question. This modular automorphism group can be considered as the time evolution of the system, i.e., in the non-commutative world a state determines the associated dynamics. The appearance of the modular automorphism group distinguishes our theory sharply from the classical theory. The modular automorphism group gives us abundant non-trivial information precisely when there is no trace on the algebra in question. Since the ordinary integration is a trace, the modular automorphism group is trivial in that case and cannot be appreciated. Furthermore, thanks to the Connes cocycle derivative theorem, Theorem VIII.3.3, the modular automorphism group is unique up to perturbation by a one unitary cocycle, which allows us to relate the structure of a von Neumann algebra of type III to that of the associated von Neumann algebra of type II_∞ equipped with a trace scaling one parameter automorphism group, cf. Chapter XII. As a byproduct of our non-commutative integration theory, a duality theorem attributed to Pontrjagin, van Kampen, Tannaka, Stinespring, Eymard, Saito and Tatsuuma, is presented in §3, Chapter VII. With this exception, no discussion of examples is presented in the second volume, Chapter VI through Chapter XII. Extensive discussions of examples and constructions of factors occupy the third volume starting in Chapter XIII and through Chapter XVIII.

The so-called Murray-von Neumann measure space construction of factors is closely investigated first in Chapter XIII yielding the Krieger construction of factors and the theory of measured groupoids. Systematic study of approximately finite dimensional factors occupies most of the third volume, cf. Chapter XIV through Chapter XIX. The theory is highlighted by the celebrated classification theorem of Alain Connes in the form of Theorems XVI.1.9, XVIII.1.1, XVIII.2.1 to which W. Krieger made a substantial contribution also, and XVIII.4.16 which requires one full section of preparation given by U. Haagerup, [550]. The last chapter, Chapter XIX, is devoted to an introduction to the theory of subfactors of an AFD factor created by V. F. R. Jones, and concludes with a classification theorem of Popa, Theorem XIX.4.16, for subfactors of an AFD factor of type II_1 with small indices.

The three volume book, "Theory of Operator Algebras", is a product of the author's research and teaching activities at the Department of Mathematics at University of California, Los Angeles, spanning the years from 1969 through the present time. It is important to mention the following: the author's visit to the University of Pennsylvania from 1968 through 1969 where the foundation of Tomita-Takesaki theory was established; the author's participation in various research activities which include several short and long visits to the University of Marseille-Aix-Luminy;

several short visits to RIMS of Kyoto University; one year participation in the Mathematical Physics Project of 1975–1976 at ZiF, University of Bielefeld; a full year participation in the operator algebra project of MSRI for 1984–1985; a one year visit to IHES, 1988–1989; two one month long participations in the one year project (1988–1989) on operator algebras at the Mittag-Leffler Institute; several visits to the University of New South Wales; and several month long visits to the Mathematics Institute of University of Warwick. The author would like to express here his sincere gratitude to these institutions and to the mathematicians who hosted him warmly and worked with him. Special thanks are due to Professor Richard V. Kadison with whom the author discussed the philosophy of the subject at length so many times, and to Professor Daniel Kastler who encouraged him in many ways and provided the opportunity to work with him and others including Alain Connes. Throughout the period of the preparation of the book, the author has been continuously supported by the National Science Foundation. Here he would like to record his appreciation of that support. The Guggenheim Foundation also gave the author support at a critical period of his career, for which the author is very grateful. The author also would like to express his gratitude to Professor Masahiro Nakamura who has constantly given his moral support to the author, to Professor Takashi Turumaru whose beautiful lectures inspired the author to be a functional analyst and to the late Professor Yoshinao Misonou under whose leadership the author started his career as a functional analyst. At the final stage of the preparation of the manuscript, Dr. Un Kit Hui and Dr. Toshihiko Masuda took pains to help the author to edit the manuscript. Although any misprints and mistakes are the author's responsibility, the author would like to thank them here.

Guidance to the Reader

Each chapter has its own introduction so that one can quickly get an overview of the content of the chapter. Theorems, Propositions, Lemmas and Definitions are numbered in one sequence, whilst formulas and equations are numbered in each section separately without reference to the section. Formulas (respectively, equations) are referred to by the formula number (respectively, equation number) alone if it is quoted in the same section, and by the section number followed by the formula number if it is quoted in a different section but in the same chapter, and finally by the chapter number, the section number and the formula number (respectively, equation number) if it is quoted in a different chapter. Some exercises are selected to help the reader to get information and techniques not covered in the main text, so they can be viewed as a supplement to the text. Those exercises taken directly literatures are marked by a †-sign, and the references are cited there.

To keep the book within a reasonable size, this three volume book does not include the materials related to the following important areas of operator algebras: K-theory for C^*-algebras, geometric theory of operator algebras such as cyclic cohomology, the classification theory of nuclear C^*-algebras, free probability theory and the advanced theory of subfactors. The interested readers are referred to the forthcoming books in this operator algebra series of encyclopedia.

Chapter XVII

Non-Commutative Ergodic Theory

§ 0 Introduction

The structure of a factor $\mathcal{M}$ is best understood through the study of symmetry of the factor, i.e. the study of the group $\mathrm{Aut}(\mathcal{M})$ of automorphisms of $\mathcal{M}$. We have been experiencing this through the structure analysis of factors of type III for instance. Apart from the modular automorphism groups, we do *not* have a systematic way of constructing an automorphism of a given factor $\mathcal{M}$. *It is still unknown if every separable factor of type* II_1 *admits an outer automorphism*. Thus we restrict ourselves to AFD factors in most cases, where we have many different ways of constructing automorphisms. The counter part in analysis of the theory of automorphisms is ergodic theory or the theory of non-singular transformations on a σ-finite standard measure space. As we have seen in Chapter XIII, the Rokhlin's tower theorem played a fundamental role in the theory of AF measured groupoids. We will present first the non-commutative analogue of this basic result in ergodic theory in §1. Unlike other parts, we need this theory for non-separable von Neumann algebras. It is interesting to note that whilst our primary interests are rest upon separable factors some results valid for non-separable von Neumann algebras are badly needed to advance our separable theory. One might be tempted to have a philosophical discussion about this irony. The results there will be applied to the analysis of outer conjugacy of single automorphisms in subsequent sections.

In §2, we will discuss the stability of outer conjugacy class of an aperiodic single automorphisms of a strongly stable factor. It is then applied to the outer conjugacy classification of a single approximately inner automorphism of a stable factor in §3. The outer conjugacy class of an approximately inner automorphism θ of an AFD factor $\mathcal{R}_0$ is determined by very simple invariants: outer period $p_0(\theta) \in \mathbf{Z}_+$ and obstruction $\mathrm{Ob}(\theta)$ a root of unity, Theorems 3.1 and 3.16.

§ 1 Non-Commutative Rokhlin Type Theorem

In the classical (or commutative) ergodic theory, Rokhlin's theorem plays a fundamental role. We extend this result to the non-commutative setting. As usual, we denote by $\mathcal{M}$ a von Neumann algebra. In this section, we **do not assume the separability** for $\mathcal{M}$, since we need the non-separable Rokhlin theorem later even if we handle only separable von Neumann algebras.

Definition 1.1. We say that $\theta \in \mathrm{Aut}(\mathcal{M})$ is *properly outer* if for every $e \in \mathrm{Proj}(\mathcal{M}^\theta)$, $e \neq 0$, the reduced automorphism $\theta^e \in \mathrm{Aut}(\mathcal{M}_e)$ is not inner.

Theorem 1.2. *For $\theta \in \mathrm{Aut}(\mathcal{M})$, the following four conditions are equivalent:*

(i) *θ is properly outer;*

(ii) *For every non-zero $e \in \mathrm{Proj}(\mathcal{M}^\theta)$, $\|\theta^e - \mathrm{id}\| = 2$;*

(iii) *For every $e \in \mathrm{Proj}(\mathcal{M})$, $e \neq 0$, and $\varepsilon > 0$, there exists $x \in \mathcal{M}_e$ such that $0 \leq x \leq 1$ and $\|x - \theta(x)\| > 1 - \varepsilon$;*

(iv) *For every $e \in \mathrm{Proj}(\mathcal{M})$, $e \neq 0$, and $\varepsilon > 0$, there exists $f \in \mathrm{Proj}(\mathcal{M})$, $0 \neq f \leq e$, such that $\|f\theta(f)\| < \varepsilon$.*

By Lemma XI.2.11, there exists a central projection $z \in \mathrm{Proj}(\mathcal{M}^\theta)$ such that θ^z is inner and θ^{1-z} is properly outer. Proposition XI.3.10 or rather its proof shows the equivalence of (i) and (ii).

Lemma 1.3. *Let* $\mathrm{Sp}(\theta)$ *be the spectrum of θ in the sense of Chapter* XI. *If $-1 \in \mathrm{Sp}(\theta)$, then for any $\varepsilon > 0$, there exists a non-zero $e \in \mathrm{Proj}(\mathcal{M})$ such that $\|e\theta(e)\| < \varepsilon$.*

PROOF: By assumption, for any $\delta > 0$, there exists $x \in \mathcal{M}$ such that $\|x\| = 1$ and $\|\theta(x)+x\| < \delta$. With $h = (x+x^*)/2$ and $k = (x-x^*)/2\mathrm{i}$, we have $\|\theta(h)+h\| < \delta$ and $\|\theta(k)+k\| < \delta$. Since $1 = \|x\| \leq \|h\|+\|k\|$, we have $\|h\| \geq 1/2$ or $\|k\| \geq 1/2$. Assume $\|h\| \geq 1/2$ (otherwise replace x by $\mathrm{i}x$), and let $a = \pm h/\|h\|$, where we choose the sign in such a way that $1 \in \mathrm{Sp}(a)$. Then we have $\|\theta(a) + a\| \leq 2\delta$ and $\|a\| = 1$. Let $e = \chi_{[1-\delta,1]}(a)$. We know $e \neq 0$ since $1 \in \mathrm{Sp}(a)$.

Representing $\mathcal{M}$ in a standard form, we assume that $\mathcal{M}$ acts on $\mathfrak{H}$ and there exists a unitary U on $\mathfrak{H}$ such that $UxU^* = \theta(x)$, $x \in \mathcal{M}$. Then we have $\theta(e)\mathfrak{H} = Ue\mathfrak{H}$. If $\xi \in e\mathfrak{H}$, then $\|a\xi - \xi\| \leq \delta\|\xi\|$. For any $\eta = U\xi' \in \theta(e)\mathfrak{H}$, we have

$$\|\theta(a)\eta - \eta\| = \|Ua\xi' - U\xi'\| \leq \delta\|\xi'\| = \delta\|\eta\|,$$

$$\|a\eta + \eta\| \leq \left\|\big[a + \theta(a)\big]\eta\right\| + \|\eta - \theta(a)\eta\| \leq 3\delta\|\eta\|.$$

Hence we have, for every $\xi \in e\mathfrak{H}$ and $\eta \in \theta(e)\mathfrak{H}$,

$$|(\xi \mid \eta) - (a\xi \mid \eta)| \leq \|(1 - a)\xi\|\|\eta\| \leq \delta\|\xi\|\|\eta\|,$$

$$|(\xi \mid a\eta) + (\xi \mid \eta)| \leq \|\xi\|\|a\eta + \eta\| \leq 3\delta\|\xi\|\|\eta\|,$$

so that we get

$$|(\xi \mid \eta)| \leq \frac{1}{2}\Big(|(\xi \mid \eta) - (a\xi \mid \eta)| + |(\xi \mid a\eta) + (\xi \mid \eta)|\Big) \leq 2\delta\|\xi\|\|\eta\|.$$

Therefore we conclude

$$\begin{aligned}\|e\theta(e)\| &= \sup\Big\{ \big|(\theta(e)e\xi \mid \eta)\big| : \xi, \eta \in \mathfrak{H},\ \|\xi\| = \|\eta\| = 1 \Big\} \\ &= \sup\Big\{ \big|(\xi \mid \eta)\big| : \xi \in e\mathfrak{H},\ \eta \in \theta(e)\mathfrak{H},\ \|\xi\| = \|\eta\| = 1 \Big\} \le 2\delta.\end{aligned}$$

Thus with $\delta = \varepsilon/2$, we get the conclusion. Q.E.D.

Lemma 1.4. *If $\theta \in \mathrm{Aut}(\mathcal{M})$ is outer, then for any $\varepsilon > 0$ there exists a non-zero $e \in \mathrm{Proj}(\mathcal{M})$ such that $\|e\theta(e)\| < \varepsilon$.*

PROOF: By the last lemma, we have only to consider the case that $-1 \notin \mathrm{Sp}(\theta)$. If the restriction of θ to the center $\mathcal{C}$ is non-trivial, then we can find a non-zero $e \in \mathrm{Proj}(\mathcal{C})$ such that $e \perp \theta(e)$. Hence we may assume that $\mathcal{C} \subset \mathcal{M}^\theta$. Suppose that θ^n is properly outer for every $n \in \mathbf{Z}$. Then the action: $n \in \mathbf{Z} \mapsto \theta^n \in \mathrm{Aut}(\mathcal{M})$ is free. Hence $\mathcal{M}' \cap (\mathcal{M} \rtimes_\theta \mathbf{Z}) = \mathcal{C}$, and therefore $\mathcal{C}$ is the center of $\mathcal{M} \rtimes_\theta \mathbf{Z}$. But $\Gamma(\theta)$ is the kernel of the restriction of $\hat{\theta}$, the dual action of $\mathbf{T}$, to $\mathcal{C}$, so that $\Gamma(\theta) = \mathbf{T}$. Hence $-1 \in \mathrm{Sp}(\theta)$, which contradicts the above assumption. Hence θ^n is not properly outer for some $n \in \mathbf{N}$. Let n be the smallest such positive integer. Here we have been assuming the proper outerness of θ by considering the reduced algebra, so that $n \ge 2$. By reducing the algebra $\mathcal{M}$, we may assume that $\theta^n \in \mathrm{Int}(\mathcal{M})$, i.e. $\theta^n = \mathrm{Ad}(u)$ for some $u \in \mathcal{U}(\mathcal{M})$. Since $\mathrm{Ad}\big(\theta(u)\big) = \theta\,\mathrm{Ad}(u)\theta^{-1} = \theta^n = \mathrm{Ad}(u)$, we have $\theta(u) = vu$ for some $v \in \mathcal{U}(\mathcal{C})$. If $v \ne 1$, then the abelian von Neumann subalgebra A generated by u and v is globally invariant under θ and $\theta|_{\mathcal{A}} \ne \mathrm{id}$. Hence $\mathcal{A}$ must contain a non-zero projection e such that $e \perp \theta(e)$. Thus, we may assume that $v = 1$, i.e. $\theta(u) = u$. We then choose an n-th root w of u in $\mathcal{M}^\theta$ to get $\bar{\theta} = \mathrm{Ad}(w)\cdot\theta$. Then we have $\bar{\theta}^n = \mathrm{id}$, and $\bar{\theta}, \bar{\theta}^2, \dots, \bar{\theta}^{n-1}$ are all properly outer by the choice of n. Let e be a spectral projection of w such that $\|we - \lambda e\| < \delta$ for a preassigned $\delta > 0$ and some $\lambda \in \mathbf{C}$ with $|\lambda| = 1$. Then we have $\|\bar{\theta}^e - \theta^e\| < 2\delta$, and therefore if $f \in \mathrm{Proj}(\mathcal{M}_e)$, $f \ne 0$, satisfies $\|f\bar{\theta}(f)\| < \delta$, then $\|f\theta(f)\| < 3\delta$. Thus, we can replace θ by $\bar{\theta}$ to look for a projection f with $\|f\theta(f)\| < 3\delta < \varepsilon$, which means that $\theta^n = \mathrm{i}$. Therefore, θ gives rise to a free action: $k \in \mathbf{Z}_n \mapsto \theta^k \in \mathrm{Aut}(\mathcal{M})$ of $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$. Thus the center $\mathcal{C}$ of $\mathcal{M}$ is the center of $\mathcal{M} \rtimes_\theta \mathbf{Z}_n$. Since $\Gamma(\theta)$ is the kernel of the restriction of the dual action $\hat{\theta}$ to the center $\mathcal{C}$ of $\mathcal{M} \rtimes_\theta \mathbf{Z}_n$, we have $\Gamma(\theta) = \hat{\mathbf{Z}}_n = \big\{\lambda \in \mathbf{T} : \lambda^n = 1\big\}$. Therefore, Proposition XI.2.26 entails the existence of a unitary $u \in \mathcal{M}$ such that $\theta(u) = \lambda u$ with $\lambda = \mathrm{e}^{2\pi\mathrm{i}/n}$. Thus, θ gives rise to a non-trivial automorphism of the abelian von Neumann subalgebra of $\mathcal{M}$ generated by u, so that there exists a non-zero spectral projection e of u such that $e \perp \theta(e)$. Q.E.D.

PROOF OF THEOREM 1.2:

(iv) $\Longrightarrow$ (iii): For any $e \in \mathrm{Proj}(\mathcal{M})$, we have $\|e - \theta(e)\| \ge \|e - e\theta(e)\| \ge 1 - \|e\theta(e)\|$. Thus (iii) follows immediately from (iv).

(iii) $\Longrightarrow$ (ii): If $\theta \le x \le e$ and $e \in \mathrm{Proj}(\mathcal{M}^\theta)$, then

$$\|\theta^e - \mathrm{id}\| \geq \left\|(\theta - \mathrm{id})\bigl(x - (e - x)\bigr)\right\| = 2\|x - \theta(x)\|;$$

so that (ii) also follows from (iii) easily.

(i) $\Longrightarrow$ (ii): This equivalence is nothing but a slight modification of Proposition XI.3.10.

(i) $\Longrightarrow$ (iv): Suppose that for some non-zero $p \in \mathrm{Proj}(\mathcal{M})$, we have

$$0 < \gamma = \inf\Bigl\{ \|e\theta(e)\| : e \in \mathrm{Proj}(\mathcal{M}_p),\ e \neq 0 \Bigr\}.$$

Choose $\varepsilon > 0$ so small that $0 < \varepsilon < \gamma/(1+\gamma)$, and $e \in \mathrm{Proj}(\mathcal{M}_p)$ such that $\gamma \leq \|e\theta(e)\| \leq \gamma + \varepsilon$. We claim that

$$\mathrm{Sp}_{\mathcal{M}_e}\bigl(e\theta(e)e\bigr) \subset \bigl[\gamma^2, (\gamma+\varepsilon)^2\bigr],$$

in particular $e\theta(e)e$ is invertible in $\mathcal{M}_e$. If this is not the case, then there exists non-zero $q \in \mathrm{Proj}(\mathcal{M}_e)$ such that $\|qe\theta(e)eq\| \leq \lambda < \gamma^2$. But we have

$$\|q\theta(q)\|^2 = \|q\theta(q)q\| \leq \|q\theta(e)q\| \leq \|eq\theta(e)qe\| = \|qe\theta(e)eq\| \leq \lambda < \gamma^2,$$

which contradicts the choice of γ. Now, consider the right polar decomposition $e\theta(e) = ku$ with $k = \bigl(e\theta(e)e\bigr)^{1/2}$. Then we have $u = k^{-1}e\theta(e),\ u^*u = \theta(e)$ and $uu^* = e$. Since $\mathrm{Sp}_{\mathcal{M}_e}(k) \subset [\gamma, \gamma+\varepsilon]$, we have $\|\gamma u - e\theta(e)\| < \varepsilon$. Set $\theta_1(x) = u\theta(x)u^*,\ x \in \mathcal{M}_e$. Then $\theta_1 \in \mathrm{Aut}(\mathcal{M}_e)$ and θ_1 is outer by the proper outerness of θ. By Lemma 1.4, there exists a non-zero $f \in \mathrm{Proj}(\mathcal{M}_e)$ with $\|f\theta_1(f)\| < \varepsilon$, which means that

$$\begin{aligned}\|f\theta(f)\| &= \|fe\theta(e)\theta(f)\| \leq \left\|f\bigl(e\theta(e) - \gamma u\bigr)\theta(f)\right\| + \gamma\|fu\theta(f)\| \\ &\leq \varepsilon + \gamma\|f\theta_1(f)\| < (1+\gamma)\varepsilon < \gamma.\end{aligned}$$

This again contradicts the choice of γ. Therefore, γ must be zero. Q.E.D.

Definition 1.5. We say that $\theta \in \mathrm{Aut}(\mathcal{M})$ is *aperiodic* if each $\theta^n,\ n \in \mathbf{N}$, is properly outer. This is equivalent to the fact that the action: $n \in \mathbf{Z} \mapsto \theta^n \in \mathrm{Aut}(\mathcal{M})$ of the integer group $\mathbf{Z}$ is free.

We now state our main result of this section, which is a non-commutative analogue of the Rokhlin tower theorem in ergodic theory.

Theorem 1.6. *Let $\mathcal{M}$ be a finite, not necessarily separable, von Neumann algebra equipped with a faithful normal tracial state τ. If $\theta \in \mathrm{Aut}(\mathcal{M})$ is aperiodic and leaves τ invariant, then for any $\varepsilon > 0$ and $n \in \mathbf{N}$, there exists a partition $\bigl\{F_j : 1 \leq j \leq n\bigr\}$ of identity in $\mathcal{M}$ such that*

$$\left.\begin{aligned}&\|\theta(F_j) - F_{j+1}\|_2 \leq \varepsilon, \quad 1 \leq j \leq n-1, \\ &\|\theta(F_n) - F_1\|_2 \leq \varepsilon.\end{aligned}\right\} \tag{1}$$

We prepare a couple of lemmas.

Lemma 1.7. *Suppose that* $0 < \varepsilon < 1/n!$ *for a fixed* $n \in \mathbf{N}$. *If* $f_1, f_2, \dots, f_n \in \mathrm{Proj}(\mathcal{M})$ *satisfy the inequality* $\| f_i f_j \| \le \varepsilon$, $1 \le i \ne j \le n$, *then there exist orthogonal* $e_1, e_2, \dots, e_n \in \mathrm{Proj}(\mathcal{M})$ *such that*

$$\begin{cases} e_j \sim f_j, \quad 1 \le j \le n, \qquad \displaystyle\sum_{j=1}^{n} e_j = \bigvee_{j=1}^{n} f_j, \\ \\ \|e_j - f_j\| < \varepsilon n!\,. \end{cases}$$

PROOF: Recall the analysis of two projections e and f in §1, Chapter V. We set

$$s(e, f) = |e - f|, \qquad c(e, f) = \big|e - (e \vee f - f)\big|. \tag{2}$$

This is a slight modification of the sine and the cosine by Definition V.1.42 to the extent that the new cosine is zero on $(e \vee f)^\perp$ whilst the old one is the identity on $(e \vee f)^\perp$. At any rate, we have

$$\left.\begin{aligned} & s(e, f)^2 + c(e, f)^2 = e \vee f, \\ & \|c(e, f)\| = \|ef\|, \qquad \|s(e, f)\| = \|e - f\|. \end{aligned}\right\} \tag{3}$$

If $\|ef\| < 1$, then $\|c(e, f)\| < 1$ and

$$F = e \vee f - e \sim f - e \wedge f = f,$$

$$\|F - f\| = \|s(F, f)\| = \Big\|s\big(e \vee f - e, f\big)\Big\| = \|c(e, f)\| = \|ef\| < 1.$$

We now assume the lemma for $n - 1$, and are going to prove the assertion for $f_1, f_2, \dots, f_n \in \mathrm{Proj}(\mathcal{M})$ with $\|f_j f_k\| < \varepsilon$, $j \ne k$. By assumption, there exist orthogonal $e_1, e_2, \dots, e_{n-1} \in \mathrm{Proj}(\mathcal{M})$ such that

$$e_j \sim f_j, \qquad \|e_j - f_j\| < (n-1)!\,\varepsilon, \qquad \sum_{j=1}^{n-1} e_j = \bigvee_{j=1}^{n-1} f_j = e.$$

Hence $\|e_j f_n - f_j f_n\| < (n-1)!\,\varepsilon$ for $j = 1, 2, \dots, n-1$, so that

$$\begin{aligned} \|ef_n\| = \Big\|\sum_{j=1}^{n-1} e_j f_n\Big\| &\le \sum_{j=1}^{n-1}\Big(\|(e_j - f_j) f_n\| + \|f_j f_n\|\Big) \\ &\le (n-1)(n-1)!\,\varepsilon + (n-1)\varepsilon < n!\,\varepsilon < 1. \end{aligned}$$

Therefore, with $e_n = e \vee f_n - e$, we get the desired projection e_n by the above arguments. Q.E.D.

