Preface

The author believes that the theory of operator algebras should be viewed as a number theory in analysis. Number theory has been attracting the interest of humans ever since civilization began. Every culture in the world throughout history has given special meanings to certain numbers.

For example, a number may represent a position, quantity and/or quality. Today's civilization would be just impossible without numbers. People have been attracted to the mysteries of numbers throughout history. Accordingly, number theory is the oldest and most developed area of mathematics. Throughout the mathematical path to the present day, people have gradually learned properties of numbers. It is surprising to find that the number zero was not recognized until Hindus found it about one thousand years ago (although it is recognized that Mayans found it as well). Compared to this old field of mathematics, the theory of operator algebras is very new; its foundation was given by the pioneering work of J. von Neumann and his collaborator F. J. Murray in the early part of the twentieth century, i.e. in the thirties. Subsequent major development occurred only a decade later in the late forties and the early fifties. But since then it has marked steady progress reaching new heights today. The theory handles self-adjoint algebras of bounded operators on a Hilbert space. The advent of quantum physics at the turn of century forced one to consider non-commutative variables. One needed to broaden the concept of numbers. Integers, rational numbers, real numbers and complex numbers are all commutative. Among the few noncommutative mathematical systems available at the beginning of quantum mechanics were matrix algebras, which did not accommodate the needs of quantum physics because the Heisenberg uncertainty principle and/or Heisenberg commutation relation do not allow one to stay in the realm of finite matrices. One needs to consider algebras of operators on a Hilbert space of infinite dimension. Some of these operators correspond to important physical quantities. One has to include operators in the list of "numbers". Number theory tells us to put numbers in a field to study them more efficiently. Similarly, the theory of operator algebras puts operators of interest in an algebra and we study the algebra and its structure first. The infinite dimensionality of the underlying Hilbert space poses big challenges and also presents interesting new phenomenon which do not occur in the classical frame work. We have already seen some of them in the first volume. For example, the continuity of dimensions in a factor of type II_1 is one of them. The infinite dimensionality of our objects forces us to create sophisticated methods to handle approximations. Simple minded counting does not lead to the heart of the matter. For example, it is impossible to introduce a simple minded coordinate system in an infinite dimensional operator algebra, thus mathematical induction based on a basis does not fly. The early part of the theory, in the period of the forties through the early sixties were spent on this issue. Luckily there is a remarkable similarity between the theory of measures on a locally compact space and the theory of operator algebras. The first volume was devoted to the pursuit of this similarity.

The second volume of “Theory of Operator Algebras” is devoted to the study of the structure of von Neumann algebras of type III and their automorphism groups, cf. Chapter VI through Chapter XII; and the third volume is devoted to the study of the fine structure analysis of approximately finite dimensional factors and their automorphism groups, cf. Chapter XIII through Chapter XVIII. The last chapter, Chapter XIX, is an introduction to the theory of subfactors and their symmetries. One should note that the class of von Neumann algebras of type III is given by exclusion, i.e., by the absence of a non-trivial trace or a non-zero finite projection. This situation presented the major obstruction for the study of von Neumann algebras from the beginning of the subject until the advent of Tomita-Takesaki theory in the late sixties whilst many examples had been found to be of type III: the infinity of non-isomorphic factors were first established for factors of type III by Powers in 1967, [670], before the discovery of infinitely many non-isomorphic factors of type II_1 or II_∞, [635, 686], and most examples from quantum physics were shown to be of type III, [430]. It was the Tomita-Takesaki theory which broke the ice. It is still amazing that the subject defined by exclusion admits such a fine structural analysis since usually exclusion does not allow one to find any alternative and is viewed as pathological. Of course, a von Neumann algebra of type III had been pathological until we discover their fine structure. We will explore this in full detail through the second volume.

Each chapter has its own introduction which describes the content of that chapter and the basic strategy so that the reader can get a quick overview of the chapter.

In the second and third volume, we present two major items in the theory of von Neumann algebras: one is the analogy with integration theory on an abstract measure space and the other is the emphatic importance of automorphisms of algebras, i.e. we emphasize the symmetries of our objects following the modern point of view of E. Galois.

In general, the theory of von Neumann algebras is considered to be non-commutative integration. In Volume I, the similarity between von Neumann algebras and measure spaces are examined from the point of view of Banach space duality. In the second and third volume, non-commutative integration goes far beyond the analogy with ordinary integration. Since it is not our main interest to examine how ordinary integration should be formulated based on commutative von Neumann algebras, it is not discussed here in detail beyond a few comments. Still it is possible to develop a theory which covers the ordinary integration theory based on the operator algebra approach. In fact, such a theory has been explored by G. K. Pedersen, [653, Chapter 6], and it does eliminate pathological uninteresting measure spaces easily. The main difference between the operator algebra approach and the conventional approach to integration theory relies on the fact that in operator algebras one considers functions first, or equivalently variables, and then one views the underlying points as the spectrum of the variables; whilst in the ordinary approach one considers points first and views variables as functions on the set of points. We would like to point out here, however, that in practice we never observe points directly only approximately by successive evaluations of coordinates. Besides this philosophical difference, there is another major difference between the ordinary integration theory and the non-commutative integration theory which rests on the fact that a weight, a non-commutative counterpart of a σ-finite measure, gives rise to a one-parameter automorphism group, called the modular automorphism group, of the von Neumann algebra in question. This modular automorphism group can be considered as the time evolution of the system, i.e., in the non-commutative world a state determines the associated dynamics. The appearance of the modular automorphism group distinguishes our theory sharply from the classical theory. The modular automorphism group gives us abundant non-trivial information precisely when there is no trace on the algebra in question. Since the ordinary integration is a trace, the modular automorphism group is trivial in that case and cannot be appreciated. Furthermore, thanks to the Connes cocycle derivative theorem, Theorem VIII.3.3, the modular automorphism group is unique up to perturbation by a one unitary cocycle, which allows us to relate the structure of a von Neumann algebra of type III to that of the associated von Neumann algebra of type II_∞ equipped with a trace scaling one parameter automorphism group, cf. Chapter XII. As a byproduct of our non-commutative integration theory, a duality theorem attributed to Pontrjagin, van Kampen, Tannaka, Stinespring, Eymard, Saito and Tatsuuma, is presented in §3, Chapter VII. With this exception, no discussion of examples is presented in the second volume, Chapter VI through Chapter XII. Extensive discussions of examples and constructions of factors occupy the third volume starting in Chapter XIII and through Chapter XVIII.

The so-called Murray-von Neumann measure space construction of factors is closely investigated first in Chapter XIII yielding the Krieger construction of factors and the theory of measured groupoids. Systematic study of approximately finite dimensional factors occupies most of the third volume, cf. Chapter XIV through Chapter XIX. The theory is highlighted by the celebrated classification theorem of Alain Connes in the form of Theorems XVI.1.9, XVIII.1.1, XVIII.2.1 to which W. Krieger made a substantial contribution also, and XVIII.4.16 which requires one full section of preparation given by U. Haagerup, [550]. The last chapter, Chapter XIX, is devoted to an introduction to the theory of subfactors of an AFD factor created by V. F. R. Jones, and concludes with a classification theorem of Popa, Theorem XIX.4.16, for subfactors of an AFD factor of type II_1 with small indices.

The three volume book, “Theory of Operator Algebras”, is a product of the author’s research and teaching activities at the Department of Mathematics at University of California, Los Angeles, spanning the years from 1969 through the present time. It is important to mention the following: the author’s visit to the University of Pennsylvania from 1968 through 1969 where the foundation