Lemma 1.8. *Let* $\{\mathcal{M}, \theta, \tau\}$ *be as in Theorem* 1.6. *If* θ *acts on the center* $\mathcal{C}$ *trivially, then for any* $n \in \mathbf{N}$, $n > 1$ *and* $\delta > 0$ *there exist orthogonal* $f_1, f_2, \dots, f_n \in \mathrm{Proj}(\mathcal{M})$ *and* $v \in \mathcal{U}(\mathcal{M})$ *such that*

$$\|v - 1\|_1 \le \delta\tau\left(\sum_{j=1}^{n} f_j\right), \qquad v\theta(f_j)v^* = f_{j+1}, \quad 1 \le j \le n, \tag{4}$$

where $f_{n+1} = f_1$.

PROOF: Let $\delta' = \delta/12(n+1)$ and choose ε and m in such a way that $m = np \ge 16/\delta'^2$ and $0 < \varepsilon < \delta'/4mm!$. The aperiodicity of θ and Theorem 1.2 enable us to find a decreasing family of projections, $E_1 \ge E_2 \ge \cdots \ge E_m$, $E_j \in \mathrm{Proj}(\mathcal{M})$, such that $\|\theta^j(E_j)E_j\| < \varepsilon$, $1 \le j \le m$. Since $E_m \le E_j$, we have

$$\|\theta^j(E_m)E_m\| \le \|\theta^j(E_j)E_j\| < \varepsilon, \quad 1 \le j \le m.$$

With $e = E_m$, we get, for $1 \le i < j \le m$,

$$\|\theta^i(e)\theta^j(e)\| = \|\theta^{j-i}(e)e\| < \varepsilon.$$

Let $E = \bigvee_{j=1}^m \theta^j(e)$ and apply the last lemma to $\{\theta^j(e) : 1 \le j \le m\}$ in $\mathcal{M}_E$ to obtain an orthogonal system $\{e_j : 1 \le j \le m\} \subset \mathrm{Proj}(\mathcal{M}_E)$ such that for each $j = 1, 2, \dots, m$,

$$e_j \sim \theta^j(e), \qquad \sum_{j=1}^m e_j = E, \qquad \|e_j - \theta^j(e)\| \le m!\,\varepsilon \le \frac{\delta'}{4m}.$$

Put now

$$F = E \vee \theta(E) = E \vee \theta^{m+1}(e), \qquad Q = \mathcal{M}_F.$$

We then consider the normalized trace $\tau_Q = \tau/\tau(F)$, and L^p-norm $\|\ \ \|'_p$ on Q with respect to τ_Q. It then follows that

$$\|x\|'_p = \tau_Q\big(|x|^p\big)^{1/p} = \tau(F)^{-1/p}\|x\|_p, \quad x \in Q.$$

With $p = m/n$, put

$$\begin{cases} f_k = \displaystyle\sum_{j=0}^{p-1} e_{nj+k}, \quad 1 \le k \le n-1, \\ f_n = \displaystyle\sum_{j=1}^{p} e_{nj}, \qquad f_{n+1} = f_1. \end{cases}$$

We then have $\sum_{k=1}^n f_k = E$. Since we have for $1 \le j \le m-1$,

$$\|\theta(e_j) - e_{j+1}\| \le \|\theta(e_j) - \theta^{j+1}(e)\| + \|\theta^{j+1}(e) - e_{j+1}\| \le \frac{\delta'}{2m},$$

we have $\left\|\theta(e_j) - e_{j+1}\right\|_2' \le \delta'/2m,\ \ j = 1, 2, \ldots, m-1$. Hence

$$\left\|\theta(f_k) - f_{k+1}\right\|_2' \le \frac{\delta'}{2}, \quad 1 \le k \le n-1.$$

Since θ leaves the center $\mathcal{C}$ fixed, the center valued trace of $\mathcal{M}$ is invariant under θ, so that $f \sim \theta(f)$ for any $f \in \operatorname{Proj}(\mathcal{M})$, which means that $\{e_j : 1 \le j \le m\}$ are mutually equivalent and therefore $\tau_Q'(e_j) \le 1/m$ and $\tau_Q'\big(\theta(e_j)\big) \le 1/m$. Hence $\|e_1\|_2' \le 1/\sqrt{m}$ and $\left\|\theta(e_{np})\right\|_2' \le 1/\sqrt{m}$, which implies that

$$\begin{aligned}
\left\|\theta(f_n) - f_1\right\|_2' &= \left\|\sum_{j=1}^{p}\theta(e_{nj}) - \sum_{j=0}^{p-1} e_{nj+1}\right\|_2' \\
&\le \sum_{j=1}^{p-1}\left\|\theta(e_{nj}) - e_{nj+1}\right\|_2' + \left\|\theta(e_{np}) - e_1\right\|_2' \\
&\le \frac{p\delta'}{2m} + \frac{2}{\sqrt{m}} \le \delta'.
\end{aligned}$$

Therefore, we get

$$\left\|\theta(f_k) - f_{k+1}\right\|_2' \le \delta', \quad 1 \le k \le n.$$

Applying Lemma XIV.2.1, we find unitaries $u_1, \ldots, u_n \in \mathcal{U}(Q)$ such that

$$u_j\theta(f_j)u_j^* = f_{j+1}, \qquad \left|u_j - F\right| \le \sqrt{2}\left|f_{j+1} - \theta(f_j)\right|.$$

With $v_j = u_j\theta(f_j)$, we get

$$\left\|v_j - f_{j+1}\right\|_2' = \left\|f_{j+1}(u_j - F)\right\|_2' \le \sqrt{2}\left\|f_{j+1} - \theta(f_j)\right\|_2' \le \sqrt{2}\delta',$$

and $v_j^*v_j = \theta(f_j),\ \ v_jv_j^* = f_{j+1}$. We next choose $v_0 \in Q$ such that $v_0^*v_0 = F - \theta(E) \precsim \theta^{m+1}(e)$ and $v_0v_0^* = F - E$. Then

$$\|v_0\|_2' \le \left\|\theta(e_m)\right\|_2' \le \frac{1}{\sqrt{m}} \le \frac{\delta'}{2}.$$

With $V = v_0 + v_1 + \cdots + v_n \in \mathcal{U}(Q)$, we have

$$\begin{cases} V\theta(f_j)V^* = f_{j+1}, \quad 1 \le j \le n, \\[1ex] \left\|V - F\right\|_2' \le \sqrt{2}(n+1)\delta' \le \dfrac{\delta}{4}. \end{cases}$$

We now set $v = V + (1 - F) \in \mathcal{U}(\mathcal{M})$ and obtain:

$$v\theta(f_j)v^* = f_{j+1}, \quad 1 \le j \le n,$$

$$\begin{aligned}
\|v - 1\|_1 &= \tau\big(|v-1|\big) = \tau\big(|V - F|\big) = \tau(F)\tau_Q\big(|V-F|\big) = \tau(F)\left\|V - F\right\|_1' \\
&\le \tau(F)\left\|V - F\right\|_2' \le \frac{\delta\tau(F)}{2} \le \delta\tau\left(\sum_{j=1}^{n} f_j\right).
\end{aligned}$$

This completes the proof. Q.E.D.

PROOF OF THEOREM 1.6: We first prove the theorem in the case that θ leaves the center $\mathcal{C}$ of $\mathcal{M}$ fixed. We fix $n > 1$ and $0 < \delta < 1$. Let X be the set of all $x = \big(F_1, F_2, \dots, F_n; V\big)$ such that

a) $\big\{F_1, \dots, F_n\big\}$ are mutually orthogonal equivalent projections;

b) $V \in \mathcal{U}(\mathcal{M})$ and $\big\|V - 1\big\|_1 \le \delta\tau\big(\sum_{j=1}^n F_j\big)$;

c) $V\theta(F_j)V^* = F_{j+1}$, where $F_{n+1} = F_1$.

We then define a relation $x = \big(F_1, F_2, \dots, F_n; V\big) \le x' = \big(F_1', F_2', \dots, F_n'; V'\big)$ by the following:

(α) $F_j \le F_j'$, $1 \le j \le n$;

(β) $\big\|V - V'\big\|_1 \le \delta\tau\big(\sum_{j=1}^n (F_j' - F_j)\big)$.

It then follows that the relation "$\le$" in X is an ordering. If $\mathfrak{Y}$ is a totally ordered subset of X, then the map: $x = \big(F_1, F_2, \dots, F_n; V\big) \in \mathfrak{Y} \mapsto \tau\big(\sum_{j=1}^n F_j\big) \in [0, 1]$ gives an order isomorphism of $\mathfrak{Y}$ into $[0, 1]$, so that $\mathfrak{Y}$ contains a cofinal sequence $\{x_m\}$. Let $x_m = \big(F_1^m, F_2^m, \dots, F_n^m; V_m\big)$. By (β), we have

$$\big\|V_{m+1} - V_m\big\|_1 \le \delta\tau\bigg(\sum_{j=1}^n \big(F_j^{m+1} - F_j^m\big)\bigg),$$

so that $\sum_{m=1}^\infty \big\|V_{m+1} - V_m\big\|_1 \le \delta$. The L^1-completeness of $\mathcal{U}(\mathcal{M})$ implies the convergence $\lim_{m\to\infty} V_m = V \in \mathcal{U}(\mathcal{M})$. Also each $\big\{F_j^m : m \in \mathbf{N}\big\}$ is an increasing sequence of projections, so that $F_j = \lim_{m\to\infty} F_j^m \in \mathrm{Proj}(\mathcal{M})$ converges. By continuity, $\big(F_1, F_2, \dots, F_n; V\big)$ satisfies (a), (b), and (c), and dominates all $\big(F_1^m, F_2^m, \dots, F_n^m; V_m\big)$. Thus, X is an inductive set, which admits therefore a maximal element $x = \big(F_1, F_2, \dots, F_n; V\big)$. We are going to show $\sum_{j=1}^n F_j = 1$.

Suppose that $E = 1 - \sum_{j=1}^n F_j \ne 0$. Let $Q = \mathcal{M}_E$. By (c), we have $V\theta(E)V^* = E$, so that $\theta' = \mathrm{Ad}(V) \circ \theta$ leaves Q globally invariant. Hence we consider θ' on Q. It then follows that θ' is aperiodic on Q too. By the last lemma, there exist $f_1, f_2, \dots, f_n \in \mathrm{Proj}(Q)$ and $v \in \mathcal{U}(Q)$ such that $f_i \perp f_j$, $i \ne j$, and $v\theta(f_j)v^* = f_{j+1}$, where $f_{n+1} = f_1$, and furthermore $\big\|v - E\big\|_1' \le \delta\tau_Q\big(\sum_{j=1}^n f_j\big)$, where $\tau_Q = \tau/\tau(E)$ and $\|\cdot\|_1'$ means the L^1-norm on Q relative to τ_Q. Put $F_j' = F_j + f_j$, $1 \le j \le n$ and $V' = \big(v + (1-E)\big)V$. Then $\big(F_1', F_2', \dots, F_n'; V'\big) = x'$ satisfies (a) and (c). Furthermore, we have

$$\big\|v + (1 - E) - 1\big\|_1 = \tau(E)\big\|v - E\big\|_1' \le \delta\tau(E)\tau_Q\bigg(\sum_{j=1}^n f_j\bigg) = \delta\tau\bigg(\sum_{j=1}^n f_j\bigg);$$

$$\big\|V' - V\big\|_1 = \Big\|\big(v + (1 - E) - 1\big)V\Big\|_1 \le \delta\tau\bigg(\sum_{j=1}^n f_j\bigg).$$

Hence we get

$$\begin{aligned}\|V'-1\|_1 &\le \|V'-V\|_1+\|V-1\|_1\\ &\le \delta\tau\Big(\sum_{j=1}^n f_j\Big)+\delta\tau\Big(\sum_{j=1}^n F_j\Big)=\delta\tau\Big(\sum_{j=1}^n F_j'\Big).\end{aligned}$$

Therefore, x' belongs to X and dominates properly x, contradicting the maximality of x. Thus we must have $\sum_{j=1}^n F_j = 1$, and now

$$\begin{aligned}\|\theta(F_j)-F_{j+1}\|_2^2 &\le \|\theta(F_j)-F_{j+1}\|\,\|\theta(F_j)-F_{j+1}\|_1\\ &\le 2\Big\|\theta(F_j)-V\theta(F_j)V^*\Big\|_1 \le 4\|V-1\|_1 \le 2\delta.\end{aligned}$$

This completes the proof for the case that θ is trivial on $\mathcal{C}$.

General Case: Let $\bar\theta = \theta|_{\mathcal{C}}$. If $\bar\theta$ is aperiodic, then the usual Rokhlin theorem, Lemma XIII.3.23, takes care of the existence of $\{F_1,\dots,F_n\}$. Hence by decomposing $\mathcal{M}$ into direct sum according to the period of $\bar\theta$, we may assume that $\bar\theta$ is periodic with period $p \ge 1$. By the previous arguments, we have only to consider the case $p > 1$. Choose $c \in \mathrm{Proj}(\mathcal{C})$ such that $\{\theta^j(c) : 0 \le j \le p-1\}$ is a partition of identity. Let $\mathcal{N} = \mathcal{M}_c$ and $\theta' = \theta^p\big|_{\mathcal{N}}$. It follows that θ' is a periodic and leaves the center $\mathcal{C}_{\mathcal{N}}$ fixed. Hence the previous arguments apply to θ' to guarantee that for any $\varepsilon' > 0$ there exists a partition $\{G_1, G_2,\dots,G_n\}$ of identity in $\mathcal{N}$ such that $\|\theta'(G_j)-G_{j+1}\|_2 < \varepsilon'$ where $G_{n+1} = G_1$. Put

$$H_{sp+r} = \theta^r(G_s),\quad 0\le r<p,\quad 0\le s<n,$$

where $G_0 = G_n$. Then $\{H_j : 0 \le j < np\}$ is a partition of identity in $\mathcal{M}$ and $\|\theta(H_k)-H_{k+1}\|_2 \le \varepsilon'$. Now put

$$F_s = \sum_{k=0}^{p-1} H_{kn+s},\quad 1\le s\le n,$$

with $H_{pn} = H_0$. We now have

$$\|\theta(F_s)-F_{s+1}\|_2 \le p\varepsilon',\quad s=1,2,\dots,n,$$

with $F_{n+1} = F_1$. Hence with $\varepsilon' = \varepsilon/p$, we complete the proof. Q.E.D.

§ 2 Stability of Outer Conjugacy Classes

Let $\mathcal{M}$ be a separable strongly stable factor unless we say specifically otherwise. By definition, we have $\mathcal{M} \cong \mathcal{M}\,\overline{\otimes}\,\mathcal{R}_0$ where $\mathcal{R}_0$ is an AFD II$_1$-factor. We study in this section tensor product perturbations of automorphisms of $\mathcal{M}$ by automorphisms of $\mathcal{R}_0$.

We fix a free ultrafilter $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$, and consider $\mathcal{M}_\omega$ as in §4, Chapter XIV. By the strong stability, $\mathcal{M}_\omega$ is a σ-finite von Neumann algebra of type II_1, Theorem XIV.4.18. Each $\theta \in \mathrm{Aut}(\mathcal{M})$ gives rise to an element $\theta_\omega \in \mathrm{Aut}(\mathcal{M}_\omega)$ naturally and the correspondence $\varepsilon_\omega \colon \theta \in \mathrm{Aut}(\mathcal{M}) \mapsto \theta_\omega \in \mathrm{Aut}(\mathcal{M}_\omega)$ is a homomorphism. The kernel of ε_ω is independent of the choice of ω and denoted by $\mathrm{Cnt}(\mathcal{M})$. We say that θ is *centrally trivial* if $\theta \in \mathrm{Cnt}(\mathcal{M})$. The *outer period* $p_0(\theta)$ is by definition the period of $\varepsilon(\theta)$ in $\mathrm{Out}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})/\mathrm{Int}(\mathcal{M})$, where ε is the quotient homomorphism of $\mathrm{Aut}(\mathcal{M})$ onto $\mathrm{Out}(\mathcal{M})$. Similarly, we define $p_a(\theta)$, the *asymptotic outer period* of $\theta \in \mathrm{Aut}(\mathcal{M})$, to be the period of $\varepsilon_\omega(\theta)$, and set $p_a(\theta) = 0$ if $\theta^n \notin \mathrm{Cnt}(\mathcal{M})$ for $n \neq 0$. We are going to use the notations from §4, Chapter XIV freely.

Lemma 2.1. *If $\mathcal{M}$ is a strongly stable separable factor, then for any separable von Neumann subalgebra $\mathcal{P}$ of $\mathcal{M}_\omega$ the relative commutant $\mathcal{P}' \cap \mathcal{M}_\omega$ is always of type* II_1 .

PROOF: Let $\{X_n\}$ be a sequence which is σ-strongly dense in the unit ball of $\mathcal{P}$. Let $\{x_n(m)\}$ be a strongly ω-central sequence representing X_n. We fix a dense sequence $\{\varphi_n\}$ in $\mathfrak{S}_* = \mathfrak{S}_*(\mathcal{M})$ and a faithful $\varphi \in \mathfrak{S}_*$. For each $m \in \mathbf{N}$, we choose a 2×2-matrix unit $\left\{ e_{i,j}(m) : 1 \leq i, j \leq 2 \right\}$ such that

$$\begin{cases} \left\| \left[e_{i,j}(m),\, x_n(m) \right] \right\|_\varphi^\sharp \leq \dfrac{1}{m}, \quad 1 \leq n \leq m, \\ \left\| \left[e_{i,j}(m),\, \varphi_k \right] \right\| \leq \dfrac{1}{m}, \quad 1 \leq k \leq m. \end{cases}$$

Such $\{e_{i,j}(m)\}$ exists by the strong stability. Then $\{e_{i,j}(m)\}$ is strongly central and with $E_{i,j} = \pi_\omega(\{e_{i,j}(n)\})$, where π_ω is the map defined in §4, Chapter XIV, we have

$$\left\| \left[E_{i,j}, X_n \right] \right\|_{2,\omega} = \lim_{m \to \omega} \left\| \left[e_{i,j}(m), x_n(m) \right] \right\|_\varphi^\sharp \leq \lim_{m \to \omega} \frac{1}{m} = 0.$$

Hence $\{E_{i,j}\}$ is a 2×2-matrix unit in $\mathcal{P}' \cap \mathcal{M}_\omega$. Therefore, the relative commutant of every separable von Neumann subalgebra of $\mathcal{M}_\omega$ contains a 2×2-matrix unit, which means by induction that $\mathcal{P}' \cap \mathcal{M}_\omega$ contains a sequence of mutually commuting 2×2-matrix units. Hence $\mathcal{P}' \cap \mathcal{M}_\omega$ must be of type II_1. Q.E.D.

Lemma 2.2. *Let $\mathcal{M}$ be a strongly stable separable factor. For $\theta \in \mathrm{Aut}(\mathcal{M})$, the following two conditions are equivalent:*

(i) $\theta \notin \mathrm{Cnt}(\mathcal{M})$;

(ii) θ_ω *is properly outer.*

PROOF:

(i) $\Longrightarrow$ (ii): We shall prove that for any sequence $\{X_n\}$ in $\mathcal{M}_\omega$ there exists $Y \in \mathcal{M}_\omega$ such that $\theta_\omega(Y) \neq Y$ and $[Y, X_n] = 0$, $n = 1, 2, \ldots$. Let $\left\{ x_n(k) : k \in \mathbf{N} \right\}$ be a representing strongly ω-central sequence for X_n. Since $\theta \notin \mathrm{Cnt}(\mathcal{M})$ there exists a strongly central sequence $\{y(k)\}$ such that $\|\theta(y(k)) - y(k)\|_\varphi \geq \delta > 0$. Since $\{y(k)\}$ is central, with faithful $\varphi \in \mathfrak{S}_*$

$$\lim_{k\to\infty}\left\|\left[y(k), x_n(m)\right]\right\|_\varphi^\sharp = 0, \quad n, m \in \mathbf{N}.$$

Let $\{k_n\}$ be an increasing sequence in $\mathbf{N}$ such that $\|[y(k_n), x_j(n)]\|_\varphi^\sharp \leq 1/n$ for $j = 1, 2, \dots, n$. Let $Y = \pi_\omega(\{y(k_n)\})$. It then follows that

$$\left\|\left[Y, X_j\right]\right\|_{2,\omega} = \lim_{n\to\omega}\left\|\left[y(k_n), x_j(n)\right]\right\|_\varphi \leq \lim_{n\to\omega}\frac{1}{n} = 0,$$

$$\left\|\theta_\omega(Y) - Y\right\|_{2,\omega} = \lim_{n\to\omega}\left\|\theta(y(k_n)) - y(k_n)\right\|_\varphi \geq \delta > 0.$$

Therefore, θ_ω is properly outer.

(ii) $\Longrightarrow$ (i): This is obvious. Q.E.D.

We continue to assume that $\mathcal{M}$ is a separable strongly stable factor.

Lemma 2.3. *If $\theta \in \mathrm{Aut}(\mathcal{M})$ has $p_a(\theta) = 0$, then for any separable von Neumann subalgebra $\mathcal{P}$ of $\mathcal{M}_\omega$ and a natural number $n \in \mathbf{N}$, there exists a partition $\{F_1, F_2, \dots, F_n\}$ of identity in $\mathcal{P}' \cap \mathcal{M}_\omega$ such that*

$$\theta_\omega(F_j) = F_{j+1}, \quad 1 \leq j \leq n, \quad (F_{n+1} = F_1). \tag{1}$$

PROOF: We fix a faithful $\varphi \in \mathfrak{S}_*$ and a dense sequence $\{\psi_j\}$ in $\mathfrak{S}_*$. Let $\{X_m\}$ be a σ-strongly dense sequence of the unit ball of $\mathcal{P}$, and $\{x_m(k)\}$ be a strongly central sequence in $\mathcal{M}$ representing X_m.

By Theorem 1.6, for any $k \in \mathbf{N}$ there exists a partition $\{\widetilde{F}_j^k\}$ of identity in $\mathcal{M}_\omega$ such that

$$\left\|\theta_\omega(\widetilde{F}_j^k) - \widetilde{F}_{j+1}^k\right\|_2 < \frac{1}{k}, \qquad \widetilde{F}_{n+1}^k = \widetilde{F}_1^k. \tag{2}$$

Let $\{F_j^k(\nu) : \nu \in \mathbf{N}\}$ be a sequence of partitions of identity representing $\{\widetilde{F}_j^k\}$. Such a sequence exists by Theorem XIV.4.6. We now choose an increasing sequence $\{\nu_m\}$ in $\mathbf{N}$ such that

a) $\left\|\left[F_j^k(\nu_k), \psi_s\right]\right\| < \dfrac{1}{k}$, $1 \leq s \leq k$, $1 \leq j \leq n$;

b) $\left\|\left[F_j^k(\nu_k), x_m(k)\right]\right\|_\varphi^\sharp < \dfrac{1}{k}$, $1 \leq m \leq k$, $1 \leq j \leq n$;

c) $\left\|\theta(F_j^k(\nu_k)) - F_{j+1}^k(\nu_k)\right\|_\varphi^\sharp < \dfrac{1}{k}$, $1 \leq j \leq n$.

We now set $F_j = \pi_\omega(\{F_j^k(\nu_k)\})$ and obtain (1). Q.E.D.

We are now ready to prove the following stability theorem for aperiodic automorphisms:

Theorem 2.4. *If $\mathcal{M}$ is a separable strongly stable factor, then every aperiodic automorphism θ of $\mathcal{M}$ gives rise to a stable action θ_ω on $\mathcal{M}_\omega$ for any free ultra filter $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$ in the sense that for every $u \in \mathcal{U}(\mathcal{M}_\omega)$ there exists $v \in \mathcal{U}(\mathcal{M}_\omega)$ such that*

$$\theta_\omega(v) = uv.$$

In other words, every inner perturbation $\mathrm{Ad}(u^*) \circ \theta_\omega$ of θ_ω is conjugate to θ_ω under $\mathrm{Int}(\mathcal{M}_\omega)$.

PROOF: Let $u \in \mathcal{U}(\mathcal{M}_\omega)$. For a fixed $\varepsilon > 0$, choose $n \in \mathbf{N}$ with $n \geq 4/\varepsilon^2$. By the last lemma, choose a partition $\{F_1, \dots, F_n\}$ of identity in $\mathcal{M}_\omega$ commuting with u such that $\theta_\omega(F_j) = F_{j+1}$. Then $\tau_\omega(F_j) = 1/n$, so that $\|F_j\|_{2,\omega} = 1/\sqrt{n} \leq \varepsilon/2$. Inductively, we put

$$v_0 = F_n, \quad v_1 = \theta_\omega^{-1}(uv_0), \dots, \quad v_{k+1} = \theta_\omega^{-1}(uv_k), \dots, \quad v_{n-1} = \theta_\omega^{-1}(uv_{n-2}).$$

We then have $v_j^* v_j = v_j v_j^* = F_{n-j}$, $0 \leq j \leq n-1$. We then put $V = \sum_{k=0}^{n-1} v_k \in \mathcal{U}(\mathcal{M}_\omega)$ and obtain

$$\theta_\omega(V) = \theta_\omega(v_0) + \sum_{k=0}^{n-2} uv_k,$$

$$uV = \sum_{k=0}^{n-2} uv_k + uv_{n-1}.$$

Hence we have

$$\left\|\theta_\omega(V) - uV\right\|_{2,\omega} = \left\|uv_{n-1} - \theta_\omega(v_0)\right\|_{2,\omega} \leq \varepsilon.$$

Therefore, we conclude that for any $\varepsilon > 0$ there exists $V_\varepsilon \in \mathcal{U}(\mathcal{M}_\omega)$ such that

$$\left\|\theta_\omega(V_\varepsilon) - uV_\varepsilon\right\|_{2,\omega} \leq \varepsilon. \tag{3}$$

We now apply the same arguments of the last lemma. Fix $\varphi \in \mathfrak{S}_*$ and $\{\psi_j\} \subset \mathfrak{S}_*$ as in the last lemma. For each $\nu \in \mathbf{N}$, let V_ν be the V_ε of (3) with $\varepsilon = 1/2\nu$. Let $\{u(k)\}$ and $\{v_\nu(k)\}$ be representing sequences of u and V_ν respectively. For each $\nu \in \mathbf{N}$, let A_ν be the set of all $k \in \mathbf{N}$ such that

a) $\left\|\left[v_\nu(k), \psi_j\right]\right\| < \dfrac{1}{\nu}, \quad j = 1, 2, \dots, \nu;$

b) $\left\|u(k)v_\nu(k) - \theta\big(v_\nu(k)\big)\right\|_\varphi < \dfrac{1}{\nu}.$

Then A_ν belongs to the filter ω. We then readjust A_ν inductively so that $A_\nu \supsetneqq A_{\nu+1}$. For each $j \in \mathbf{N}$, put

$$\nu(j) = \max\left\{ \nu \in \mathbf{N} : j \in A_\nu \right\}.$$

Since $\bigcap A_\nu = \emptyset$, $\nu(j)$ is finite. Put

$$v_j = v_{\nu(j)}(j), \quad j \in \mathbf{N}.$$

Then $\{v_j\}$ is strongly central and $v = \pi_\omega\big(\{v_j\}\big)$ satisfies $\theta_\omega(u) = vu$. Q.E.D.

Theorem 2.5. *Let $\mathcal{M}$ be a strongly stable separable factor. Let $\mathcal{P}$ and $\mathcal{Q}$ be* AFD II_1*-subfactors of $\mathcal{M}$ such that $\mathcal{P} \vee \mathcal{P}^c = \mathcal{M} = \mathcal{Q} \vee \mathcal{Q}^c$ and $\mathcal{P}^c \cong \mathcal{M} \cong \mathcal{Q}^c$. If the decompositions $\mathcal{P} \vee \mathcal{P}^c = \mathcal{M} = \mathcal{Q} \vee \mathcal{Q}^c$ are both tensor product factorizations, then $\mathcal{P}$ and $\mathcal{Q}$ are conjugate under $\overline{\mathrm{Int}}(\mathcal{M})$, i.e. there exists $\sigma \in \overline{\mathrm{Int}}(\mathcal{M})$ such that $\sigma(\mathcal{P}) = \mathcal{Q}$.*

PROOF: Since $\mathcal{M}_* = \mathcal{P}_* \widehat{\otimes} \mathcal{P}^c_*$, every central sequence of $\mathcal{P}$ or $\mathcal{Q}$ is strongly central in $\mathcal{M}$. Let $\left\{ e_{i,j}(k) : 1 \le i, j \le 2 \right\}$ (resp. $\{ f_{i,j}(k) : 1 \le i, j \le 2 \}$) be mutually commuting sequence of 2×2-matrix units which generates $\mathcal{P}$ (resp. $\mathcal{Q}$). Let $\{\psi_j\}$ be a dense sequence of $\mathfrak{S}_*$.

By induction, we are going to construct sequences $\{n_\nu\} \subset \mathbf{N}$ and $\{u_\nu\} \subset \mathcal{U}(\mathcal{M})$ such that

a) $\big[u_\nu,\ f_{i,j}(n_k)\big] = 0, \quad k = 1, 2, \dots, \nu - 1;$

b) with $v_\nu = u_\nu u_{\nu-1} \cdots u_1$,

$$v_\nu e_{i,j}(n_k) v_\nu^* = f_{i,j}(n_k), \quad k = 1, 2, \dots, \nu;$$

c)
$$\left\| \psi_j \circ \mathrm{Ad}(v_\nu) - \psi_j \circ \mathrm{Ad}(v_{\nu-1}) \right\| < 2^{-\nu},$$
$$\left\| \psi_j \circ \mathrm{Ad}(v_\nu^*) - \psi_j \circ \mathrm{Ad}(v_{\nu-1}^*) \right\| < 2^{-\nu}, \quad j = 1, 2, \dots, \nu.$$

Suppose n_k and u_k have been found for $k \le \nu - 1$. Let $\mathcal{R} = \mathcal{M} \cap \left\{ f_{i,j}(n_k) : 1 \le i, j \le 2,\ 1 \le k \le \nu - 1 \right\}'$. Since

$$v_{\nu-1} e_{i,j}(n_k) v_{\nu-1}^* = f_{i,j}(n_k), \quad 1 \le k \le \nu - 1,$$

$v_{\nu-1} e_{i,j}(n) v_{\nu-1}^* \in \mathcal{R}$ for any $n > n_{\nu-1}$, and also $f_{i,j}(n) \in \mathcal{R}$ for $n > n_{\nu-1}$. Let $E_{i,j} = \pi_\omega\big(\{v_{\nu-1} e_{i,j}(n) v_{\nu-1}^*\}\big) \in \mathcal{R}_\omega$ and $F_{i,j} = \pi_\omega\big(\{ f_{i,j}(n)\}\big) \in \mathcal{R}_\omega$. Since $\mathcal{R}_\omega \subset \mathcal{M}_\omega$ and $\mathcal{R}_\omega$ is of type II_1, there exists $W \in \mathcal{R}_\omega$ such that $W^*W = E_{11}$ and $WW^* = F_{11}$. By Theorem XIV.4.6, we choose a representing sequence $\{w_n\} \subset \mathcal{R}$ of W such that

$$w_n^* w_n = v_{\nu-1} e_{11}(n) v_{\nu-1}^*, \qquad w_n w_n^* = f_{11}(n), \quad n > n_{\nu-1},$$

and put

$$x_n = \sum_{j=1}^{2} f_{j,1}(n) w_n v_{\nu-1} e_{1j}(n) v_{\nu-1}^*.$$

Then $\{x_n\} \subset \mathcal{U}(\mathcal{R})$ is strongly ω-central. If n is sufficiently large along the ultra filter ω, $u_\nu = x_n$ satisfies the above (a), (b) and (c).

By (c), $\{\mathrm{Ad}(v_\nu)\}$ is a Cauchy sequence in $\mathrm{Aut}(\mathcal{M})$, so that it converges to $\sigma \in \mathrm{Aut}(\mathcal{M})$. Let

$$\mathcal{P}_1 = \{ e_{i,j}(n_k) : k \in \mathbf{N} \}'', \qquad Q_1 = \{ f_{i,j}(n_k) : k \in \mathbf{N} \}''.$$

Then we have $\sigma(\mathcal{P}_1) = Q_1$. By construction, we have

$$\mathcal{P} = \mathcal{P}_1 \mathbin{\overline{\otimes}} \big(\mathcal{P}_1' \cap \mathcal{P}\big), \qquad Q = Q_1 \mathbin{\overline{\otimes}} \big(Q_1' \cap Q\big).$$

Let α and β be isomorphisms of $\mathcal{M}$ onto $\mathcal{P}^c$ and Q^c respectively, and put $\mathcal{P}_2 = \alpha(\mathcal{P})$ and $Q_2 = \beta(Q)$. It then follows that

$$\begin{aligned} \mathcal{M} &\cong \mathcal{P}_1 \mathbin{\overline{\otimes}} \big(\mathcal{P}_1' \cap \mathcal{P}\big) \mathbin{\overline{\otimes}} \mathcal{P}_2 \mathbin{\overline{\otimes}} \big(\mathcal{P}_2' \cap \mathcal{P}' \cap \mathcal{M}\big) \\ &\cong Q_1 \mathbin{\overline{\otimes}} \big(Q_1' \cap Q\big) \mathbin{\overline{\otimes}} Q_2 \mathbin{\overline{\otimes}} \big(Q_2' \cap Q' \cap \mathcal{M}\big). \end{aligned}$$

In $\mathcal{P} \mathbin{\overline{\otimes}} \mathcal{P}_2$ (resp. $Q \mathbin{\overline{\otimes}} Q_2$), $\mathcal{P}_1$ and $\mathcal{P}$ (resp. Q_1 and Q) are conjugate under $\mathrm{Aut}(\mathcal{P} \mathbin{\overline{\otimes}} \mathcal{P}_2) = \overline{\mathrm{Int}}\big(\mathcal{P} \mathbin{\overline{\otimes}} \mathcal{P}_2\big)$ (resp. $\mathrm{Aut}(Q \mathbin{\overline{\otimes}} Q_2)$). Therefore, $\mathcal{P}$ and Q are conjugate under $\overline{\mathrm{Int}}(\mathcal{M})$. Q.E.D.

We are now going to construct a model automorphism of $\mathcal{R}_0$. Fix $p \in \mathbf{N}$, $p \geq 2$, and set

$$\left.\begin{aligned} &M_n = M(p;\mathbf{C}), \quad n \in \mathbf{N}; \\ &\mathcal{R}_0 = \prod_{n=1}^{\infty}{}^{\otimes} M_n. \end{aligned}\right\} \tag{4}$$

Let $\{ e_{i,j}(n) : 1 \leq i, j \leq p \}$ be the standard matrix unit of M_n and put

$$\left.\begin{aligned} &u_n = \sum_{k=1}^{p} \mathrm{e}^{2\pi \mathrm{i}k/p} e_{kk}(n) \in \mathcal{U}(M_n), \\ &\sigma_p = \prod_{n=1}^{\infty}{}^{\otimes} \mathrm{Ad}(u_n) \in \mathrm{Aut}(\mathcal{R}_0). \end{aligned}\right\} \tag{5}$$

Since $u_n^p = 1$, and $u_n^k \notin \mathbf{T}$ for $1 \leq k \leq p-1$, σ_p has period p and $p_0(\sigma_p) = p$. Furthermore, we set $\mathcal{R}_n = \mathcal{R}_0$, $n \in \mathbf{N}$, and identify $\mathcal{R}_0$ with $\prod_{n=2}^{\infty\,\otimes} \mathcal{R}_n$, then we put

$$\left.\begin{aligned} &\sigma_0 = \prod_{n=2}^{\infty}{}^{\otimes} \sigma_n \in \mathrm{Aut}(\mathcal{R}_0), \\ &\sigma_1 = \mathrm{id} \in \mathrm{Aut}(\mathcal{R}_0). \end{aligned}\right\} \tag{6}$$

Thus we get a sequence $\{ \sigma_n : n = 0, 1, 2, \dots \}$ in $\mathrm{Aut}(\mathcal{R}_0)$. It follows that σ_0 is aperiodic. Hence we have

$$p_0(\sigma_k) = k, \quad k = 0, 1, 2, \dots. \tag{7}$$

Lemma 2.6. *Let $\mathcal{M}$ be a von Neumann algebra with no direct summand of finite type* I*, and $\alpha \in \mathrm{Aut}(\mathcal{M})$. If α is either stable or gives a free action of $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$, then for each $p \in \mathbf{N}$, $p \geq 2$, and an n-th root λ of 1 there exists a $p \times p$-matrix unit $\left\{ f_{i,j} : 1 \leq i, j \leq p \right\}$ in $\mathcal{M}$ such that*

$$\alpha(f_{i,j}) = \lambda^{i-j} f_{i,j}, \quad 1 \leq i, j \leq p. \tag{8}$$

PROOF: The case that α is stable: Let $\left\{ e_{i,j} : 1 \leq i, j \leq p \right\}$ be a $p \times p$-matrix unit in $\mathcal{M}$. Since $e_{ii} \sim \alpha(e_{ii})$, there exists $W \in \mathcal{U}(\mathcal{M})$ such that $We_{11}W^* = \alpha(e_{11})$. Put $V = \sum_{i=1}^{p} \alpha(e_{i1})We_{1i}$. Then $\alpha(e_{ij}) = Ve_{ij}V^*$. Now, set $U = \sum_{j=1}^{p} \lambda^j e_{jj}$. Then $UV^*\alpha(e_{ij})VU^* = \lambda^{i-j}e_{i,j}$. By the stability of α, we can find $v \in \mathcal{U}(\mathcal{M})$ such that $UV^* = v^*\alpha(v)$. Then we get $\mathrm{Ad}(UV^*) \circ \alpha = \mathrm{Ad}(v^*) \circ \alpha \circ \mathrm{Ad}(v)$. Hence $f_{ij} = ve_{ij}v^*$ gives the required matrix unit.

The periodic case: We view α as a free action of $\mathbf{Z}_n$. By the assumption on $\mathcal{M}$, $\mathcal{M}^\alpha$ contains a $p \times p$-matrix unit $\left\{ e_{i,j} : 1 \leq i, j \leq p \right\}$. Put $U = \sum_{k=1}^{p} \lambda^k e_{kk} \in \mathcal{M}^\alpha$. Then $\left\{ U^k : 0 \leq k \leq n-1 \right\}$ is an α-cocycle of G. By Proposition XI.2.26, there exists $V \in \mathcal{U}(\mathcal{M})$ such that $U^k = V^*\alpha^k(V)$, $0 \leq k \leq n-1$. Hence $\mathrm{Ad}(U) \circ \alpha = \mathrm{Ad}(V^*) \circ \alpha \circ \mathrm{Ad}(V)$. Therefore, $f_{ij} = Ve_{ij}V^*$ satisfies (8). Q.E.D.

We apply the last lemma to $\left\{\mathcal{M}_\omega, \theta_\omega\right\}$.

Lemma 2.7. *Let $\mathcal{M}$ be a strongly stable separable factor and $\theta \in \mathrm{Aut}(\mathcal{M})$. For each $p \in \mathbf{N}$, $\lambda \in \mathbf{T}$ with $\lambda^{p_a(\theta)} = 1$, $\psi_1, \dots, \psi_q \in \mathfrak{S}_*$, a faithful $\varphi \in \mathfrak{S}_*$ and $\varepsilon > 0$, there exist $u \in \mathcal{U}(\mathcal{M})$ and a $p \times p$-matrix unit $\{e_{i,j}\}$ such that*

a) $\left\|\left[e_{i,j}, \psi_k\right]\right\| < \varepsilon$, $1 \leq i$, $j \leq p$, $1 \leq k \leq q$;

b) $\mathrm{Ad}(u) \circ \theta(e_{ij}) = \lambda^{i-j}e_{ij}$, $1 \leq i, j \leq p$;

c) $\left\|u - 1\right\|_\varphi^\sharp < \varepsilon$.

PROOF: We apply the last lemma to $\left\{\mathcal{M}_\omega, \theta_\omega\right\}$ with a fixed free ultra filter $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$, to find a $p \times p$-matrix unit $\left\{ E_{ij} : 1 \leq i, j \leq p \right\}$ in $\mathcal{M}_\omega$ such that $\theta_\omega(E_{ij}) = \lambda^{i-j}E_{i,j}$. Let $\left\{e_{ij}(n)\right\}$ be a strongly ω-central sequence of $p \times p$-matrix units representing $\{E_{ij}\}$.

Since $e_{11}(n) \sim \theta\big(e_{11}(n)\big)$ and $E_{11} = \theta_\omega(E_{11})$, we find a strongly ω-central sequence $\left\{u(n)\right\}$ of partial isometries such that

$$\pi_\omega\big(\left\{u(n)\right\}\big) = E_{11}, \qquad u(n)^*u(n) = e_{11}(n), \qquad u(n)u(n)^* = \theta\big(e_{11}(n)\big)$$

by Theorem XIV.4.6.(iv). Set

$$v(n) = \sum_{j=1}^{p} \lambda^{1-j}\theta\big(e_{j1}(n)\big)u(n)e_{1j}(n) \in \mathcal{U}(\mathcal{M}).$$

Then $\left\{v(n)\right\}$ is strongly ω-central, and $\pi_\omega\big(\left\{v(n)\right\}\big) = 1$ because $\theta_\omega(E_{j1}) = \lambda^{j-1}E_{j1}$ and $\pi_\omega\big(\left\{u(n)\right\}\big) = E_{11}$. Hence $\lim_{n\to\omega}\left\|v(n) - 1\right\|_\varphi^\sharp = 0$. By construction, we have

$$\mathrm{Ad}\big(v(n)^*\big) \circ \theta\big(e_{ij}(n)\big) = \lambda^{i-j} e_{ij}(n), \quad 1 \le i, j \le p, \quad n \in \mathbf{N}.$$

Hence if we choose a sufficiently large n along the filter ω, $\{e_{ij}(n)\}$ and $v(n)$ satisfy (a), (b) and (c). Q.E.D.

Definition 2.8. Two automorphisms θ_1 of $\mathcal{M}_1$ and θ_2 of $\mathcal{M}_2$ are said to be *outer conjugate* and written $\theta_1 \sim \theta_2$ if there exist an isomorphism π of $\mathcal{M}_1$ onto $\mathcal{M}_2$ and a unitary $u \in \mathcal{M}_1$ such that $\pi \circ \mathrm{Ad}(u) \circ \theta_1 \circ \pi^{-1} = \theta_2$.

Remark 2.9. It is possible to define outer conjugacy for group actions similarly. But in this case we should not confuse with cocycle conjugacy. In the cocycle conjugacy, one requires one cocycle condition in the perturbing unitaries, while arbitrary inner automorphism perturbations are allowed in the outer conjugacy. In our case, since $\mathbf{Z}$ is cohomologically trivial, the outer conjugacy and the cocycle conjugacy coincide. But when we consider periodic automorphisms and want to pass to the cyclic group $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$, the subtle difference will appear.

Theorem 2.10. *Let $\mathcal{M}$ be a strongly stable separable factor. For $\theta \in \mathrm{Aut}(\mathcal{M})$ and $p \in \mathbf{N}$, the following three conditions are equivalent:*

(i) $p_a(\theta) \equiv 0 \bmod p$;
(ii) *θ and $\theta \otimes \sigma_p$ are outer conjugate;*
(iii) *For any faithful $\varphi \in \mathfrak{S}_*$ and $\delta > 0$, there exists $u \in \mathcal{U}(\mathcal{M})$ such that*

$$\mathrm{Ad}(u) \circ \theta \cong \theta \otimes \sigma_p, \qquad \|u - 1\|_\varphi^\sharp < \delta.$$

PROOF:

(i) $\Longrightarrow$ (iii): Assume (i). Fix $\delta > 0$. We choose sequences $\{n_\nu\} \subset \mathbf{N}$ and $\{\lambda_\nu\} \subset \mathbf{T}$ as follows: If $p = 0$, then $n_\nu > 1$ and every $q \in \mathbf{N}, q > 1$, should appear in $\{n_\nu\}$ infinitely many times and $\lambda_\nu = \mathrm{e}^{2\pi \mathrm{i}/n_\nu}$; if $p = 1$, then $n_\nu = 2$ and $\lambda_\nu = 1$; if $p > 1$, then $n_\nu = p$ and $\lambda_\nu = \mathrm{e}^{2\pi\mathrm{i}/p}$. We further fix a faithful $\varphi \in \mathfrak{S}_*$ and a dense sequence $\{\psi_j\}$ in $\mathfrak{S}_*$.

Applying Lemma 2.7 inductively, we will choose a sequence $\{u_\nu\} \subset \mathcal{U}(\mathcal{M})$ and a sequence $\{e_{i,j}(\nu)\}$ of mutually commuting $n_\nu \times n_\nu$-matrix units satisfying the conditions:

a) $\left\|\big[e_{i,j}(\nu), \psi_k\big]\right\| \le \dfrac{1}{2^\nu n_\nu^2}$, $1 \le k \le \nu$, $1 \le i, j \le n_\nu$;

b) $\big[u_\nu, e_{i,j}(k)\big] = 0$, $1 \le k \le \nu - 1$, $1 \le i, j \le n_\nu$;

c) $\theta_\nu = \mathrm{Ad}\big(u_\nu u_{\nu-1} \cdots u_1\big) \circ \theta$ satisfies

$$\theta_\nu\big(e_{i,j}(\nu)\big) = \lambda_\nu^{i-j} e_{i,j}(\nu), \quad 1 \le i, j \le n_\nu,$$

d) $\left\|u_\nu u_{\nu-1} \cdots u_1 - u_{\nu-1} u_{\nu-2} \cdots u_1\right\|_\varphi^\sharp < \dfrac{\delta}{2^\nu}$.

Suppose that $\{u_1, \dots, u_\nu\}$ and $\{e_{i,j}(1), \dots, e_{i,j}(\nu)\}$ have been chosen. Let $M_k = \sum \mathbf{C}e_{i,j}(k)$, and $N_\nu = M_1 \vee M_2 \vee \cdots \vee M_\nu$. Put $\mathcal{M}_\nu = N_\nu' \cap \mathcal{M}$. Then $\mathcal{M} = N_\nu \otimes \mathcal{M}_\nu$. We choose $\varepsilon > 0$ and $\tilde{\psi}_1, \tilde{\psi}_2, \dots, \tilde{\psi}_r \in \mathfrak{S}_*(\mathcal{M}_\nu)$ such that

$$\left\|[x, \psi_j]\right\| \leq \frac{1}{n_{\nu+1}^2 2^{\nu+1}}, \quad 1 \leq j \leq \nu + 1$$

whenever $x \in \mathcal{M}_\nu$, $\|x\| \leq 1$, and $\left\|[x, \tilde{\psi}_i]\right\| < \varepsilon$, $1 \leq i \leq r$. We also choose $\eta > 0$ so that $u \in \mathcal{U}(\mathcal{M}_\nu)$, $\|u - 1\|_\varphi^\sharp < \eta$ implies

$$\left\|u\left(u_\nu u_{\nu-1} \cdots u_1\right) - u_\nu u_{\nu-1} \cdots u_1\right\|_\varphi^\sharp < \frac{\delta}{2^{\nu+1}}.$$

Let $\tilde{\theta} = \theta_\nu\big|_{\mathcal{M}_\nu}$. By (c), $\tilde{\theta} \in \mathrm{Aut}(\mathcal{M}_\nu)$ and $p_a(\tilde{\theta}) = p_a(\theta)$. We apply Lemma 2.7 to $\{\mathcal{M}_\nu, \tilde{\theta}, \lambda_{\nu+1}, n_{\nu+1}\}$ to obtain an $n_{\nu+1} \times n_{\nu+1}$-matrix unit $\{e_{i,j}\}$ in $\mathcal{M}_\nu$ and $\tilde{u} \in \mathcal{U}(\mathcal{M}_\nu)$ such that

$$\begin{cases} \left\|\left[e_{i,j}, \tilde{\psi}_k\right]\right\| < k, & 1 \leq k \leq r, \quad 1 \leq i, j \leq n_{\nu+1}; \\ \mathrm{Ad}(\tilde{u}) \circ \tilde{\theta}(e_{ij}) = \lambda_{\nu+1}^{i-j} e_{ij}, & 1 \leq i, j \leq n_{\nu+1}; \\ \left\|\tilde{u} - 1\right\|_\varphi^\sharp < \eta. & \end{cases}$$

We then set $e_{ij}(\nu+1) = e_{i,j}$ and $u_{\nu+1} = \tilde{u}$. It is now clear that (a), (b), (c) and (d) hold for $\nu + 1$.

By condition (d), $u = \lim u_\nu u_{\nu-1} \cdots u_1 \in \mathcal{U}(\mathcal{M})$ converges in the σ-strong* topology. Let $\mathcal{R}_0 = \bigvee_{\nu=1}^\infty M_\nu$. By Lemmas XIV.4.9 and XIV.4.10, we have $\mathcal{M} = \mathcal{R}_0 \mathbin{\overline{\otimes}} (\mathcal{R}_0' \cap \mathcal{M})$. The restriction of $\theta_\infty = \mathrm{Ad}(u) \circ \theta$ to $\mathcal{R}_0$ is conjugate to σ_p by (c) and $\theta_\infty \cong \sigma_p \otimes (\theta_\infty|_{\mathcal{M}_\infty})$ where $\mathcal{M}_\infty = \mathcal{R}_0' \cap \mathcal{M}$. Since $\sigma_p \cong \sigma_p \otimes \sigma_p$, we have $\theta_\infty \cong \theta_\infty \otimes \sigma_p$. Finally we have

$$\|u - 1\|_\varphi^\sharp \leq \sum_{\nu=1}^\infty \left\|u_\nu u_{\nu-1} \cdots u_1 - u_{\nu-1} \cdots u_1\right\|_\varphi^\sharp < \delta.$$

(iii) $\Longrightarrow$ (ii): This is trivial.

(ii) $\Longrightarrow$ (i): Suppose $q \not\equiv 0 \bmod p$ and $\theta \sim \theta \otimes \sigma_p$. We work with $\theta \otimes \sigma_p$ on $\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{R}_0$ instead. By construction, $\mathcal{R}_0$ admits a central sequence $\{y_n\}$ such that $\left\|\sigma_p^q(y_n) - y_n\right\|_2 = 1$, $\|y_n\| = 1$. Then $x_n = 1 \otimes y_n$ is strongly central in $\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{R}_0$ and $\left\|(\theta \otimes \sigma_p)^q(x_n) - x_n\right\|_{\varphi \otimes \tau} = 1$ for any $\varphi \in \mathfrak{S}_*(\mathcal{M})$, where τ is the canonical trace of $\mathcal{R}_0$. Hence $(\theta \otimes \sigma_p)^q$ does not belong to $\mathrm{Cnt}(\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{R}_0)$. Hence $q \neq p_a(\theta)$.

Q.E.D.

Corollary 2.11. *Let $\mathcal{M}$ be a strongly stable separable factor, and ε be the canonical quotient map of* $\mathrm{Aut}(\mathcal{M})$ *onto* $\mathrm{Out}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})/\mathrm{Int}(\mathcal{M})$. *Then $\varepsilon\big(\mathrm{Cnt}(\mathcal{M})\big)$ is precisely the centralizer of $\varepsilon\big(\overline{\mathrm{Int}}(\mathcal{M})\big)$.*

PROOF: By Lemma XIV.4.14, $\varepsilon\big(\mathrm{Cnt}(\mathcal{M})\big)$ and $\varepsilon\big(\overline{\mathrm{Int}}(\mathcal{M})\big)$ commute.

Suppose that $\theta \in \mathrm{Aut}(\mathcal{M})$ and $\theta \notin \mathrm{Cnt}(\mathcal{M})$. By definition, we have $p = p_a(\theta) \neq 1$. By the last theorem, we may assume that $\mathcal{M} = \mathcal{M}_1 \overline{\otimes} \mathcal{R}_0$ and $\theta = \theta_1 \otimes \sigma_p$. In the construction of σ_p of (5), choose $v_n \in \mathcal{U}(M_n)$ so that $u_n v_n u_n^* v_n^* \notin \mathbf{T}$ and set $\alpha_0 = \prod_{n=1}^{\infty}{}^{\otimes} \mathrm{Ad}(v_n)$. Then we have $\sigma_p \alpha_0 \sigma_p^{-1} \alpha_0^{-1} \notin \mathrm{Int}(\mathcal{R}_0)$ and hence $\varepsilon(\theta)$ and $\varepsilon\big(\mathrm{id} \otimes \alpha_0\big)$ do not commute in $\mathrm{Out}(\mathcal{M})$, while $\mathrm{id} \otimes \alpha_0$ belongs to $\overline{\mathrm{Int}}(\mathcal{M})$. Q.E.D.

Proposition 2.12. *For any separable von Neumann algebra $\mathcal{M}$ the modular automorphism group $\{\sigma_t^\varphi\}$ of any faithful semi-finite normal weight φ on $\mathcal{M}$ acts trivially on strongly central sequences, i.e. $\sigma_t^\varphi \in \mathrm{Cnt}(\mathcal{M})$.*

PROOF: It is sufficient to prove that σ_t^φ acts on a strongly central sequence $\{u_n\}$ of unitaries. Let ψ be a fixed faithful normal state on $\mathcal{M}$. Then we have

$$\big(\mathrm{D}\psi \circ \mathrm{Ad}(u_n) : \mathrm{D}\psi\big)_t = u_n^* \sigma_t^\psi(u_n), \quad t \in \mathbf{R}.$$

The strong centrality of $\{u_n\}$ means that $\lim_{n\to\infty} \|\psi \circ \mathrm{Ad}(u_n) - \psi\| = 0$. By Theorem IX.1.19.(iii), $\big(\mathrm{D}\psi \circ \mathrm{Ad}(u_n) : \mathrm{D}\psi\big)_t$ converges σ-strongly to 1 for every $t \in \mathbf{R}$, so that $\big\{u_n^* \sigma_t^\psi(u_n)\big\}$ converges to 1 σ-strongly. Thus σ_t^ψ acts on $\{u_n\}$ trivially. Since σ_t^φ, $\varphi \in \mathfrak{W}_0(\mathcal{M})$, is an inner perturbation of σ_t^ψ, σ_t^φ acts trivially on $\{u_n\}$ as well. Q.E.D.

Exercise XVII.2

1) Let $\mathcal{M}$ be an AFD factor of type II_1 and $\alpha \in \mathrm{Aut}(\mathcal{M})$. Suppose that α is aperiodic. Fix a free ultra filter $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$ and consider $\mathcal{M}_\omega$ as usual. Following the steps suggested below, show that if $\beta \in \mathrm{Aut}(\mathcal{M})$ has the property that $\beta_\omega(x) = x$ for every $x \in (\mathcal{M}_\omega)^{\alpha_\omega}$ then $\beta_\omega = \alpha_\omega^n$ for some $n \in \mathbf{Z}$.

(**a**) Show first that $\Gamma(\alpha_\omega) = \mathbf{T}$.

(**b**) Show that $\big(\mathcal{M}_\omega^{\alpha_\omega}\big)' \cap \mathcal{M}_\omega = \mathbf{C}$.

(**c**) Show that for each $s \in]-\pi, \pi] = \mathbf{T}$ there exists a unitary $U(s) \in \mathcal{U}(\mathcal{M}_\omega)$ such that $\alpha_\omega\big(U(s)\big) = \mathrm{e}^{\mathrm{i}s} U(s)$.

(**d**) Show that $U(s)^* \beta_\omega\big(U(s)\big) \in \big(\mathcal{M}_\omega^{\alpha_\omega}\big)'$.

(**e**) Show that $\beta_\omega\big(U(s)\big) = \lambda(s) U(s)$, $s \in \mathbf{T}$.

(**f**) Show that $\lambda(s) = \mathrm{e}^{ins}$, $s \in \mathbf{T}$, for some $n \in \mathbf{Z}$.

(**g**) Show that $\beta_\omega = \alpha_\omega^n$.

2) Let $\mathcal{M}$ be a strongly stable separable factor and $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$ a free ultrafilter. Consider $\mathcal{M}_\omega$ as in §4, Chapter XIV. Let $\alpha, \beta \in \mathrm{Aut}(\mathcal{M})$ be two automorphisms commuting modulo $\mathrm{Int}(\mathcal{M})$ in the sense that there exists $w \in \mathcal{U}(\mathcal{M})$ such that $\alpha \circ \beta = \mathrm{Ad}(w) \circ \beta \circ \alpha$. Observe that $\alpha_\omega \circ \beta_\omega = \beta_\omega \circ \alpha_\omega$. Assume that α_ω and β_ω generate an isomorphism of $\mathbf{Z}^2$ into $\mathrm{Aut}(\mathcal{M}_\omega)$, i.e. $\alpha^m \beta^n \notin \mathrm{Cnt}(\mathcal{M})$ for $(m, n) \neq (0, 0)$.

§ 3 Outer Conjugacy of Approximately Inner Automorphisms of Strongly Stable Factors

In this section, we determine completely the outer conjugacy classes of approximately inner automorphisms of strongly stable factors.

We fix a strongly stable separable factor $\mathcal{M}$, and denote by $\mathcal{R}_0$ an AFD factor of type II_1. Let φ be a fixed faithful normal state on $\mathcal{M}$. We also fix a free ultra filter $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$. Let ε denote the canonical quotient map of $\mathrm{Aut}(\mathcal{M})$ onto $\mathrm{Out}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})/\,\mathrm{Int}(\mathcal{M})$.

We announce here the first main result of the section:

Theorem 3.1. *If $\mathcal{M}$ is a strongly stable separable factor and if $\theta_1, \theta_2 \in \overline{\mathrm{Int}}(\mathcal{M})$ are strongly aperiodic in the sense that $p_a(\theta_1) = p_a(\theta_2) = 0$, then there exists $\sigma \in \overline{\mathrm{Int}}(\mathcal{M})$ such that*

$$\varepsilon(\theta_2) = \varepsilon\big(\sigma\theta_1\sigma^{-1}\big).$$

More precisely, for any $\varepsilon > 0$, there exists $w \in \mathcal{U}(\mathcal{M})$ such that

$$\mathrm{Ad}(w) \circ \theta_1 \cong \theta_2, \qquad \|w - 1\|_\varphi < \varepsilon.$$

We need some preparation for the proof. We decompose $\mathcal{M} = \mathcal{M}_1 \overline{\otimes} \mathcal{R}_0$ by the strong stability of $\mathcal{M}$, where $\mathcal{M}_1 \cong \mathcal{M}$. We are going to compare θ and $\mathrm{id} \otimes \sigma_0$.

Lemma 3.2. *If $\theta \in \overline{\mathrm{Int}}(\mathcal{M})$ and $p_a(\theta) = 0$, then there exists a sequence $\{u_n\}$ in $\mathcal{U}(\mathcal{M})$ such that*

a) $\theta = \lim \mathrm{Ad}(u_n)$;

b) $\lim_{n\to\infty} \big\|\theta(u_n^k) - u_n^k\big\|_\varphi^\sharp = 0$, $\quad k \in \mathbf{Z}$.

PROOF: By assumption, $\theta = \lim_{n\to\infty} \mathrm{Ad}(v_n)$ for some $\{v_n\}$ in $\mathcal{U}(\mathcal{M})$. Let $w_n = v_n^*\theta(v_n)$. Then $\{w_n\}$ is strongly central. Let $W = \pi_\omega\big(\{w_n\}\big)$ and apply Theorem 2.4 to $\big\{W, \theta_\omega\big\}$ to obtain $X \in \mathcal{M}_\omega$ such that $W = X^*\theta_\omega(X)$. Let $\{x_n\}$ be a representing sequence of X in $\mathcal{U}(\mathcal{M})$. We then have $\lim_{n\to\omega} \mathrm{Ad}(x_n) = \mathrm{id}$, and $\lim_{n\to\omega} \big\|x_n^*\theta(x_n) - v_n^*\theta(v_n)\big\|_\varphi^\sharp = 0$. Passing to a subsequence, we can choose $\{y_n\}$ from $\{x_n\}$ such that

$$\mathrm{id} = \lim_{n\to\infty} \mathrm{Ad}(y_n) \qquad \text{and} \qquad \lim_{n\to\infty} \big\|y_n^*\theta(y_n) - v_n^*\theta(v_n)\big\|_\varphi^\sharp = 0.$$

With $u_n = v_n y_n^*$,[1] we have

$$\theta = \lim_{n\to\infty} \mathrm{Ad}(u_n); \qquad \lim_{n\to\infty} \big\|\theta(u_n) - u_n\big\|_\varphi^\sharp = 0$$

[1] Since $\{y_n\}$ is strongly central, the multiplication by $\{y_n\}$ from the right is almost isometry in the φ-norm $\|\cdot\|_\varphi$.

We are going to prove $\theta\big(u_n^k\big) - u_n^k \to 0$ σ-strongly* by induction. Suppose $\theta\big(u_n^k\big) - u_n^k \to 0$ σ-strongly*. Then $u_n^{-k}\theta\big(u_n^k\big) \to 1$ σ-strongly*, and so $u_n^{-1}\theta(u_n)u_n^{-k}\theta\big(u_n^k\big) \to 1$ σ-strongly*. Since $\psi \circ \theta^{-1} = \lim_{n\to\infty} u_n\psi u_n^*$ for any $\psi \in \mathfrak{S}_*$, we have $\lim_{n\to\infty}\big\|u_n^*(\psi \circ \theta^{-1}) - \psi u_n^*\big\| = 0$. Hence we have, for any $\psi \in \mathfrak{S}_*$,

$$
\begin{aligned}
&\left|\left\langle u_n^{-(k+1)}\theta\big(u_n^{k+1}\big),\ \psi\right\rangle - \langle 1,\ \psi\rangle\right| \\
&\quad = \left|\left\langle u_n^* u_n^{-k}\theta\big(u_n^k\big)\theta(u_n),\ \psi\right\rangle - \big\langle 1,\ \psi\circ\theta^{-1}\big\rangle\right| \\
&\quad = \left|\left\langle u_n^{-k}\theta\big(u_n^k\big)\theta(u_n),\ \psi u_n^*\right\rangle - \big\langle 1,\ \psi\circ\theta^{-1}\big\rangle\right| \\
&\quad \le \left\|\psi u_n^* - u_n^*\big(\psi\circ\theta^{-1}\big)\right\| + \left|\left\langle u_n^{-k}\theta\big(u_n^k\big)\theta(u_n)u_n^* - 1,\ \psi\circ\theta^{-1}\right\rangle\right| \ \to\ 0
\end{aligned}
$$

as $n \to \infty$. Thus $u_n^{-(k+1)}\theta\big(u_n^{k+1}\big)$ converges to 1 σ-weakly. On $\mathcal{U}(\mathcal{M})$, the σ-strong* topology and the σ-weak topology agree, so that $\theta\big(u_n^{k+1}\big) - u_n^{k+1} \to 0$ σ-strongly* as $n \to \infty$. Q.E.D.

Lemma 3.3. *Let $\big\{\mathcal{M},\theta\big\}$ be as above. For any $\psi_1,\psi_2,\ldots,\psi_q \in \mathcal{M}_*^+$, $n,k \in \mathbf{N}$ and $\delta > 0$ there exist a partition $\big\{F_1,F_2,\ldots,F_n\big\}$ of identity and $u,w \in \mathcal{U}(\mathcal{M})$ such that*

a) $\big\|[F_j,\psi_s]\big\| \le \delta$, $1 \le j \le n$, $1 \le s \le q$;

b) $uF_ju^* = F_{j+1}$, $j = 1,2,\ldots,n$, *where* $F_{n+1} = F_1$;

c) $\big\|\psi_s\circ\theta^{-1} - \psi_s\circ \operatorname{Ad} u^{-1}\big\| \le \delta$, $1 \le s \le q$;

d) *With* $\theta' = \operatorname{Ad} w \circ\theta$, $\big\|\varphi\circ\theta' - \varphi\circ\operatorname{Ad} u\big\| < \delta$;

e) $\big\|\theta'(u^\ell) - u^\ell\big\|_\varphi \le \delta$, $|\ell| \le k$;

f) $\theta'(F_j) = F_{j+1}$, $1 \le j \le n$;

g) $\|w - 1\|_\varphi^\sharp \le \delta$.

PROOF: The map: $v \in \mathcal{U}(\mathcal{M}) \mapsto \operatorname{Ad}(v) \in \operatorname{Aut}(\mathcal{M})$ is continuous, and the metric: $(v_1,v_2) \in \mathcal{U}(\mathcal{M}) \times \mathcal{U}(\mathcal{M}) \mapsto \big\|v_1 - v_2\big\|_\varphi^\sharp$ gives the topology of $\mathcal{U}(\mathcal{M})$, so that there exists an $\eta > 0$ such that

$$\left\|\psi_s\circ\theta^{-1}\circ\operatorname{Ad}(v^*) - \psi_s\circ\theta^{-1}\right\| \le \frac{\delta}{4}, \quad s \le q$$

whenever $\|v - 1\|_\varphi^\sharp < \eta$, $v \in \mathcal{U}(\mathcal{M})$. We take such an $\eta < \delta$. Applying Theorem 2.10 to $\big\{\mathcal{M},\theta\big\}$ and η, we find $w \in \mathcal{U}(\mathcal{M})$ such that $\operatorname{Ad}(w)\circ\theta \cong \theta\otimes\sigma_0$

and $\|w-1\|_\varphi^\sharp < \eta$. Thus, $\theta' = \mathrm{Ad}(w)\circ\theta$ is of the form: $\theta_1 \otimes \sigma_0$ in a factorization $\mathcal{M} = \mathcal{N} \mathbin{\overline{\otimes}} \mathcal{R}$ with $\mathcal{R} \cong \mathcal{R}_0$. Once θ' is fixed, the construction of σ_0 allows us to choose a partition $\left\{ F_j : j = 1, 2, \dots, n \right\}$ of identity in $\mathbf{C} \otimes \mathcal{R}$ such that (a) and (f) hold. We now set

$$\varphi_\ell = \varphi \circ \theta'^{-\ell}, \qquad |\ell| \le k, \qquad \psi = \frac{1}{2k+1} \sum_{|\ell|\le k} \varphi_\ell,$$

and choose $0 < \varepsilon < \delta/2$ so small that

$$3\varepsilon + 2k\left(2\varepsilon + (2k+1)^{\frac{1}{2}}\sqrt{2}n\varepsilon\right) \le \delta; \qquad \sqrt{2}\,(2k+1)^{\frac{1}{2}} n\varepsilon \le \eta',$$

where $\eta' > 0$ is another small number such that whenever $\|v-1\|_\varphi^\sharp < \eta'$ we have

$$\left\| \psi_s \circ \theta'^{-1} \circ \mathrm{Ad}\, v^{-1} - \psi_s \circ \theta'^{-1} \right\| \le \frac{\delta}{4}, \quad 1 \le s \le q.$$

By the last lemma, we can find $a \in \mathcal{U}(\mathcal{M})$ such that

$$\left\| \psi_s \circ \theta'^{-1} - \psi_s \circ \mathrm{Ad}(a) \right\| \le \varepsilon, \qquad 1 \le s \le q; \tag{α}$$

$$\left\| \varphi_\ell - \varphi \circ \mathrm{Ad}(a^*)^\ell \right\| \le \varepsilon^2, \quad |\ell| \le k; \tag{β}$$

$$\left\| aF_j a^* - \theta'(F_j) \right\|_\psi^\sharp \le \varepsilon, \qquad 1 \le j \le n; \tag{γ}$$

$$\left\| \theta'(a^\ell) - a^\ell \right\|_\varphi^\sharp \le \varepsilon, \qquad |l| \le k. \tag{δ}$$

Since $\theta'(F_j) = F_{j+1},\ 1 \le j \le n$, by (f), Lemma XIV.2.1 guarantees the existence of unitaries $b_j \in \mathcal{U}(\mathcal{M})$ such that

$$b_j a F_j a^* b_j^* = F_{j+1}, \qquad |b_j - 1| \le \sqrt{2}\left|aF_j a^* - F_{j+1}\right|.$$

Putting $b = \sum_{j=1}^n F_{j+1} b_j \in \mathcal{U}(\mathcal{M})$, we obtain a unitary b such that

$$baF_j a^* b^* = F_{j+1};$$

$$\begin{aligned}(b-1)^*(b-1) &= \sum_{j=1}^n (b_j-1)^* F_{j+1} (b_j - 1)\\ &\le \sum_{j=1}^n (b_j-1)^*(b_j-1) \le 2\sum_{j=1}^n \left(aF_j a^* - F_{j+1}\right)^2,\end{aligned}$$

so that

$$\|b-1\|_\varphi^\sharp \le \sqrt{2}\, n\varepsilon.$$

For each $\ell,\ |\ell| \le k$, we have $\|b-1\|_{\varphi_\ell}^2 \le (2k+1)\|b-1\|_\psi^2$ and so $\|b-1\|_{\varphi_\ell} \le \sqrt{2k+1}\sqrt{2}\, n\varepsilon$. By (β) and $\|b-1\| \le 2$, we get

$$\begin{aligned}\left\|(b-1)a^\ell\right\|_\varphi = \|b-1\|_{\varphi\circ\mathrm{Ad}(a^*)^\ell} &\le \|b-1\|_{\varphi_\ell} + 2\left\|\varphi_\ell - \varphi\circ\mathrm{Ad}(a^*)^\ell\right\|^{\frac12}\\ &\le 2\varepsilon + \sqrt{2}\sqrt{2k+1}\,n\varepsilon = \lambda.\end{aligned}$$

Since $(ba)^{\ell+1} - a^{\ell+1} = (b-1)a^{\ell+1} + ba\big((ba)^\ell - a^\ell\big)$ for $\ell \ge 0$, we have

$$\left\|(ba)^{\ell+1} - a^{\ell+1}\right\|_\varphi \le \left\|(b-1)a^{\ell+1}\right\|_\varphi + \left\|(ba)^\ell - a^\ell\right\|_\varphi,$$

so that for $0 \le \ell \le k$, we get

$$\left\|(ba)^\ell - a^\ell\right\|_\varphi \le \ell\lambda.$$

Similarly, for $0 > \ell \ge -k$, we have

$$\left\|(ba)^\ell - a^\ell\right\|_\varphi \le |\ell|\lambda.$$

Considering $\varphi\circ\mathrm{Ad}(a)$ instead of φ, we get

$$\left\|a\,(ba)^\ell a^* - a^\ell\right\|_\varphi \le |\ell|\lambda, \quad |\ell| \le k-1.$$

This together with (β) implies that, for $|\ell| \le k-1$

$$\begin{aligned}\left\|\theta'\big((ba)^\ell\big) - \theta'(a^\ell)\right\|_\varphi &= \left\|(ba)^\ell - a^\ell\right\|_{\varphi\circ\theta'}\\ &\le \sqrt{2}\left\|\varphi\circ\theta' - \varphi\circ\mathrm{Ad}(a)\right\|^{\frac12} + \left\|(ba)^\ell - a^\ell\right\|_{\varphi\circ\mathrm{Ad}(a)}\\ &< 2\varepsilon + |\ell|\lambda.\end{aligned}$$

With $u = ba$, we get, for $|\ell| < k$, using (δ),

$$\begin{aligned}\left\|\theta'(u^\ell) - u^\ell\right\|_\varphi &\le \left\|\theta'\big((ba)^\ell\big) - \theta'(a^\ell)\right\|_\varphi + \left\|\theta'(a^\ell) - a^\ell\right\|_\varphi + \left\|(ba)^\ell - a^\ell\right\|_\varphi\\ &\le 2\varepsilon + |\ell|\lambda + \varepsilon + |\ell|\lambda \;<\; 3\varepsilon + 2k\left(2\varepsilon + \sqrt{2}\sqrt{2k+1}\,n\varepsilon\right) \;\le\; \delta.\end{aligned}$$

What remain to be proven are conditions (c) and (d). Since $\|b-1\|_\psi \le \sqrt{2}n\varepsilon$ and $(2k+1)^{1/2}\sqrt{2}n\varepsilon \le \eta'$, we have $\|b-1\|_\varphi \le \eta'$, so that for $1 \le s \le q$

$$\begin{aligned}&\left\|\psi_s\circ\mathrm{Ad}\,u^* - \psi_s\circ\theta'^{-1}\right\|\\ &\quad= \left\|\psi_s\circ\mathrm{Ad}(a^*)\circ\mathrm{Ad}(b^*) - \psi_s\circ\theta'^{-1}\right\|\\ &\quad\le \left\|\psi_s\circ\mathrm{Ad}(a^*) - \psi_s\circ\theta'^{-1}\right\| + \left\|\psi_s\circ\theta'^{-1}\circ\mathrm{Ad}(b^*) - \psi_s\circ\theta'^{-1}\right\|\\ &\quad\le \varepsilon + \frac{\delta}{4} \;\le\; \delta.\end{aligned}$$

Finally, we get

$$\begin{aligned}\|\varphi\circ\theta' - \varphi\circ\mathrm{Ad}(u)\| &\le \|\varphi\circ\theta' - \varphi\circ\mathrm{Ad}(a)\| + \|\varphi\circ\mathrm{Ad}(ba) - \varphi\circ\mathrm{Ad}(a)\| \\ &\le \varepsilon^2 + \|\varphi\circ\mathrm{Ad}(b) - \varphi\| \\ &\le \varepsilon^2 + 2\|b-1\|_\varphi \ \le\ \varepsilon^2 + 2\eta' \ <\ \delta.\end{aligned}$$

Thus (d) holds. Q.E.D.

We have still some mileage to go to finish the proof Theorem 3.1. Let us find out our position and set the general direction of our arguments. Given $\theta \in \overline{\mathrm{Int}}(\mathcal{M})$ with $p_a(\theta) = 0$, we can find $w \in \mathcal{U}(\mathcal{M})$ near 1 such that $\theta' = \mathrm{Ad}(w)\circ\theta = \theta_1 \otimes \sigma_0$ by Theorem 2.10. Furthermore, by the last lemma θ' is approximated by $\mathrm{Ad}(u)$, $u \in \mathcal{U}(\mathcal{M})$, and $\mathrm{Ad}(u)$ behaves with respect to the partition $\{F_1, \ldots, F_n\}$ of identity like the I_n-component of σ_n. However we do not know the behavior of u itself. If u can be adjusted so that $u^n = 1$, then $\{F_1, \ldots, F_n\}$ and u generate a I_n-subfactor. If θ' leaves u fixed in addition, then θ' behaves on $\{F_1, \ldots, F_n, u\}''$ almost like σ_n on the I_n-component. Namely, $\mathrm{Ad}(w)\circ\theta$ is approximated by $\mathrm{Ad}(u)$ and $\mathrm{Ad}(u)$ on $\{F_1, \ldots, F_n, u\}$ is the I_n-component of σ_n. Since w was chosen from a neighborhood of 1, we will show inductively that the product of w's converges to $w_\infty \in \mathcal{U}(\mathcal{M})$ and the product of $\mathrm{Ad}(u)$'s converges to $\mathrm{Ad}(w_\infty)\circ\theta$ which completes the proof.

Identifying $\mathbf{T}$ with $\mathbf{R}/2\pi\mathbf{Z}$, we set

$$J(\lambda, q) = \left[\lambda - \frac{2\pi}{4^q},\ \lambda + \frac{2\pi}{4^q}\right] \bmod (2\pi\mathbf{Z}), \quad \lambda \in \mathbf{T}. \tag{1}$$

For $\varphi \in \mathfrak{S}_*(\mathcal{M})$ and $u \in \mathcal{U}(\mathcal{M})$, let

$$\Lambda(\varphi, u) = \left\{\lambda \in \mathbf{T} : \varphi\Big(e\big(J(\lambda, q)\big)\Big) \le \frac{1}{2^q},\ q \in \mathbf{N},\ q > 2\right\}, \tag{2}$$

where $e(\lambda)$ is the spectral measure of u.

Lemma 3.4. $\Lambda(\varphi, u) \ne \emptyset$, $u \in \mathcal{U}(\mathcal{M})$.

PROOF: Let m denote the normalized Haar measure on $\mathbf{T}$, i.e. $\mathrm{d}m(\lambda) = 1/2\pi\ \mathrm{d}\lambda$, and μ be the probability measure on $\mathbf{T}$ given by $\mu(J) = \varphi\big(e(J)\big)$ for any Borel set $J \subset \mathbf{T}$. Put for $q > 2$

$$A_q = \left\{\lambda \in \mathbf{T} : \mu\big(J(\lambda, q)\big) \le 2^{-q}\right\},$$

$$B_q = \left\{\big(\lambda_1, \lambda_2\big) \in \mathbf{T}^2 : \big|\lambda_1 - \lambda_2\big| \le \frac{2\pi}{4^q}\right\}.$$

Then we have,

$$\big(m \otimes \mu\big)(B_q) = \int_{\mathbf{T}} m\big(J(\lambda, q)\big)\ \mathrm{d}\mu(\lambda) = 2\cdot 4^{-q}\mu(\mathbf{T}) = 2\cdot 4^{-q},$$

and by Fubini's theorem,

$$\big(m \otimes \mu\big)(B_q) = \int_{\mathbf{T}} \mu\big(J(\lambda, q)\big)\, \mathrm{d}\lambda \geq \int_{\complement A_q} \mu\big(J(\lambda, q)\big)\, \mathrm{d}\lambda > 2^{-q} m\big(\complement A_q\big).$$

Hence we get $m\big(\complement A_q\big) < 2^{-q+1}$, so that

$$m\Big(\bigcup_{q>2} \complement A_q\Big) < \sum_{q=3}^{\infty} 2^{-q+1} = \frac{1}{2}.$$

Therefore, we get

$$m\big(\Lambda(\varphi, u)\big) = m\Big(\bigcap_{q>2} A_q\Big) > 1 - \frac{1}{2} = \frac{1}{2},$$

so that $\Lambda(\varphi, u) \neq \emptyset$. Q.E.D.

We now define a Borel function f_n on $\mathbf{T}$ by

$$f_n\big(\mathrm{e}^{\mathrm{i}\theta}\big) = \mathrm{e}^{\mathrm{i}\theta/n}, \quad \theta \in\,]-\pi, \pi] \cong \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}. \tag{3}$$

Lemma 3.5. *Let* $\big\{\mathcal{M}, \theta, \varphi, \psi_1, \ldots, \psi_q\big\}$ *be as before. For any* $\varepsilon > 0$ *and* $n \in \mathbf{N}$, *there exists a partition* $\big\{F_1, \ldots, F_n\big\}$ *of identity and* $u, v \in \mathcal{U}(\mathcal{M})$ *with the following properties:*

a) $\big\|[F_j, \psi_k]\big\| < \varepsilon$, $1 \leq j \leq n$, $1 \leq k \leq q$;

b) $uF_j u^* = F_{j+1}$, $1 \leq j \leq n$, $\big(F_{n+1} = F_1\big)$;

c) $\big\|\psi_k \circ \theta^{-1} - \psi_k \circ \mathrm{Ad}(u^*)\big\| \leq \varepsilon$, $1 \leq k \leq q$;

d) $-1 \in \Lambda(\varphi, u^n)$;

e) $\mathrm{Ad}(v) \circ \theta(x) = uxu^*$ *for any* x *in the* I_n*-subfactor* $\mathcal{K}$ *generated by* $\tilde{u} = uf_n(u^n)^*$ *and* $\big\{F_j : j = 1, \ldots, n\big\}$;

f) $\|v - 1\|_\varphi < \varepsilon$.

PROOF: Choose a large $m \in \mathbf{N}$ such that $3\,(2^{-m})^{\frac{1}{2}} \leq \varepsilon/8n$. For $p = 1, 2, \ldots, n$, choose polynomials $R_p(z) = \sum_{|t| \leq k} a_{p,t} z^t$ of z and z^{-1} such that

$$\left.\begin{aligned} \Big|R_p(z) - \big(zf_n(z^n)^{-1}\big)^p\Big| &\leq \frac{\varepsilon}{8n}, \quad z \in \mathbf{T}, \quad z^n \notin J(-1, m); \\ \big|R_p(z)\big| &\leq 2, \qquad z \in \mathbf{T}. \end{aligned}\right\} \tag{4}$$

Let $A = \sum_{p,t} |a_{p,t}|$ and choose a small $\delta > 0$ to be specified. Applying Lemma 3.3 to $\big\{\mathcal{M}, \theta, \varphi, \psi_1, \ldots, \psi_q, \delta, n, k\big\}$, we obtain a partition $\big\{F_1, F_2, \ldots, F_n\big\}$ of identity and $u, w \in \mathcal{U}(\mathcal{M})$. Since we can replace u by λu, $\lambda \in \mathbf{T}$, if necessary, Lemma 3.4 allows us to assume $-1 \in \Lambda(\varphi, u^n)$. Put $\theta' = \mathrm{Ad}(w) \circ \theta$, and let e be the spectral projection of u^n corresponding to $J(-1, m)$. Since $\varphi(e) \leq 2^{-m}$, we have, by (4),

$$\left\| R_p(u) - \tilde{u}^p \right\|_\varphi \leq \frac{\varepsilon}{8n} + 3 \cdot 2^{-m/2} < \frac{\varepsilon}{4n} \tag{5}$$

for $p = 1, 2, \ldots, n$ when m is so large that $3 \cdot 2^{-m/2} < \varepsilon/8n$.

With $\varphi_1 = \varphi \circ \mathrm{Ad}(u)$, we also have

$$\left\| R_p(u) - \tilde{u}^p \right\|_{\varphi_1} < \frac{\varepsilon}{4n}$$

because u commutes with both $R_p(u)$ and $\tilde{u}$. Lemma 3.3.(d) and the inequality $\left\| R_p(u) - \tilde{u}^p \right\| \leq 3$ imply that

$$\left\| R_p(u) - \tilde{u}^p \right\|_{\varphi \circ \theta'} < \left(\left(\frac{\varepsilon}{4n} \right)^2 + 9\delta \right)^{\frac{1}{2}}, \quad 1 \leq p \leq n. \tag{6}$$

By the condition (e) of Lemma 3.3 and the choice of A, we get

$$\left\| R_p\big(\theta'(u)\big) - R_p(u) \right\|_\varphi \leq A\delta, \quad 1 \leq p \leq n. \tag{7}$$

Combining (5), (6) and (7), we get

$$\left\| \tilde{u}^p - \theta'\big(\tilde{u}^p\big) \right\|_\varphi \leq \frac{\varepsilon}{4n} + \left(\left(\frac{\varepsilon}{4n} \right)^2 + 9\delta \right)^{\frac{1}{2}} + A\delta.$$

Hence, with a small enough $\delta > 0$, we get

$$\left\| \tilde{u}^p - \theta'\big(\tilde{u}^p\big) \right\|_\varphi \leq \frac{\varepsilon}{n}, \quad 1 \leq p \leq n. \tag{8}$$

As $\delta < \varepsilon$, the conditions (a), (b), (c) and (d) are now automatic. We are going to construct $a \in \mathcal{U}(\mathcal{M})$ so that $v = aw$ satisfies (e) and (f). By construction, $\tilde{u}^n = 1$. Since u^n commutes with F_j's, so does $f_n(u^n)^*$ which means

$$\tilde{u} F_j \tilde{u}^* = u F_j u^* = F_{j+1}, \quad 1 \leq j \leq n.$$

Hence $\tilde{u}$ and F_j's generate a subfactor M of type I_n. We then set $e_{i,j} = \tilde{u}^{i-j} F_j$ to obtain a matrix unit $\{e_{i,j}\}$ of M. We note that u^n and $f_n(u^n)$ both belong to M'. Hence $\tilde{u} e_{i,j} \tilde{u} = e_{i+1,j+1}$ entails that $u e_{i,j} u^* = e_{i+1,j+1}$ for all i, j also, where the indices i, j are considered in $\bmod n$. We then put

$$a = \sum_{j=1}^{n} e_{j+1,2} \theta'(e_{1,j}).$$

By the condition (f) of Lemma 3.3, a is a unitary of $\mathcal{M}$, and

$$a\theta'(e_{i,j})a^* = e_{i+1,2}\theta'(e_{11})e_{2,j+1} = e_{i+1,j+1} = u e_{i,j} u^*.$$

We now set $v = aw$, and see that v satisfies (e). To prove (f), we note that for $1 \leq j \leq n$

$$e_{j+1,2}\theta'(e_{1,j}) = e_{j+1,2}\theta'(F_1)\theta'(\tilde{u}^{1-j}) = e_{j+1,2}\theta'(\tilde{u}^{1-j}),$$

so that (8) entails

$$\left\| e_{j+1,2}\theta'(e_{1,j}) - e_{j+1,2}\tilde{u}^{1-j} \right\|_\varphi \leq \frac{\varepsilon}{n}.$$

Since $e_{j+1,2}\tilde{u}^{1-j} = \tilde{u}^{j-1}F_2\tilde{u}^{1-j} = F_{j+1}$, we have

$$\|a - 1\|_\varphi \leq \varepsilon,$$

so that $\|v - 1\|_\varphi = \|aw - 1\|_\varphi \leq 2\varepsilon$. This completes the proof by replacing ε by $\varepsilon/2$. Q.E.D.

We choose a sequence $\{n_\nu\}$ in $\mathbf{N}$ such that

$$\sum_{\nu=1}^{\infty} \frac{1}{n_\nu} < +\infty \tag{9}$$

and put

$$\delta_\nu = \frac{1}{2^\nu (n_\nu + 1)^3}. \tag{10}$$

Lemma 3.6. *If $\left\{ F_j : 1 \leq j \leq n_\nu \right\}$ is a partition of identity in $\mathcal{M}$ and u is a unitary of $\mathcal{M}$ such that $u^{n_\nu} = 1$ and $uF_ju^* = F_{j+1}$ $(F_{n_{\nu+1}} = F_1)$, then*

$$\left\| \psi - \psi|_{M^c} \otimes \tau_M \right\| < 2^{-\nu}, \quad \psi \in \mathcal{M}_*,$$

whenever $\left\| [u, \psi] \right\| \leq \delta_\nu$ and $\left\| [F_j, \psi] \right\| \leq \delta_\nu$, where M is the I_{n_ν}-subfactor of $\mathcal{M}$ generated by u and $\{F_j\}$.

PROOF: This follows immediately from Lemma XIV.4.9 and the fact that $e_{i,j} = u^{i-j}F_j$ is a matrix unit of M and $\left\| [e_{i,j}, \psi] \right\| \leq (n_\nu + 1)\delta$ if $\left\| [u, \psi] \right\| < \delta$ and $\left\| [F_j, \psi] \right\| < \delta$. Q.E.D.

As before, we keep a fixed faithful state φ.

Lemma 3.7. *For any $n \in \mathbf{N}$ and $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta, \eta) > 0$ such that if $u \in \mathcal{U}(\mathcal{M})$ and $-1 \in \Lambda(\varphi, u^n)$, then with $\tilde{u} = uf_n(u^n)^*$*

$$\left\| [\tilde{u}, \psi] \right\| \leq \delta, \quad \psi \in \mathcal{M}_*^+$$

whenever $\left\| [u, \psi] \right\| \leq 2\varepsilon$ and $0 \leq \psi \leq \varphi$.

PROOF: Let $R(z) = \sum_{k=-m}^{m} a_m z^k$ be such that $\left| R(z) \right| \leq 2,\ z \in \mathbf{T}$ and

$$\left| R(z) - \overline{f_n(z^n)} z \right|^2 \leq \frac{\delta^2}{8}, \quad z \in \mathbf{T} \setminus J(-1, q),$$

where $q \geq 3$ is chosen so that $9/2^q \leq \delta^2/8$. Since $-1 \in \Lambda(\varphi, u^n)$, we have $\left(\left\| R(u) - u f_n(u^n)^* \right\|_\varphi^\sharp\right)^2 \leq \delta^2/8 + 9 \cdot 2^{-q}$. Hence $\| R(u) - \tilde{u} \|_\psi^\sharp \leq \delta/2$ whenever $0 \leq \psi \leq \varphi$. If $\left\| [u, \psi] \right\| < \varepsilon$, then $\left\| [u^k, \psi] \right\| \leq |k|\varepsilon$, $k \in \mathbf{Z}$, so that we set

$$\varepsilon = \varepsilon(\delta, n) = \frac{\delta}{4}\left(\sum_{k=-m}^{m} |k||a_k| \right).$$

It is straightforward to check that this ε works. Q.E.D.

We now choose a decreasing sequence $\{\varepsilon_\nu\}$ such that

$$0 < \varepsilon_\nu \leq \min\left\{ \varepsilon\left(\delta_{\nu+1},\ n_{\nu+1}\right),\ \frac{1}{n_\nu} \right\}. \tag{11}$$

Lemma 3.8. *Let N be a type I_n-subfactor of $\mathcal{M}$ and decompose $\mathcal{M}$ into the tensor product $\mathcal{M} = N \otimes N^c$. Let $\left\{ e_{i,j} : 1 \leq i, j \leq n \right\}$ be a matrix unit of N and $\{\omega_{i,j}\}$ be the basis of N_* dual to $\{e_{i,j}\}$. Then every $\psi \in \mathcal{M}_*$ is uniquely written in the form:*

$$\psi = \sum_{i,j=1}^{n} \omega_{i,j} \otimes \psi_{i,j}, \quad \psi_{i,j} \in N_*^c, \quad 1 \leq i, j \leq n. \tag{12}$$

We then have

a) $\left\| [1 \otimes x, \psi] \right\| \leq n^2 \sup \left\| [\psi_{i,j}, x] \right\|$, $\quad x \in N^c$;

b) *for any $u \in \mathcal{U}(N)$, $v \in \mathcal{U}(N^c)$ and $\theta \in \mathrm{Aut}(N)$,*

$$\left\| \psi \circ \left((\mathrm{Ad}\ u) \otimes \theta \right) - \psi \circ \left(\mathrm{Ad}(u \otimes v) \right) \right\| \leq n^2 \sup \left\| \psi_{i,j} \circ \theta - \psi_{i,j} \circ \mathrm{Ad}(v) \right\|.$$

PROOF: We know that $\|\omega_{i,j}\| = 1$. For every $x \in N^c$,

$$\left[1 \otimes x, \psi \right] = \sum_{i,j} \omega_{i,j} \otimes \left[x, \psi_{i,j} \right],$$

which shows (a). For (b), we have

$$\psi \circ \left(\mathrm{Ad}(u) \otimes \theta \right) = \sum_{i,j} \left(\omega_{i,j} \circ \mathrm{Ad}(u) \right) \otimes \left(\psi_{i,j} \circ \theta \right),$$

$$\psi \circ \left(\mathrm{Ad}(u \otimes v) \right) = \sum_{i,j} \left(\omega_{i,j} \circ \mathrm{Ad}(u) \right) \otimes \left(\psi_{i,j} \circ \mathrm{Ad}(v) \right).$$

Since $\left\| \omega_{ij} \circ \mathrm{Ad}(u) \right\| = \|\omega_{ij}\| = 1$, we get (b). Q.E.D.

We finally come to the following last lemma:

Lemma 3.9. *Let* $\{\mathcal{M}, \theta, \varphi\}$ *be as before, and* $\{\psi_j\}$ *be a sequence in* $\mathcal{M}_*$ *with* $0 \le \psi_j \le \varphi$. *Then there exist a sequence* $\{M_\nu\}$ *of finite type* I *subfactors of* $\mathcal{M}$ *and a sequence* $\{a_\nu\}$ *of unitaries in* $\mathcal{M}$ *such that the following conditions hold:*

a) M_j *and* M_ν *commute,* $j < \nu$, $\nu \in \mathbf{N}$;

b) M_ν *is generated by a partition* $\left\{ F_j^\nu : 1 \le j \le n_\nu \right\}$ *of identity and a unitary* u_ν *such that*

$$u_\nu F_j^\nu u_\nu^* = F_{j+1}^\nu, \quad \left(F_{n_\nu+1}^\nu = F_1^\nu\right), \quad 1 \le j \le n_\nu, \quad u_\nu^{n_\nu} = 1;$$

c) $\left\|[u_\nu, \psi_\ell]\right\| \le \delta_\nu$, $\ell < \nu$, $\left\|[F_j^\nu, \psi_\ell]\right\| \le \delta_\nu$, $\ell < \nu$;

d) $a_\nu \in \left(M_1 \vee M_2 \vee \cdots \vee M_{\nu-1}\right)^c$;

e) $\left\|(a_\nu - 1)a_{\nu-1} \cdots a_1\right\|_\varphi^\sharp \le 8/n_\nu$;

f) *With* $\theta_\nu = \mathrm{Ad}\left(a_\nu a_{\nu-1} \ldots a_1\right) \circ \theta$, θ_ν *leaves* M_j, $1 \le j \le \nu$, *globally invariant and agrees with* $\mathrm{Ad}(u_j)$ *on* M_j;

g) $\left\|\psi_j \circ \theta_\nu^{-1} - \psi_j \circ \mathrm{Ad}(u_\nu u_{\nu-1} \cdots u_1)^{-1}\right\| \le \varepsilon_\nu$.

PROOF: We construct $\{a_\nu\}$ and $\{M_\nu\}$ by induction. Suppose a_j and M_j, $1 \le j \le \nu$, have been constructed.

Let $N = M_1 \vee M_2 \vee \cdots \vee M_\nu$ and $n = \prod_{j=1}^{n_\nu} n_j$. Then N is of type I_n. Put $Q = N^c$ and $U = u_\nu u_{\nu-1} \cdots u_1$. By assumption, $\theta_\nu = \mathrm{Ad}\left(a_\nu a_{\nu-1} \cdots a_1\right) \circ \theta$ leaves N globally invariant and agrees with $\mathrm{Ad}(U)$. Let $\tilde{\theta}$ be the restriction of θ_ν to N^c. Let $\left\{ \omega_{i,j} : 1 \le i, j \le n \right\}$ be the basis of N_* dual to a matrix unit of N and for $\ell = 1, 2, \ldots, \nu + 1$, decompose ψ_ℓ in the form: $\psi_\ell = \sum_{i,j=1}^n \omega_{ij} \otimes \psi_\ell^{ij}$ with $\psi_\ell^{i,j} \in N_*^c$.

By Theorem 2.10, $\tilde{\theta}$ is outer conjugate to θ, so that $\tilde{\theta} \in \overline{\mathrm{Int}}(N^c)$ and $p_a(\tilde{\theta}) = 0$. We apply Lemma 3.5 to $\left\{N^c, \tilde{\theta}, \varphi_{N^c} = \varphi|_{N^c}, \psi_l^{i,j}, n_{\nu+1}\right\}$ to obtain a partition $\left\{ F_j : 1 \le j \le n_{\nu+1} \right\}$ of identity in N^c and unitaries $u, v \in N^c$ such that

$$\left\|[F_j, \psi_\ell^{i,k}]\right\| \le \delta_{\nu+1}/n^2, \quad 1 \le j \le n_{\nu+1}, \quad 1 \le i, k \le n, \tag{13}$$

$$uF_j u^* = F_{j+1}, \quad 1 \le j \le n_{\nu+1}, \tag{14}$$

$$\left\|\psi_\ell^{i,j} \circ \tilde{\theta}^{-1} - \psi_\ell^{i,j} \circ \mathrm{Ad}(u^{-1})\right\| < \frac{\varepsilon_{\nu+1}}{2n^2}, \quad 1 \le \ell \le \nu + 1, \tag{15}$$

$$-1 \in \Lambda\left(\varphi_{N^c}, u^{n_{\nu+1}}\right), \tag{16}$$

$$\mathrm{Ad}(v) \circ \theta(x) = uxu^*, \quad x \in M, \tag{17}$$

where M is the $\mathrm{I}_{n_{\nu+1}}$-subfactor generated by the F_j's and

$$\tilde{u} = u f_{n_{\nu+1}}(u^{n_{\nu+1}})^*, \tag{18}$$

$$\left.\begin{aligned} &\|v-1\|_\varphi < \frac{\varepsilon_{\nu+1}}{2}, \qquad \left\|\tilde\theta^{-1}(v)-1\right\|_\varphi < \frac{\varepsilon_{\nu+1}}{4} \\ &\left\|(v-1)a_\nu a_{\nu-1}\cdots a_1\right\|_\varphi < \frac{\varepsilon_{\nu+1}}{2}. \end{aligned}\right\} \tag{19}$$

Here we are applying Lemma 3.5 with $\varepsilon \leq \min\{\delta_{\nu+1}, \varepsilon_{\nu+1}/2\}$ which is so small that $\|v-1\|_{\varphi|_{N^c}} < \varepsilon$ for any $v \in \mathcal{U}(N^c)$ entails the above condition (19). Since $\theta_\nu = \operatorname{Ad} U \otimes \tilde\theta$ in the decomposition $\mathcal{M} = N \otimes N^c$, (15) and the last lemma imply that

$$\left\|\psi_\ell \circ \theta_\nu^{-1} - \psi_\ell \circ \operatorname{Ad}(uU)^{-1}\right\| \leq \frac{\varepsilon_{\nu+1}}{2}, \quad 1 \leq \ell \leq \nu+1. \tag{20}$$

The induction hypothesis (g) means

$$\left\|\psi_\ell \circ \theta_\nu^{-1} - \psi_\ell \circ \operatorname{Ad}(U)^{-1}\right\| \leq \varepsilon_\nu, \quad 1 \leq \ell \leq \nu.$$

The commutativity of u and U then gives, for $1 \leq \ell \leq \nu$,

$$\left\|\psi_\ell \circ \operatorname{Ad}(u)^{-1} - \psi_\ell\right\| \leq \varepsilon_\nu + \frac{\varepsilon_{\nu+1}}{2} \leq 2\varepsilon_\nu,$$

so that

$$\left\|[u, \psi_\ell]\right\| \leq 2\varepsilon_\nu, \quad 1 \leq \ell \leq \nu. \tag{21}$$

As $0 \leq \psi_\ell \leq \varphi$, Lemma 3.7 entails by (11) that

$$\left\|[\tilde u, \psi_\ell]\right\| \leq \delta_{\nu+1}, \quad 1 \leq \ell \leq \nu. \tag{22}$$

Put $b = f_{n_{\nu+1}}(u^{n_{\nu+1}})^* = \tilde u u^*$. Then $\|b-1\| \leq \pi/n_{\nu+1}$, and with $a = bv$ we have by (19) and (11)

$$\left\|(1-a)a_\nu a_{\nu-1}\cdots a_1\right\|_\varphi \leq \frac{\pi}{n_{\nu+1}} + \frac{\varepsilon_{\nu+1}}{2} \leq \frac{8}{n_{\nu+1}},$$

$$\left\|a_1^* a_2^* \cdots a_\nu^*(1-a^*)\right\|_\varphi \leq \left\|(1-v^*)\right\|_\varphi + \frac{\pi}{n_{\nu+1}} < \frac{\varepsilon_{\nu+1}}{2} + \frac{\pi}{n_{\nu+1}} \leq \frac{8}{n_{\nu+1}},$$

so that

$$\left\|(1-a)a_\nu a_{\nu-1}\cdots a_1\right\|_\varphi^\sharp < \frac{8}{n_{\nu+1}}. \tag{23}$$

As $0 \leq \psi_\ell \leq \varphi$, we have by (19)

$$\begin{aligned} \left\|\psi_\ell \circ \theta_\nu^{-1} \circ \operatorname{Ad}(v^*) - \psi_\ell \circ \theta_\nu^{-1}\right\| &= \left\|\psi_\ell \circ \operatorname{Ad}\big(\theta_\nu^{-1}(v^*)\big) - \psi_\ell\right\| \\ &\leq 2\left\|\theta_\nu^{-1}(v) - 1\right\|_{\psi_\ell} \ \leq\ 2\left\|\theta_\nu^{-1}(v)-1\right\|_\varphi \\ &= 2\left\|\tilde\theta^{-1}(v) - 1\right\|_\varphi \ \leq\ \frac{\varepsilon_{\nu+1}}{2}. \end{aligned}$$

Hence, we get by (20)

$$\left\|\psi_\ell \circ \theta_\nu^{-1} \circ \operatorname{Ad}(v^*) - \psi_\ell \circ \operatorname{Ad}(uU)^{-1}\right\| \leq \varepsilon_{\nu+1}, \quad 1 \leq \ell \leq \nu+1.$$

Therefore we obtain the estimate:

$$\begin{aligned}
&\left\|\psi_\ell \circ \theta_\nu^{-1} \circ \mathrm{Ad}(a^*) - \psi_\ell \circ \mathrm{Ad}(\tilde{u}U)^{-1}\right\| \\
&\quad = \left\|\psi_\ell \circ \theta_\nu^{-1} \circ \mathrm{Ad}(v^*) \circ \mathrm{Ad}(b^*) - \psi_\ell \circ \mathrm{Ad}(U^{-1}) \circ \mathrm{Ad}(u^*b^*)\right\| \\
&\quad = \left\|\psi_\ell \circ \theta_\nu^{-1} \circ \mathrm{Ad}(v^*) - \psi_\ell \circ \mathrm{Ad}(uU)^{-1}\right\| \;\le\; \varepsilon_{\nu+1}.
\end{aligned}$$

Hence we have

$$\left\|\psi_\ell \circ \theta_\nu \circ \mathrm{Ad}(a)^{-1} - \psi_\ell \circ \mathrm{Ad}(\tilde{u}U)^{-1}\right\| \le \varepsilon_{\nu+1}, \quad 1 \le \ell \le \nu + 1. \tag{24}$$

We now set

$$\begin{cases}
F_j^{\nu+1} = F_j, \quad 1 \le j \le n_{\nu+1}, \\
u_{\nu+1} = \tilde{u}, \\
M_{\nu+1} = \left\{ F_j, u_{\nu+1} : 1 \le j \le n_{\nu+1} \right\}'', \\
a_{\nu+1} = a.
\end{cases}$$

Conditions (a) and (b) are automatic. Inequality (22) gives the first half of (c). Inequality (13) and Lemma 3.7 give the second half of (c). Condition (d) is automatic by the construction of a. Inequality (23) is precisely (e). With

$$\theta_{\nu+1} = \mathrm{Ad}(a) \circ \theta_\nu = \mathrm{Ad}\, U \otimes \mathrm{Ad}(a) \circ \tilde{\theta},$$

we get (f) for $1 \le j \le \nu$. To establish (f) for $j = \nu + 1$, we observe that $M_{\nu+1} = M$, and must show that $\mathrm{Ad}(a) \circ \tilde{\theta}(x) = \tilde{u}x\tilde{u}^*$ for every $x \in M$. But we have (17) and $a = bv$, so that for every $x \in M$

$$a\tilde{\theta}(x)a^* = bv\theta(x)v^*b^* = buxu^*b^* = \tilde{u}x\tilde{u}^*.$$

Thus (f) for $j = \nu + 1$ follows.

Finally, inequality (24) is precisely (g) for $\nu + 1$. Q.E.D.

PROOF OF THEOREM 3.1: Let $\{\mathcal{M}, \theta\}$ be as before and choose a faithful $\varphi \in \mathfrak{S}_*$ and a sequence $\{\psi_\nu\}$ which is dense in $\left\{\psi \in \mathcal{M}_*^+ : 0 \le \psi \le \varphi\right\}$. For a given $\varepsilon > 0$, we choose $\{n_\nu\} \subset \mathbf{N}$ such that $\sum_{\nu=1}^\infty 1/n_\nu < \varepsilon/8$. We then construct $\{M_\nu\}$, $\{a_\nu\}$ and $\{u_\nu\}$ by Lemma 3.9, and set $\mathcal{R} = \bigvee_{\nu=1}^\infty M_\nu$. We now observe that

a) $\mathcal{R}$ is an AFD II_1-subfactor of $\mathcal{M}$ and with $Q = \mathcal{R}^c$

$$\mathcal{M} = \mathcal{R} \mathbin{\overline{\otimes}} Q \tag{25}$$

by Lemmas XIV.4.9 and XIV.4.10;

b) $w_\nu = a_\nu a_{\nu-1} \cdots a_1$ converges σ-strongly* to a unitary $w \in \mathcal{M}$ since $\|w_\nu - w_{\nu-1}\|_\varphi^\sharp \le 8/n_\nu$ by the condition (e) of the lemma.

Since we may set $w_0 = 1$, we have

$$\|w - 1\|_\varphi \le \sum_{\nu=1}^\infty \left\|w_\nu - w_{\nu-1}\right\|_\varphi \le \sum_{\nu=1}^\infty \frac{8}{n_\nu} < \varepsilon.$$

Let $\theta_\infty = \mathrm{Ad}(w) \circ \theta = \lim_{\nu\to\infty} \theta_\nu$. Then θ_∞ leaves each M_ν globally invariant and agrees with $\mathrm{Ad}(u_\nu)$ on M_ν. Therefore $\theta_\infty|_{\mathcal{R}} \cong \prod_{\nu=1}^{\infty\,\otimes} \mathrm{Ad}(u_\nu)$. Let $\alpha = \prod_{\nu=1}^{\infty\,\otimes} \mathrm{Ad}(u_\nu)$ on $\mathcal{R}$. By (g) of Lemma 3.9, we get, relative to the decomposition (25),

$$\theta_\infty \cong \alpha \otimes \mathrm{id}_Q, \tag{26}$$

where id_Q is the identity automorphism of Q.

By Theorem 2.10, a small perturbation α' of α by $\mathrm{Int}(\mathcal{R})$ is conjugate to $\alpha \otimes \mathrm{id}_{\mathcal{R}}$, so that there exists $u \in \mathcal{U}(\mathcal{R})$ such that $\|u - 1\|_\varphi < \varepsilon$ and $\mathrm{Ad}(u) \circ \mathrm{Ad}(w) \circ \theta = \alpha \otimes \mathrm{id}_{\mathcal{R}\otimes Q}$ with respect to the decomposition $\mathcal{M} = \mathcal{R} \,\overline{\otimes}\, \mathcal{R} \,\overline{\otimes}\, Q$. The behavior of α on $\mathcal{R}$ depends only on the choice of $\{n_\nu\}$ not on θ itself. Thus, we conclude that a small perturbation θ' of θ by $\mathrm{Int}(\mathcal{M})$ conjugate to $\sigma_o \otimes \mathrm{id}_{\mathcal{N}}$ with respect to a decomposition $\mathcal{M} = \mathcal{R}_0 \,\overline{\otimes}\, \mathcal{N}$ where $\mathcal{N} \cong \mathcal{M}$. This completes the proof of Theorem 3.1. Q.E.D.

Corollary 3.10. *The* AFD *factor* $\mathcal{R}_0$ *of type* II_1 *has only one outer conjugacy class of aperiodic automorphisms. More precisely, if* θ_1 *and* θ_2 *are aperiodic automorphisms of* $\mathcal{R}_0$*, then for any* $\varepsilon > 0$ *there exist* $u \in \mathcal{U}(\mathcal{R}_0)$ *and* $\sigma \in \mathrm{Aut}(\mathcal{R}_0)$ *such that*

a) $\|u - 1\|_2 < \varepsilon$;

b) $\sigma \circ \mathrm{Ad}(u) \circ \theta_1 \circ \sigma^{-1} = \theta_2$.

PROOF: By Theorem XIV.2.16, we have $\mathrm{Aut}(\mathcal{R}_0) = \overline{\mathrm{Int}}(\mathcal{R}_0)$ and also $\mathrm{Cnt}(\mathcal{R}_0) = \mathrm{Int}(\mathcal{R}_0)$ by Theorem XIV.4.16. Hence Theorem 3.1 whose proof was just completed gives the above conclusion. Q.E.D.

Lemma 3.11. *If* $\mathcal{R}_{0,1}$ *is the* AFD *factor of type* II_∞*, then*

$$\mathrm{Cnt}\big(\mathcal{R}_{0,1}\big) = \mathrm{Int}\big(\mathcal{R}_{0,1}\big).$$

PROOF: Let $\theta \in \mathrm{Cnt}\big(\mathcal{R}_{0,1}\big)$, so that $p_a(\theta) = 1$. By Theorem 2.10, θ is outer conjugate to $\theta \otimes \mathrm{id}_{\mathcal{R}_0}$ with a decomposition: $\mathcal{R}_{0,1} \cong \mathcal{R}_{0,1} \,\overline{\otimes}\, \mathcal{R}_0$. Furthermore, θ is outer conjugate to $\theta \otimes \mathrm{id}_{\mathcal{B}}$ with a decomposition: $\mathcal{R}_{0,1} \cong \mathcal{R}_{0,1} \,\overline{\otimes}\, \mathcal{B}$, where $\mathcal{B} = \mathcal{L}\big(\ell^2(\mathbf{Z})\big)$. Hence we have $\theta \sim \theta \otimes \mathrm{id}_{\mathcal{R}_{0,1}}$, so that

$$\theta \otimes \mathrm{id}_{\mathcal{R}_{0,1}} \in \mathrm{Cnt}\big(\mathcal{R}_{0,1} \,\overline{\otimes}\, \mathcal{R}_{0,1}\big).$$

Let s be the symmetry: $s(x \otimes y) = y \otimes x$ on $\mathcal{R}_{0,1} \,\overline{\otimes}\, \mathcal{R}_{0,1}$. It follows that $s \in \overline{\mathrm{Int}}\big(\mathcal{R}_{0,1} \,\overline{\otimes}\, \mathcal{R}_{0,1}\big)$. Hence s and $\theta \otimes \mathrm{id}_{\mathcal{R}_{0,1}}$ commute modulo $\mathrm{Int}\big(\mathcal{R}_{0,1} \,\overline{\otimes}\, \mathcal{R}_{0,1}\big)$ by Corollary 2.11 or Lemma XIV.4.14. Thus we have

$$\theta \otimes \theta^{-1} = \big(\theta \otimes \mathrm{id}_{\mathcal{R}_{0,1}}\big) s \big(\theta^{-1} \otimes \mathrm{id}_{\mathcal{R}_{0,1}}\big) s^{-1} \in \mathrm{Int}\big(\mathcal{R}_{0,1} \otimes \mathcal{R}_{0,1}\big),$$

so that θ itself is inner. Q.E.D.

Theorem 3.12. *If* $\mathcal{R}_{0,1}$ *is the* AFD *factor of type* II_∞*, then the conjugacy class of non-unimodular automorphism* θ *of* $\mathcal{R}_{0,1}$*, i.e.* $\theta \in \mathrm{Aut}\big(\mathcal{R}_{0,1}\big)$ *with* $\mathrm{mod}(\theta) \neq 1$*, is uniquely determined by* $\mathrm{mod}(\theta)$*.*

PROOF: Since $\mathcal{R}_{0,1} \cong \mathcal{R}_0 \overline{\otimes} \mathcal{L}(\ell^2)$, where $\mathcal{R}_0$ is the AFD II_1-factor, $\mathcal{R}_{0,1}$ is strongly stable. Let θ_1 and θ_2 be automorphisms of $\mathcal{R}_{0,1}$ with $\mathrm{mod}(\theta_1) = \mathrm{mod}(\theta_2) = \lambda \neq 1$. Then θ_1 and θ_2 are both aperiodic, so that $p_a(\theta_1) = p_0(\theta_1) = 0 = p_0(\theta_2) = p_a(\theta_2)$. Hence $\theta_1 \sim \theta_1 \otimes \sigma_0$ and $\theta_2 \sim \theta_2 \otimes \sigma_0$ by Theorem 2.10. Put $\theta = \theta_1 \otimes \theta_2^{-1}$. Then we have $\mathrm{mod}(\theta) = 1$. Identify $\mathcal{R}_{0,1}$ with $\mathcal{R}_{0,1} \overline{\otimes} \mathcal{R}_{0,1}$. Choose $e \in \mathrm{Proj}(\mathcal{R}_{0,1})$ with $\tau(e) = 1$. Then $e \sim \theta(e)$, so that there exists $u \in \mathcal{U}(\mathcal{R}_{0,1})$ such that $e = \mathrm{Ad}(u) \circ \theta(e)$. Let $\big\{ e_{i,j} : i, j \in \mathbf{N} \big\}$ be a matrix unit of $\mathcal{R}_{0,1}$ such that $e_{1,1} = e$, and let $\mathcal{B}$ be the type I_∞ subfactor of $\mathcal{R}_{0,1}$ generated by $\{e_{i,j}\}$. Let $v = \sum_{i=1}^n e_{i,1} u \theta(e_{1,i})$ and $\theta' = \mathrm{Ad}(v) \circ \theta$. Then we have $\mathcal{R}_{0,1} \cong (\mathcal{R}_{0,1})_e \overline{\otimes} \mathcal{B}$ and $\theta' = \theta_0 \otimes \mathrm{id}$, where $\theta_0 = (\theta')^e$. Since $\theta_0 \in \mathrm{Aut}\big((\mathcal{R}_{0,1})_e\big) = \overline{\mathrm{Int}}\big((\mathcal{R}_{0,1})_e\big)$, $(\mathcal{R}_{0,1})_e$ being isomorphic to $\mathcal{R}_0$, we have $\theta' \in \overline{\mathrm{Int}}(\mathcal{R}_{0,1})$; consequently $\theta \in \overline{\mathrm{Int}}(\mathcal{R}_{0,1})$. Therefore, we conclude that $\theta \sim \sigma_0 \otimes \mathrm{id}$ on $\mathcal{R}_{0,1} = \mathcal{R}_0 \overline{\otimes} \mathcal{L}(\ell^2)$. Finally, we get on $\mathcal{R}_{0,1} \cong \mathcal{R}_{0,1} \overline{\otimes} \mathcal{R}_0$

$$\begin{aligned} \theta_2 &\sim \theta_2 \otimes \sigma_0 \sim \theta_2 \otimes \sigma_0 \otimes \mathrm{id}_{\mathcal{L}(\mathfrak{H})} \sim \theta_2 \otimes \big(\theta_1 \otimes \theta_2^{-1}\big) \sim \theta_2 \otimes \big(\theta_2^{-1} \otimes \theta_1\big) \\ &\sim \big(\theta_2 \otimes \theta_2^{-1}\big) \otimes \theta_1 \sim \mathrm{id}_{\mathcal{L}(\mathfrak{H})} \otimes \sigma_0 \otimes \theta_1 \sim \theta_1. \end{aligned}$$

Therefore, θ_1 and θ_2 are outer conjugate. By Theorem XII.1.11, θ_1 and θ_2 are stable, so that they are conjugate. Q.E.D.

Proposition 3.13. *Let* $\mathcal{M}$ *be a factor and* $\theta = \mathrm{Aut}(\mathcal{M})$*. Let* $p = p_a(\theta) > 0$*. Choose* $u \in \mathcal{U}(\mathcal{M})$ *with* $\theta^p = \mathrm{Ad}(u)$*. Then there exists a* p*-th root* γ *of unity such that* $\theta(u) = \gamma u$*. The number* γ *is an outer conjugacy invariant of* θ*.*

PROOF: First, we have

$$\mathrm{Ad}\big(\theta(u)\big) = \theta\, \mathrm{Ad}(u) \theta^{-1} = \theta \theta^p \theta^{-1} = \theta^p = \mathrm{Ad}(u),$$

so that $u^* \theta(u) = \gamma \in \mathbf{C}$, and $|\gamma| = 1$. Next,

$$\gamma^p u = \theta^p(u) = u\, u\, u^* = u,$$

so that $\gamma^p = 1$. Since u is unique up to a scalar multiple, γ does not depend on the choice of u.

The number γ is clearly conjugacy invariant. Suppose $\overline{\theta} = \mathrm{Ad}(v) \circ \theta$. Inductively set $v_k = v\theta(v_{k-1})$ with $v_1 = v$ and $v_s = 1$. Then we have $\overline{\theta}^k = \mathrm{Ad}(v_k) \circ \theta^k$, so that $\overline{\theta}^p = \mathrm{Ad}(v_p u)$; and so

$$\overline{\theta}(v_p u) = v\theta(v_p) \gamma u v^* = \gamma v_{p+1} u v^* = \gamma v_p \theta^p(v) \theta^p(v^*) u = \gamma v_p u.$$

Therefore, the same γ works for $\overline{\theta}$. Thus, γ is an outer conjugacy invariant of θ. Q.E.D.

Definition 3.14. The p-th root γ of unity in the last proposition is called the *obstruction* of θ and denoted by $\mathrm{Ob}(\theta)$.

Proposition 3.15. *For any $p \in \mathbf{N}$ and a p-th root of unity γ, the* AFD *factor $\mathcal{R}_0$ of type* II_1 *admits an automorphism σ_p^γ with $p = p_0(\sigma_p^\gamma)$ and $\gamma = \mathrm{Ob}(\sigma_p^\gamma)$.*

PROOF: Let $\gamma = \mathrm{e}^{2\pi \mathrm{i}k/p}$, $0 \leq k \leq p-1$. Let α be an aperiodic automorphism of an AFD factor $\mathcal{M}$ of type II_1. Set $\mathcal{R}_0 = \mathcal{M} \rtimes_\alpha (p\mathbf{Z})$. Since $\mathcal{R}_0$ is injective, it is AFD and of type II_1. Let u be the unitary of $\mathcal{R}_0$ implementing α^p on $\mathcal{M}$. Consider the dual action $\hat{\alpha}$ of $(p\mathbf{Z})^\frown$ in the pairing: $\hat{\alpha}_s(u) = \mathrm{e}^{2\pi \mathrm{i}s}u$, $0 \leq s < 1$, and $\hat{\alpha}_s(x) = x$, $x \in \mathcal{M}$. Observe that the automorphism α extends to $\mathcal{R}_0$ naturally, and $\alpha^p = \mathrm{Ad}(u)$. Set $\theta = \hat{\alpha}_{k/p} \circ \alpha$. Since α and $\hat{\alpha}$ commute, we can compute θ^p easily and in fact $\theta^p = \alpha^p = \mathrm{Ad}(u)$. Also we have $\theta(u) = \mathrm{e}^{2\pi \mathrm{i}k/p}u = \gamma u$.

Suppose $\theta^j = \mathrm{Ad}(v)$ for $1 \leq j < p$ with $v \in \mathcal{U}(\mathcal{R}_0)$. Since θ and α agree on $\mathcal{M}$, we have $\alpha^j(x) = vxv^*$ for every $x \in \mathcal{M}$. For every $n \in \mathbf{Z}$, we have also

$$u^n vxv^* u^{-n} = \alpha^{pn+j}(x), \quad x \in \mathcal{M}.$$

Let ε be the canonical conditional expectation of $\mathcal{R}_0$ onto $\mathcal{M}$. Then we have

$$\varepsilon(u^n v)x = \alpha^{pn+j}(x)\varepsilon(u^n v), \quad x \in \mathcal{M}.$$

Since α^{pn+j} is free, we have $\varepsilon(u^n v) = 0$, $n \in \mathbf{Z}$. But we have, in $L^2(\mathcal{R}_0, \tau)$,

$$v = \sum_{n \in \mathbf{Z}} u^n \varepsilon(u^{-n}v).$$

Thus, this is impossible. Hence $p_0(\theta) = p$. Q.E.D.

Theorem 3.16. *Let $\mathcal{M}$ be a strongly stable separable factor, and $\theta_1, \theta_2 \in \overline{\mathrm{Int}}(\mathcal{M})$. If $p = p_0(\theta_1) = p_0(\theta_2) = p_a(\theta_1) = p_a(\theta_2) > 0$, then the following two conditions are equivalent:*

(i) $\mathrm{Ob}(\theta_1) = \mathrm{Ob}(\theta_2)$;

(ii) *θ_1 and θ_2 are outer conjugate, i.e. there exist $\pi \in \mathrm{Aut}(\mathcal{M})$ and a unitary $u \in \mathcal{U}(\mathcal{M})$ such that*

$$\pi \circ \mathrm{Ad}(u) \circ \theta_1 \circ \pi^{-1} = \theta_2.$$

PROOF: The implication: (ii) $\Longrightarrow$ (i) was already proved in Proposition 3.13.

First, we reduce the proof to the case of trivial obstruction. Namely, we assume that if $\theta \in \overline{\mathrm{Int}}(\mathcal{M})$ satisfies the condition that $p_0(\theta) = p_a(\theta) = p$ and $\mathrm{Ob}(\theta) = 1$, then $\theta \sim \mathrm{id} \otimes \sigma_p$ on $\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{R}_0$ where $\sigma_p \in \mathrm{Aut}(\mathcal{R}_0)$ is the automorphism given by (2.5).

Suppose that $\theta \in \overline{\mathrm{Int}}(\mathcal{M})$ satisfies the condition that $p_0(\theta) = p_a(\theta) = p$ and $\mathrm{Ob}(\theta) = \gamma$. By Theorem 2.10, we have $\theta \sim \theta \otimes \sigma_p$ on $\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{R}_0$. Since $\mathrm{Ob}(\alpha \otimes \beta) = \mathrm{Ob}(\alpha)\,\mathrm{Ob}(\beta)$ for any $\alpha \in \mathrm{Aut}(\mathcal{M})$ and $\beta \in \mathrm{Aut}(\mathcal{R}_0)$ with $p_0(\alpha) = p_0(\beta) = p$, we have $\mathrm{Ob}\big(\theta \otimes \sigma_p^{\overline{\gamma}}\big) = \gamma\,\overline{\gamma} = 1$. Since $\theta \otimes \sigma_p^{\overline{\gamma}} \in \overline{\mathrm{Int}}\big(\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{R}_0\big)$ and $\theta \otimes \sigma_p^{\overline{\gamma}} \sim \mathrm{id} \otimes \sigma_p$.

Therefore, we have

$$\begin{aligned}\theta &\sim \theta \otimes \sigma_p \sim \theta \otimes \mathrm{id} \otimes \sigma_p \sim \theta \otimes \big(\sigma_p^{\overline{\gamma}} \otimes \sigma_p^{\gamma}\big)\\ &\sim \big(\theta \otimes \sigma_p^{\overline{\gamma}}\big) \otimes \sigma_p^{\gamma} \sim \big(\mathrm{id} \otimes \sigma_p\big) \otimes \sigma_p^{\gamma} \sim \mathrm{id} \otimes \big(\sigma_p \otimes \sigma_p^{\gamma}\big) \sim \mathrm{id} \otimes \sigma_p^{\gamma}.\end{aligned}$$

Thus, all we have to do is to prove the statement that if $\theta \in \overline{\mathrm{Int}}(\mathcal{M})$ satisfies the conditions: $p_0(\theta) = p_a(\theta) = p > 0$ and $\mathrm{Ob}(\theta) = 1$, then $\theta \sim \mathrm{id} \otimes \sigma_p$ on $\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{R}_0$.

Since $\mathrm{Ob}(\theta) = 1$, we can choose $u \in \mathcal{U}(\mathcal{M}^\theta)$ with $\theta^p = \mathrm{Ad}(u)$. Let $u = \mathrm{e}^{\mathrm{i}h}$ for some self-adjoint $h \in \mathcal{M}^\theta$. Set $v = \mathrm{e}^{-\mathrm{i}h/p}$. Then $\big(\mathrm{Ad}(v) \circ \theta\big)^p = \mathrm{id}$. We now replace θ by $\mathrm{Ad}(v) \circ \theta$, so that we have $\theta^p = \mathrm{id}$. Thus θ can be interpreted as a free action of the cycle group $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z} = \big\{0, 1, \dots, p-1\big\}$. We fix a pairing of $\mathbf{Z}_p$ and its dual $\widehat{\mathbf{Z}}_p = \mathbf{Z}_p$ as follows:

$$\langle j, k\rangle = \mathrm{e}^{2\pi\mathrm{i}jk/p}, \quad j, k \in \mathbf{Z}_p.$$

We then split the proof into a few steps. [STOP]

Lemma 3.17. *There exists a sequence $\{u_n\}$ in $\mathcal{U}(\mathcal{M}^\theta)$ such that*

$$\theta = \lim_{n\to\infty} \mathrm{Ad}(u_n) \qquad \textit{and} \qquad u_n^p = 1, \quad n \in \mathbf{N}.$$

PROOF: By the assumption, there exists a sequence $\{v_n\}$ in $\mathcal{U}(\mathcal{M})$ with $\theta = \lim_{n\to\infty} \mathrm{Ad}(v_n)$ in the topology of $\mathrm{Aut}(\mathcal{M})$. Let ω be a free ultra filter on $\mathbf{N}$, and consider $\big\{v_n^*\theta^k(v_n)\big\} = V_k$ as an element of $\mathcal{U}(\mathcal{M}_\omega)$. By the assumption on $p_a(\theta)$, θ_ω is a free action of $\mathbf{Z}_p$ and $\{V_k\}$ is a one cocycle for θ_ω, so that there exists an element $x = \{x_n\}$ in $\mathcal{U}(\mathcal{M}_\omega)$ and that $V_k = X^*\theta_\omega^k(X)$. Passing to a subsequence of $\{x_n\}$, we obtain a strongly central sequence $\{x_n\}$ in $\mathcal{U}(\mathcal{M})$ such that

$$\lim_{n\to\infty}\left\| x_n^*\theta^k(x_n) - v_n^*\theta^k(v_n)\right\|_\varphi^\sharp = 0, \quad k \in \mathbf{Z}_p,$$

for any faithful $\varphi \in \mathfrak{S}_*$. The strong centrality of $\{x_n\}$ means that the multiplication of x_n from either side is almost unitary in $\|\cdot\|_\varphi$-norm. The convergence of $\mathrm{Ad}(v_n)$ to θ in $\mathrm{Aut}(\mathcal{M})$ implies that the set $\big\{\varphi \circ \mathrm{Ad}\big(\theta^k(v_n)\big)^{\pm 1} : n \in \mathbf{N},\ k \in \mathbf{Z}_p\big\}$ is relatively compact in norm. Thus we conclude that

$$\lim_{n\to\infty}\left\| v_n x_n^* - \theta^k\big(v_n x_n^*\big)\right\|_\varphi^\sharp = 0, \quad k \in \mathbf{Z}_p.$$

Then with $\varepsilon = 1/p\sum_{k=0}^{p-1}\theta^k$, the conditional expectation to $\mathcal{M}^\theta$ invariant under θ, $\big\{v_n x_n^*\big\}$ is equivalent to the sequence $\big\{\varepsilon(v_n x_n^*)\big\}$ in $\mathcal{M}^\theta$. since $v_n x_n^*$ is a unitary, $\varepsilon(v_n x_n^*)$ can be readjusted to a unitary w_n in $\mathcal{M}^\theta$, so that $\big\{v_n x_n^*\big\}$ is equivalent to the sequence $\{w_n\}$ in $\mathcal{U}(\mathcal{M}^\theta)$. Hence we obtain

$$\lim_{n\to\infty}\mathrm{Ad}(w_n) = \lim_{n\to\infty}\mathrm{Ad}(v_n)\,\mathrm{Ad}(x_n)^{-1} = \lim_{n\to\infty}\mathrm{Ad}(v_n)\lim\mathrm{Ad}(x_n)^{-1} = \theta.$$

Since $\theta^p = \mathrm{id}$, $\{w_n^p\}$ is strongly central in $\mathcal{M}$. Write $w_n^p = \mathrm{e}^{\mathrm{i}k_n}$ in $\mathcal{M}^\theta$ with $-\pi \leq h_n \leq \pi$. Set $h_n' = (1-1/n)h_n$ and $y_n = \mathrm{e}^{\mathrm{i}h_n'}$. Then we have $\|h_n - h_n'\| \leq 2\pi/n$ and $\|w_n^p - y_n\| \leq \sin(\pi/n)$. Hence $\{y_n\}$ is also strongly central. Since $y_n \to \log y_n = h_n'$ is a C^*-algebraic functional calculus because $\mathrm{Sp}(y_n)$ does not contain -1, $\{h_n'\}$ is also strongly central, so that $\{h_n\}$ itself is strongly central. Set $z_n = \mathrm{e}^{\mathrm{i}h_n/p}$ and $u_n = z_n w_n$. Since w_n and h_n commute, $u_n^p = z_n^p w_n^p = 1$, and we have

$$\lim_{n\to\infty} \mathrm{Ad}(u_n) = \lim_{n\to\infty} \mathrm{Ad}(z_n) \lim_{n\to\infty} \mathrm{Ad}(w_n) = \theta. \qquad \text{Q.E.D.}$$

Lemma 3.18. *If θ gives a free action of $\mathbf{Z}_p$, then θ is conjugate to $\theta \otimes \sigma_p$.*

PROOF: By Theorem 2.10, there exists $w \in \mathcal{U}(\mathcal{M})$ such that $\mathrm{Ad}(w) \circ \theta \simeq \theta \otimes \sigma_p$ on $\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{R}_0$. For $1 \leq k \leq p$, we define w_k inductively as follows: $w_1 = w$ and $w_k = w\theta(w_{k-1})$. Then $\big(\mathrm{Ad}(w) \circ \theta\big)^k = \mathrm{Ad}(w_k) \circ \theta^k$. Since $\big(\mathrm{Ad}(w) \circ \theta\big)^p \simeq \big(\theta \otimes \sigma_p\big)^p = \mathrm{id}$, and $\theta^p = \mathrm{id}$, w_p must be a scalar of absolute value one. Let γ be a p-th root of w_p and set $v_k = \gamma^{-k} w_k$, $k \in \mathbf{Z}_p$. Then $\{v_k\}$ is a θ-cocycle over $\mathbf{Z}_p$. By the stability of free action of a finite group. Proposition XI.2.26, the cocycle $\{v_k\}$ is a coboundary, i.e. $v_k = a^*\theta^k(a)$ for some $a \in \mathcal{U}(\mathcal{M})$. Hence we have

$$\begin{aligned} \theta \otimes \sigma_p &\simeq \mathrm{Ad}(w) \circ \theta = \mathrm{Ad}(v_1) \circ \theta = \mathrm{Ad}\big(a^*\theta(a)\big) \circ \theta \\ &= \mathrm{Ad}(a)^{-1} \circ \theta \circ \mathrm{Ad}(a) \simeq \theta. \end{aligned} \qquad \text{Q.E.D.}$$

Let $\lambda = \mathrm{e}^{2\pi\mathrm{i}/p}$. Then from the above it follows that there exists a strongly central sequence $\{w_n\}$ of unitaries such that

$$\theta(w_n) = \lambda w_n \qquad \text{and} \qquad w_n^p = 1.$$

Lemma 3.19. *Fix a faithful $\varphi \in \mathfrak{S}_*$ and $\varepsilon > 0$. If $w \in \mathcal{U}(\mathcal{M})$ satisfies*

$$\theta(w) = \lambda w \qquad \textit{and} \qquad w^p = 1,$$

then there exist $v \in \mathcal{U}(\mathcal{M})$ and a sequence $\{u_n\}$ in $\mathcal{U}(\mathcal{M}^\theta)$ such that

$$\begin{gathered} \|w - v\|_\varphi < \varepsilon; \\ \theta(v) = \lambda v, \quad v^p = 1; \\ u_n v u_n^* = \lambda v, \quad u_n^p = 1; \\ \theta = \lim_{n\to\infty} \mathrm{Ad}(u_n). \end{gathered}$$

PROOF: Let $\{a_n\}$ be a sequence in $\mathcal{U}(\mathcal{M}^\theta)$ such that $\theta = \lim \mathrm{Ad}(a_n)$ and $a_n^p = 1$. We then split the arguments according to the type of $\mathcal{M}$.

Type III case: Suppose $\mathcal{M}$ is of type III. Then $\mathcal{M}^\theta$ is also of type III. Let $\alpha = \mathrm{Ad}(w) \in \mathrm{Aut}(\mathcal{M}^\theta)$. Since $\theta = \lim \mathrm{Ad}(a_n)$, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|a_n w^* a_n^* - \bar{\lambda} w^*\|_\varphi = \lim_{n\to\infty} \|a_n w^* a_n^* w - \bar{\lambda}\|_{\varphi\circ\alpha} \\ &= \lim_{n\to\infty} \|w^* a_n^* w - \bar{\lambda} a_n^*\|_{\varphi\circ\alpha}; \end{aligned}$$

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|a_n^* w^* a_n - \lambda w^*\|_\varphi \\ &= \lim_{n\to\infty} \|a_n^* w^* a_n w - \lambda\|_{\varphi\circ\alpha} = \lim_{n\to\infty} \|w^* a_n w - \lambda a_n\|_{\varphi\circ\alpha}. \end{aligned}$$

Hence $w^* a_n w - \lambda a_n$ converges σ^*-strongly to 0, equivalently, $\lim_{n\to\infty}\big(\alpha(a_n) - \bar{\lambda} a_n\big) = 0$ σ^*-strongly. Let $a_n = \sum_{k\in\mathbf{Z}_p} \lambda^k e_k(n)$ be the spectral decomposition. Then $\alpha\big(e_k(n)\big) \sim e_{k+1}(n)$ under $\mathcal{U}(\mathcal{M}^\theta)$ since $\mathcal{M}^\theta$ is of type III, and also $\lim_{n\to\infty}\big[\alpha\big(e_k(n)\big) - e_{k+1}(n)\big] = 0$ σ^*-strongly. By Lemma XIV.2.1, there exists a unitary $b_k(n) \in \mathcal{M}^\theta$ such that

$$b_k(n)\alpha\big(e_k(n)\big)b_k(n)^* = e_{k+1}(n); \qquad \big|1 - b_k(n)\big| \le \sqrt{2}\,\Big|\alpha\big(e_k(n)\big) - e_{k+1}(n)\Big|.$$

Setting $b(n) = \sum_{k\in\mathbf{Z}_p} e_{k+1}(n) b_k(n)$, we have

$$b(n)\alpha(e_k)b(n)^* = e_{k+1}(n), \quad k \in \mathbf{Z}_p;$$

$$\|1 - b(n)\|_\varphi^2 \le 2 \sum_{k\in\mathbf{Z}_p} \Big\|\alpha\big(e_k(n)\big) - e_{k+1}(n)\Big\|_\varphi^2 \to 0.$$

Hence $\{b(n)\}$ converges to 1 in $\mathcal{U}(\mathcal{M}^\theta)$. We also have $\theta\big(b(n)w\big) = \lambda b(n)w$, and $b(n)wa_n = \bar{\lambda} a_n b(n)w$. Thus $(b(n)w)^p$ belongs to $\mathcal{U}(\mathcal{M}^\theta)$ and commutes with a_n. Since $w^p = 1$ and $b(n)$ is near 1, $\big(b(n)w\big)^p$ is also close to 1 in the topology of $\mathcal{U}(\mathcal{M}^\theta)$. With the function f_p of (3), we set $v(n) = b(n)wf_p\big(b(n)w^p\big)^*$. Then we have $v(n)a_n = \bar{\lambda} a_n v(n)$, $v(n)^p = 1$ and $v(n)$ converges to w in $\mathcal{U}(\mathcal{M})$.

Type II_∞ case: Suppose $\mathcal{M}$ is of type II_∞. Since θ has period p, θ preserve the trace τ of $\mathcal{M}$. Hence τ is also semi-finite on $\mathcal{M}^\theta$. Furthermore, $\mathcal{M}^\theta$ is also of type II_∞. Therefore, factoring $\mathcal{M}^\theta = \mathcal{N} \mathbin{\overline{\otimes}} \mathcal{B}$ into the tensor product of a factor $\mathcal{N}$ of type II_1 and a factor $\mathcal{B}$ of type I_∞, we obtain a decomposition of $\mathcal{M} = \mathcal{M}_1 \mathbin{\overline{\otimes}} \mathcal{B}$ so that θ also decomposes $\theta = \theta_1 \otimes \mathrm{id}$, where $\{\mathcal{M}_1, \theta_1\}$ is a free covariant system of type II_1 over $\mathbf{Z}_p$. Thus, the proof is reduced to the case of type II_1.

Type II_1 case: Suppose $\mathcal{M}$ is a factor of II_1 with the normalized trace τ. The arguments for the type III case breaks down at the point of the equivalence: $\alpha\big(e_k(n)\big) \sim e_{k+1}(n)$. Thus, we need a slight detour. In any case, we have $\lim_{n\to\infty} \|\alpha(a_n) - \bar{\lambda} a_n\|_\tau = 0$, so that

$$\lim_{n\to\infty} \Big\|\alpha\big(e_k(n)\big) - e_{k+1}(n)\Big\|_\tau = 0, \quad k \in \mathbf{Z}_p.$$

Hence $\big|\tau\big(\alpha^k(e_0(n)\big) - \tau\big(e_k(n)\big)\big| \to 0$, $k \in \mathbf{Z}_p$, thus we have $\big|\tau\big(e_k(n)\big) - 1/p\big| \to 0$, $k \in \mathbf{Z}_p$. Thus we can find a sequence $\{f_k(n)\}_{k\in\mathbf{Z}_p}$ of partition of unity such that $\tau\big(f_k(n)\big) = 1/p$ and either $f_k(n) \leq e_k(n)$ or $f_k(n) \geq e_k(n)$. Then we have $\lim_{n\to\infty}\big\| f_k(n) - e_k(n)\big\|_\tau = 0$. Let

$$u_n = \sum_{k\in\mathbf{Z}_p} \bar{\lambda}^k f_k(n) \in \mathcal{U}(\mathcal{M}^\theta).$$

We then have

$$\lim_{n\to\infty}\Big\|\alpha\big(f_k(n)\big) - f_{k+1}(n)\Big\|_\tau = 0;$$

$$\alpha\big(f_k(n)\big) \sim f_{k+1}(n);$$

$$\lim\big\|u_n - a_n\big\|_\tau = 0.$$

From this point one, we can return to the arguments of the type III case. Q.E.D.

Thus, we have obtained a sequence $\{a_n\}$ of $\mathcal{U}(\mathcal{M}^\theta)$ and a strongly central sequence $\{b_n\}$ in $\mathcal{U}(\mathcal{M})$ such that

$$\left.\begin{aligned} \theta(b_n) &= a_n b_n a_n^* = \lambda b_n, \\ a_n^p &= b_n^p = 1; \\ \theta &= \lim_{n\to\infty} \mathrm{Ad}(a_n). \end{aligned}\right\} \tag{27}$$

We now prove the following result which implies Theorem 3.16 as seen in the first part of the above arguments:

Theorem 3.20. *Let $\mathcal{M}$ be a separable strongly stable factor. If $\theta \in \overline{\mathrm{Int}}(\mathcal{M})$ gives rise to a free action of the cyclic group $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$, $p > 0$, and $p_a(\theta) = p$ also, then θ is conjugate to $\mathrm{id}\otimes\sigma_p$ on $\mathcal{M}\,\overline{\otimes}\,\mathcal{R}_0$ where σ_p is the automorphism of the* AFD *factor $\mathcal{R}_0$ of type II_1 constructed in (2.5).*

PROOF: Let $\varphi \in \mathfrak{S}_*$ be faithful, and put $\delta_\nu = 2^{-\nu}(p+1)^{-3}$ as in (10) with $n_\nu = p$. Let $\big\{\psi_j : j \in \mathbf{N}\big\}$ is a dense sequence in $\mathcal{M}_*$. We are going to construct a sequence $\{M_\nu\}$ of type I_p subfactors of $\mathcal{M}$ and two sequences $\{u_\nu\}$ and $\{v_\nu\}$ of unitaries such that for $1 \leq \ell \leq \nu$, $1 \leq j \leq \nu$ and $s \in \mathbf{Z}_p$

a) M_ℓ and M_ν commute,

b) M_ν is generated by u_ν and v_ν which satisfy

$$u_\nu^p = v_\nu^p = 1, \qquad u_\nu v_\nu = \lambda v_\nu u_\nu;$$

c) $\big\|[u_\nu^s, \psi_\ell]\big\| \leq \delta_\nu$, $\big\|[v_\nu^s, \psi_\ell]\big\| \leq \delta_\nu$;

d) $\theta(M_j) = M_j$ and $\theta|_{M_j} = \mathrm{Ad}(u_j)\big|_{M_j}$;

e) $\big\|\psi_j \circ \theta^s - \psi_j \circ \big(\mathrm{Ad}(u_\nu u_{\nu-1}\cdots u_1)^s\big)\big\| \leq \delta_\nu/4$.

Suppose $u_j,\ v_j$ and $M_j,\ 1 \leq j \leq \nu$, have been constructed. Let $N = M_1 \vee M_2 \vee \cdots \vee M_\nu$, and $n = p^\nu$. Then N is of type I_{p^ν}. Put $Q = N^c$ and $U = u_\nu u_{\nu-1} \cdots u_1$. Then θ leaves N globally invariant and agrees with $\mathrm{Ad}(U)$ on N. Consider the restriction $\widetilde{\theta}$ of θ to Q. Let $\left\{ \omega_{i,j} : 1 \leq i, j \leq n = p^\nu \right\}$ be the basis of N_* dual to a matrix unit of N. For $\ell = 1, 2, \ldots, \nu + 1$, write

$$\psi_\ell = \sum_{i,j=1}^{n} \omega_{ij} \otimes \psi_\ell^{ij}$$

with $\psi_\ell^{i,j} \in Q_*$.

Since $\theta \simeq \theta \otimes \sigma_p$, $\widetilde{\theta}$ is conjugate to θ. Thus, we can apply the previous arguments to $\widetilde{\theta}$ to find unitaries u and v in Q such that

$$u^p = v^p = 1; \qquad \theta(u) = u;$$

$$\widetilde{\theta}(v) = \lambda v = u v u^*;$$

$$\left\|[v, \psi_\ell^{j,k}]\right\| \leq \frac{\delta_{\nu+1}}{n^2}, \quad 1 \leq j, k \leq n, \quad 1 \leq \ell \leq \nu + 1$$

$$\left\|\psi_\ell^{j,k} \circ \widetilde{\theta}^s - \psi_\ell^{j,k} \circ \mathrm{Ad}(u)^s\right\| < \frac{\delta_{\nu+1}}{4n^2}, \quad s \in \mathbf{Z}_p.$$

Let $M_{\nu+1} = \left\{u, v\right\}''$, $u_{\nu+1} = u$ and $v_{\nu+1} = v$. Then we have $\widetilde{\theta}(x) = \mathrm{Ad}(u)(x)$, $x \in M_{\nu+1}$. Since $\theta = \mathrm{Ad}(U) \otimes \widetilde{\theta}$ relative to $\mathcal{M} = N \otimes Q$, we have, by Lemma 3.8,

$$\left\|\psi_\ell \circ \theta^s - \psi_\ell \circ \mathrm{Ad}(uU)^s\right\| < \frac{\delta_{\nu+1}}{4}, \quad 1 \leq \ell \leq \nu + 1;$$

$$\left\|[v^s, \psi_\ell]\right\| \leq \delta_{\nu+1}, \quad 1 \leq \ell \leq \nu + 1.$$

By the induction hypothesis (e), we have for $1 \leq \ell \leq \nu$

$$\left\|\psi_\ell \circ \mathrm{Ad}(U)^s - \psi_\ell \circ \theta^s\right\| \leq \frac{\delta_\nu}{4}, \quad s \in \mathbf{Z}_p;$$

so that for $1 \leq \ell \leq \nu$

$$\begin{aligned}
\left\|\psi_\ell \circ \mathrm{Ad}(u)^s - \psi_\ell\right\| &= \left\|\psi_\ell \circ \mathrm{Ad}(uU)^s - \psi_\ell \circ \mathrm{Ad}(U)^s\right\| \\
&\leq \left\|\psi_\ell \circ \mathrm{Ad}(uU)^s - \psi_\ell \circ \theta^s\right\| + \left\|\psi_\ell \circ \theta^s - \psi_\ell \circ \mathrm{Ad}(U)^s\right\| \\
&\leq \frac{\delta_{\nu+1} + \delta_\nu}{4} \;=\; \frac{3}{4}\delta_{\nu+1} \;<\; \delta_{\nu+1}.
\end{aligned}$$

Therefore, we get

$$\left\|\left[u_{\nu+1}^s, \psi_\ell\right]\right\| < \delta_{\nu+1}, \quad 1 \leq \ell \leq \nu;$$

$$\left\|\left[v_{\nu+1}^s, \psi_\ell\right]\right\| < \delta_{\nu+1}, \quad 1 \leq \ell \leq \nu.$$

the other conditions for $\left\{M_{\nu+1}, u_{\nu+1}, v_{\nu+1}\right\}$ have been already proved. Thus, the induction process is complete.

The normalized trace τ_ν on M_ν is given by

$$\tau_\nu = \frac{1}{p^2} \sum_{k,\ell \in \mathbf{Z}_p} \mathrm{Ad}\big(u_\nu^k v_\nu^\ell\big).$$

so that the conditional expectation $\mathcal{E}_\nu$ from $\mathcal{M}$ to $M_\nu^c = M_\nu' \cap \mathcal{M}$ satisfies the inequality: $\left\| \psi_j \circ \mathcal{E}_\nu - \psi_j \right\| \leq \delta_\nu,\ 1 \leq j \leq \nu$. Hence Lemma XIV.4.10 yields that $\mathcal{R} = \bigvee_{j=1}^{\infty} M_j$ is an AFD factor of type II_1 such that $\mathcal{M} = \mathcal{R}^c \,\overline{\otimes}\, \mathcal{R}$. By construction, we have $\theta = \mathrm{id} \otimes \theta\big|_{\mathcal{R}}$ relative to this factorization and $\theta|_{\mathcal{R}} \simeq \sigma_p$.
Q.E.D.

Thus, we have completed the proof of Theorem 3.16 as well.

Corollary 3.21. *If* $\mathcal{R}_0$ *is an* AFD *factor of type* II_1, *then* $\mathrm{Out}(\mathcal{R}_0)$ *is a simple group with only countably many conjugacy classes labelled by the outer period* $p_0(\theta)$ *and the obstruction* $\mathrm{Ob}(\theta)$.

PROOF: Let N be a normal subgroup of $\mathrm{Out}(\mathcal{R}_0)$, and set $G = \varepsilon^{-1}(N)$. If G contains an element $\theta \in \mathrm{Aut}(\mathcal{R}_0)$ with $p_0(\theta) = p > 1$ and $\gamma = \mathrm{Ob}(\theta)$, then $\theta \sim \sigma_p^\gamma$, and $\sigma_p^\gamma \sim \mathrm{id} \otimes \sigma_p^\gamma$ on $\mathcal{R}_0 \,\overline{\otimes}\, \mathcal{R}_0$, so that $\sigma_p^\gamma \otimes \sigma_p^\gamma = \big(\sigma_p^\gamma \otimes \mathrm{id}\big)\big(\mathrm{id} \otimes \sigma_p^\gamma\big)$ belongs to G and $\mathrm{Ob}\big(\sigma_p^\gamma \otimes \sigma_p^\gamma\big) = \gamma^2$. Repeating this, we see that $\sigma_p^\gamma \otimes \cdots \otimes \sigma_p^\gamma$, p-times tensor product, belongs to G and $\sigma_p \sim \sigma_p^\gamma \otimes \cdots \otimes \sigma_p^\gamma$. Hence G contains $\theta \in \mathrm{Aut}(\mathcal{R}_0)$ with $p_0(\theta) = p$ and $\mathrm{Ob}(\theta) = 1$.

Suppose now G contains $\theta \in \mathrm{Aut}(\mathcal{R}_0)$ with $p_0(\theta) = p > 1$ and $\mathrm{Ob}(\theta) = 1$. Recall the proof of Proposition 3.15, and observe that $p_0(\alpha) = p,\ \mathrm{Ob}(\alpha) = 1$, $p_0\big(\hat{\alpha}_{1/p}\big) = p$ and $\mathrm{Ob}\big(\hat{\alpha}_{1/p}\big) = 1$. Hence $\alpha \in G$ and $\hat{\alpha}_{1/p} \in G$, whilst $\sigma_p^\gamma = \hat{\alpha}_{k/p} \circ \alpha = \big(\hat{\alpha}_{1/p}\big)^k \circ \alpha$; thus $\sigma_p^\gamma \in G$.

Let n be a sufficiently large positive integer and write $\mathcal{R}_0 = \prod_{k=1}^{\infty \otimes} M_k$ with $M_k = M_n(\mathbf{C})$. On $M_n(\mathbf{C})$, take a pair of unitaries u and v such that $u^p = v^p = 1$, $u^k \notin \mathbf{T}$ and $v^k \notin \mathbf{T}$ for $1 \leq k \leq p-1$, and $w = uvu^*v^*$ is an aperiodic unitary in the sense that $w^k \notin \mathbf{T}$ for all $k \neq 0$. Let $u_k = u$ and $v_k = v$, and $\alpha = \prod_{k=1}^{\infty \otimes} \mathrm{Ad}(u_k)$ and $\beta = \prod_{k=1}^{\infty \otimes} \mathrm{Ad}(v_k)$. Then we have $p_0(\alpha) = p_0(\beta) = p$ and $\mathrm{Ob}(\alpha) = \mathrm{Ob}(\beta) = 1$, so that α and β both belong to G. But $\alpha\beta\alpha^{-1}\beta^{-1} = \prod_{k=1}^{\infty \otimes} \mathrm{Ad}(w_k)$ with $w_k = w$ is aperiodic, so that G contains σ_0.

If G contains aperiodic element, then it contains σ_0. But $\sigma_0 \sim \sigma_p \otimes \sigma_0$ and $\sigma_0 \sim \mathrm{id} \otimes \sigma_0$, so that $\sigma_p \otimes \sigma_0$ and $\mathrm{id} \otimes \sigma_0$ on $\mathcal{R}_0 \,\overline{\otimes}\, \mathcal{R}_0$ belong to G. Hence $\sigma_p \otimes \mathrm{id} = \big(\sigma_p \otimes \sigma_0\big)\big(\mathrm{id} \otimes \sigma_0\big)^{-1}$ belongs to G. Thus G contains σ_p. Thus, if G contains an element of any type representing a conjugacy class, then G contains all others. Therefore G must be the entire $\mathrm{Aut}(\mathcal{R}_0)$, which means that $\mathrm{Out}(\mathcal{R}_0)$ is simple.
Q.E.D.

We now close this section with the following result which will be used to describe the structure of $\mathrm{Out}(\mathcal{R})$ of an AFD factor of type $\mathrm{III}_\lambda,\ 0 < \lambda < 1$, which is in turn needed to the uniqueness of AFD factors of type III_1.

Proposition 3.22. *Let $\mathcal{M}$ be a separable strongly stable factor. If α is an action of a discrete countable abelian group G such that $\alpha^{-1}\big(\mathrm{Cnt}(\mathcal{M})\big) = H$ then for any free ultra filter ω and a character $p \in (G/H)\widehat{\ } = H^{\perp}$, there exists $u \in \mathcal{U}(\mathcal{M}_\omega)$ such that*

$$\alpha_g^\omega(u) = \langle g, p\rangle u, \quad g \in G,$$

where α^ω is of course the natural action of G on $\mathcal{M}_\omega$ induced by α.

In fact, we can do slightly better. Namely, if $\mathcal{P}$ is a separable von Neumann subalgebra of $\mathcal{M}_\omega$, then we can choose the above u from the relative commutant $\mathcal{P}' \cap \mathcal{M}_\omega$ of $\mathcal{P}$.

PROOF: By Lemma 2.2, the action α^ω of G/H on $\big(\bigcup_{g\in G} \alpha_g^\omega(\mathcal{P})\big)' \cap \mathcal{M}_\omega$ is free. Therefore, for any $\varepsilon > 0$, and a finite subset F of G, there exists $U = U(\varepsilon, F) \in \mathcal{U}\big(\mathcal{M}_\omega \cap \mathcal{P}'\big)$ such that

$$\left\|\alpha_g^\omega(U) - \langle g, p\rangle U\right\|_{\tau,\omega} < \varepsilon, \quad g \in F.$$

Let $\{X_n\}$ be a σ^*-strongly dense sequence of $\mathcal{P}$ and each X_n be represented by $\big\{x_n(k) : k \in \mathbf{N}\big\}$. Then we have with $U(\varepsilon, F) = \big\{u_k(\varepsilon, F)\big\}$,

$$\lim_{n\to\omega}\left\|\big[x_n(k),\ u_k(\varepsilon, F)\big]\right\|_\varphi^\sharp = 0;$$

$$\lim_{n\to\omega}\left\|\alpha_g\big(u_k(\varepsilon, F)\big) - \langle g, p\rangle u_k(\varepsilon, F)\right\|_\varphi^\sharp < \varphi, \quad g \in F;$$

$$\lim_{k\to\omega}\left\|\big[u_k(\varepsilon, F), \psi_j\big]\right\| = 0.$$

where φ is a faithful normal state on $\mathcal{M}$ and $\{\psi_j\}$ is a dense sequence in $\mathcal{M}_*$. Let $\big\{F_\nu : \nu \in \mathbf{N}\big\}$ be an increasing sequence of finite subsets of G with $G = \bigcup_{\nu=1}^\infty F_\nu$. Let $\big\{A_\nu : \nu \in \mathbf{N}\big\}$ be a strictly decreasing sequence of subsets of $\mathbf{N}$ belonging to the ultra filter ω such that for any $k \in A_\nu$

$$\left\|\left[x_n(k), u_k\left(\frac{1}{\nu}, F_\nu\right)\right]\right\|_\varphi^\sharp < \frac{1}{\nu}, \quad 1 \le n \le \nu,$$

$$\left\|\alpha_g\left(u_k\left(\frac{1}{\nu}, F_\nu\right)\right) - \langle g, p\rangle u_k\left(\frac{1}{\nu}, F_\nu\right)\right\|_\varphi^\sharp < \frac{1}{\nu}, \quad g \in F_\nu;$$

$$\left\|\left[u_k\left(\frac{1}{\nu}, F_\nu\right)^a, \psi_j\right]\right\| < \frac{1}{\nu}, \quad 1 \le j \le \nu.$$

Let k_ν be the first element of A_ν and set $u_\nu = u_{k_\nu}\big(1/\nu, F_\nu\big)$. Then $u = \pi_\omega\big(\{u_\nu\}\big)$ satisfies the requirement. Q.E.D.

Exercise XVII.3

1) Let $\mathcal{R}_0$ be an AFD factor of type II_1.

(**a**) Show that the adjoint map $\mathrm{Ad}\colon u \in \mathcal{U}(\mathcal{R}_0) \mapsto \mathrm{Ad}(u) \in \mathrm{Int}\,\mathcal{R}_0)$ admits a Borel cross-section $\alpha \in \mathrm{Int}(\mathcal{R}_0) \mapsto u(\alpha) \in \mathcal{U}(\mathcal{R}_0)$ so that $\alpha = \mathrm{Ad}\big(u(\alpha)\big),\ \alpha \in \mathrm{Int}\,\mathcal{R}_0)$.

(**b**) Show that there exist $\mathbf{T}$-valued Borel functions λ_u on $\mathrm{Int}(\mathcal{R}_0) \times \mathrm{Aut}(\mathcal{R}_0)$ and μ_u on $\mathrm{Int}(\mathcal{R}_0) \times \mathrm{Int}(\mathcal{R}_0)$ such that

$$\gamma\big(u(\gamma^{-1}\alpha\gamma)\big) = \lambda_u(\alpha, \gamma)u(\alpha), \quad \alpha \in \mathrm{Int}(\mathcal{R}_0), \quad \gamma \in \mathrm{Aut}(\mathcal{R}_0)$$

and

$$u(\alpha)u(\beta) = \mu_u(\alpha, \beta)u(\alpha\beta), \quad \alpha, \beta \in \mathrm{Int}(\mathcal{R}_0).$$

(**c**) Show that the pair $(\lambda, \mu) = (\lambda_u, \mu_u)$ satisfies the following identities:

$$\begin{cases} \mu(\alpha, \beta)\mu(\alpha\beta, \gamma) = \mu\big(\alpha, \beta\gamma\big)\mu(\beta, \gamma), & \alpha, \beta, \gamma \in \mathrm{Int}(\mathcal{R}_0); \\ \lambda\big(\alpha, \gamma_1\gamma_2\big) = \lambda(\alpha, \gamma_1)\lambda\big(\gamma_1\alpha\gamma_1^{-1}, \gamma_2\big), & \alpha \in \mathrm{Int}(\mathcal{R}_0), \quad \gamma_1, \gamma_2 \in \mathrm{Aut}(\mathcal{R}_0); \\ \lambda(\alpha, \gamma)\lambda(\beta, \gamma)\lambda\big(\alpha\beta, \gamma\big)^{-1} = \mu\big(\gamma^{-1}\alpha\gamma, \gamma^{-1}\beta\gamma\big)\mu(\alpha, \beta)^{-1}; & \\ \lambda(\alpha, \gamma) = \mu(\alpha, \beta) = 1 & \\ \qquad \text{if any of } \alpha, \beta \in \mathrm{Int}(\mathcal{R}_0), \text{ or } \gamma \in \mathrm{Aut}(\mathcal{R}_0) \text{ is the identity.} & \end{cases}$$

(**d**) Observe that the set $\mathrm{Z}\big(\mathrm{Aut}(\mathcal{R}_0), \mathrm{Int}(\mathcal{R}_0), \mathbf{T}\big)$ consisting of all pairs (λ, μ) of $\mathbf{T}$-valued Borel functions λ on $\mathrm{Int}(\mathcal{R}_0) \times \mathrm{Aut}(\mathcal{R}_0)$ and μ on $\mathrm{Int}(\mathcal{R}_0) \times \mathrm{Int}(\mathcal{R}_0)$ satisfying the identities of (c) form an abelian group relative to the pointwise multiplication. With $\mathrm{Map}\big(\mathrm{Int}(\mathcal{R}_0), \mathbf{T}\big) = \big\{f\colon\ \mathrm{Int}(\mathcal{R}_0) \mapsto \mathbf{T}\ \text{Borel},\ \ f(\mathrm{id}) = 1\big\}$, show that if

$$\begin{cases} \partial_1 f(\alpha, \gamma) = f\big(\gamma^{-1}\alpha\gamma\big) f(\alpha)^{-1}; \\ \partial_2 f(\alpha, \beta) = f(\alpha) f(\beta) f(\alpha\beta)^{-1}, \quad \alpha, \beta \in \mathrm{Int}(\mathcal{R}_0), \quad \gamma \in \mathrm{Aut}(\mathcal{R}_0). \end{cases}$$

then the pair $\partial f = \big(\partial_1 f, \partial_2 f\big)$ satisfies the identities of (c).

Set $\mathrm{B}\big(\mathrm{Aut}(\mathcal{R}_0), \mathrm{Int}(\mathcal{R}_0), \mathbf{T}\big) = \partial\big(\mathrm{Map}(\mathrm{Int}(\mathcal{R}_0), \mathbf{T})\big)$ and consider the quotient group:

$$\Lambda\big(\mathrm{Aut}(\mathcal{R}_0), \mathrm{Int}(\mathcal{R}_0), \mathbf{T}\big) = \mathrm{Z}\big(\mathrm{Aut}(\mathcal{R}_0), \mathrm{Int}(\mathcal{R}_0), \mathbf{T}\big) \big/ \mathrm{B}\big(\mathrm{Aut}(\mathcal{R}_0), \mathrm{Int}(\mathcal{R}_0), \mathbf{T}\big).$$

(**e**) Show that the class $\chi_{\mathcal{R}_0} \in \Lambda\big(\mathrm{Aut}(\mathcal{R}_0), \mathrm{Int}(\mathcal{R}_0), \mathbf{T}\big)$ of the pair (λ_u, μ_u) of (c) does not depend on the choice of the Borel cross-section u. Thus $\chi_{\mathcal{R}_0}$ is an intrinsic quantity of $\mathcal{R}_0$. This was called the intrinsic invariant of $\mathcal{R}_0$ in Definition XII.6.18.

2) Let G be a countable discrete group and α be an action of G on $\mathcal{R}_0$. For a normal subgroup $N \lhd G$, construct the group $\Lambda(G, N, \mathbf{T})$ of characteristic invariants following the construction in Problem 1.

(**a**) Show that $\Lambda(G, N, \mathbf{T})$ is a compact abelian group.

(**b**) Show that with $N(\alpha) = \alpha^{-1}\big(\mathrm{Int}(\mathcal{R}_0)\big)$ the element $\chi(\alpha) = \alpha^*(\chi_{\mathcal{R}_0}) \in \Lambda\big(G, N(\alpha), \mathbf{T}\big)$ is a cocycle conjugacy invariant of α.

(**c**) Show that if $G = \mathbf{Z}$ and $N = p\mathbf{Z}$, then $\Lambda(\mathbf{Z}, p\mathbf{Z}, \mathbf{T}) \cong \mathbf{Z}/p\mathbf{Z}$.

3) Let $\theta \in \mathrm{Aut}(\mathcal{R}_0)$ with $\mathcal{R}_0$ an AFD factor of type II_1 as before. Viewing θ as an action of $\mathbf{Z}$ on $\mathcal{R}_0$, show that $p_0(\theta)\mathbf{Z} = \theta^{-1}\big(\mathrm{Int}(\mathcal{R}_0)\big)$ and that $\mathrm{Ob}(\theta)$ represent the class $\chi(\theta) = \theta^*(\chi_{\mathcal{R}_0}) \in \Lambda\big(\mathbf{Z}, p_0(\theta)\mathbf{Z}, \mathbf{T}\big) \cong \mathbf{Z}/p_0(\theta)\mathbf{Z}$.

4) Let G be a countable discrete infinite group and fix λ, $0 < \lambda \le 1$. To each element $g \in G$ associate a 2×2-matrix algebra $M_g = M_2(\mathbf{C})$ and a state ω_g^λ on M_g given by the matrix:

$$\omega_g = \begin{pmatrix} \dfrac{\lambda^{\frac{1}{2}}}{\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}} & 0 \\ 0 & \dfrac{\lambda^{-\frac{1}{2}}}{\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}} \end{pmatrix}.$$

Observe that $\big\{\mathcal{R}_\lambda, \omega_\lambda\big\} = \prod_{g\in G}^{\otimes}\big\{M_g, \omega_g^\lambda\big\}$ is an AFD factor of type III_λ for $\lambda \ne 1$ and type II_1 for $\lambda = 1$.

(**a**) Show that to each $h \in G$ there corresponds an automorphism α_h of $\mathcal{R}_\lambda$ such that

$$\alpha_h\bigg(\prod_{g\in G}^{\otimes} x_g\bigg) = \prod_{g\in G}^{\otimes} x_{hg}, \quad \prod_{g\in G}^{\otimes} x_g \in \mathcal{R}_\lambda.$$

(**b**) Show that α is a free action of G on $\mathcal{R}_\lambda$ such that $\alpha^{-1}\big(\mathrm{Cnt}(\mathcal{R}_\lambda)\big) = \{e\}$.

(**c**) Show that the crossed product $\mathcal{R}_\lambda \rtimes_\alpha G$ is a factor of type III_λ for $0 < \lambda < 1$ and type II_1 for $\lambda = 1$.

5) Let G be a countable discrete group and $N \lhd G$ be a normal subgroup. Fix $(\lambda, \mu) \in Z(G, N, \mathbf{T})$. Construct an action α of G on an AFD factor of type II_1 such that $\chi(\alpha) = [\lambda, \mu] \in \Lambda(G, N, \mathbf{T})$ following the steps suggested below.

(**a**) Fix a free action β of G on an AFD factor $\mathcal{P}_0$. Construct the twisted crossed product $\mathcal{P}_0 \rtimes_{\beta,\mu} N$ as follows: represent $\mathcal{P}_0$ on a Hilbert space $\mathfrak{K}$ then set $\mathfrak{H} = \ell^2(G, \mathfrak{K})$ on which define operators:

$$\begin{cases} \big(\pi_\beta(a)\xi\big)(m) = \beta_m(a)\xi(m), & \xi \in \mathfrak{H}, \quad a \in \mathcal{P}_0; \\ \big(u(n)\xi\big)(m) = \mu(m, n)\xi(mn), & m, n \in N; \end{cases}$$

then set $\mathcal{R}_0 = \mathcal{P}_0 \rtimes_{\beta,\mu} N = \big(\pi_\beta(\mathcal{P}_0) \cup u(N)\big)''$. Observe that $u(m)u(n) = \mu(m, n)u(mn)$, $m, n \in N$. Prove using Theorem XIV.1.9 that $\mathcal{R}_0$ is an AFD factor of type II_1.

(**b**) To each $g \in G$ associate an operator $U_\lambda(g)$ defined by

$$\big(U_\lambda(g)\xi\big)(m) = \lambda(m, g)\xi(g^{-1}mg), \quad m \in N.$$

Show that $\alpha_g^{\lambda,\mu} = \mathrm{Ad}\big(U_\lambda(g)\big)\big|_{\mathcal{R}_0}$, $g \in G$, is an action of G on $\mathcal{R}_0$ such that $\chi(\alpha) = [\lambda, \mu] \in \Lambda(G, N, \mathbf{T})$.

(**c**) To each $f \in \mathrm{Map}(N, \mathbf{T})$ associate an operator defined by:

$$(U_f\xi)(m) = f(m)\xi(m), \quad \xi \in \mathfrak{H}, \quad m \in N.$$

Observe that $U_f U_g = U_{fg}$, $f, g \in \mathrm{Map}(G, \mathbf{T})$, and show that

$$\begin{cases} U_f\big(\mathcal{P}_0 \rtimes_{\beta,\mu} N\big)U_f^* = \mathcal{P}_0 \rtimes_{\beta,\mu\partial_2 f} N; \\ \mathrm{Ad}(U_f) \circ \alpha_g^{\lambda,\mu} \circ \mathrm{Ad}(U_f)^{-1} = \alpha_g^{(\lambda,\mu)\partial f}, \quad g \in G. \end{cases}$$

Notes on Chapter XVII

As mentioned earlier, the study of the automorphism group $\mathrm{Aut}(\mathcal{M})$ of a von Neumann algebra occupied the central place in the entire history of operator algebras. It was already noted in the mid fifties by I. M. Singer, [332], that the double commutation theorem of von Neumann, the fundamental theorem of the subject, Theorem II.3.9, should be viewed as a kind of the Galois correspondence, or that the lack of the double commutation theorem for subfactors of a factor relative to the relative commutant can be restored by the Galois type correspondence at least in some cases. In fact, in the late fifties and the early sixties, M. Nakamura and Z. Takeda, independently N. Suzuki, laid down the basics of the theory of finite groups of automorphisms of a factor of type II_1 by establishing subgroups-subfactors Galois correspondence, [702, 644, 645]. But one had to wait until the work of A. Connes in 1975 for the true picture of the structure of the automorphism group $\mathrm{Aut}(\mathcal{R})$ of an AFD factor $\mathcal{R}$, [466, 469]. The weapon he used in this endeavor was the analysis of asymptotic commutativity, i.e. the study of central sequences, which came into the subject from two sources: one from the property Γ of Murray and von Neumann, and the other from mathematical physics which asserts that the Einstein causality law implies the commutativity of two observables localized in regions of space-like separation. The property Γ of a factor $\mathcal{M}$ of type II_1 is equivalent to the non-closedness of $\mathrm{Int}(\mathcal{M})$ in $\mathrm{Aut}(\mathcal{M})$, cf. Theorem XIV.3.8. The study of asymptotic commutativity was intensified in the mid sixties through the mid seventies due partly to the influence from mathematical physics, which yielded a number of results including the discovery of non-isomorphic factors of Powers and McDuff. The characterization of the strong stability of a factor was obtained by D. McDuff, [636], and H. Araki, [423]. In a way, the time was ripe for A. Connes. As we have seen through this chapter as well as previous chapters, the analysis was carried out through ultra filters, which allowed him to peek through ultra filters an elaborated mathematical structure hidden in the narrow gap of the σ-weak topology and the σ-strong* topology. The group $\mathrm{Out}(\mathcal{R}_0)$ of an AFD factor $\mathcal{R}_0$ is the quotient group of a Polish group $\mathrm{Aut}(\mathcal{R}_0)$ by a dense subgroup $\mathrm{Int}(\mathcal{R}_0)$, so that as a topological group $\mathrm{Out}(\mathcal{R}_0)$ is non-manageable. He discovered that $\mathrm{Out}(\mathcal{R}_0)$ can be faithfully represented in $\mathrm{Aut}(\mathcal{R}_0)_\omega$. This was the key for his success in this analysis as we have seen throughout the second half of this volume.

After Connes' work, V. F. R. Jones classified actions of finite groups on an AFD factor $\mathcal{R}_0$ of type II_1 in his thesis, [570], by extending the concept of the obstruction of a single automorphism to the *characteristic invariant* $\chi(\alpha)$ of an action of a group on a factor. It was then further extended by A. Ocneanu to the cocycle conjugacy classification of actions of a countable discrete amenable group on a semifinite AFD factor and provided analytical tools for the later study of such group actions on AFD factors by proving two cohomology vanishing theorem and Rokhlin's tower and/or paving for the group, [648]. Today, one can summarize the cocycle conjugacy classification of countable discrete amenable group actions on an AFD factor $\mathcal{R}$ in the following one theorem:

Theorem. *Let $\mathcal{R}$ be an approximately finite dimensional factor and G a countable discrete amenable group. The cocycle conjugacy class $[\alpha]$ of an action α of G on $\mathcal{R}$ is completely determined by the pull back:*

$$\widetilde{\chi}(\alpha) = \alpha^*\big(\Theta(\mathcal{R})\big) \in \Lambda\big(G \times \mathbf{R}, N(\alpha), \mathcal{U}(\mathcal{C})\big)$$

of the intrinsic invariant $\Theta(\mathcal{R})$ of $\mathcal{R}$ defined in Definition XII.6.18, *where $N(\alpha) = \alpha^{-1}\big(\mathrm{Cnt}(\mathcal{R})\big)$ and $\big\{\mathcal{C}, \mathbf{R}, \theta\big\}$ is the flow of weights of $\mathcal{R}$.*

This was the result of many hands: Jones-Takesaki, [574], C. E. Sutherland–M. Takesaki, [699, 700], Y. Kawahigashi–C. E. Sutherland–M. Takesaki, [593] and finally Y. Katayama–C. E. Sutherland–M. Takesaki, [590, 591].

The conjugacy classification of compact abelian group actions on AFD factors is also now completed by the hand of V. F. R. Jones–M. Takesaki, [574], for the semi-finite case and Y. Kawahigashi–M. Takesaki for the type III case, [594].

We should note however that the above fine classifications are valid only for discrete group actions or compact abelian group actions. Toward continuous group actions, there are only few works. Y. Kawahigashi classified the cocycle conjugacy of product type one parameter automorphism groups on semifinite AFD factors, [592], which covers all known examples but not all actions. Hui, [565], extended the work of Kawahigashi to the case of type III.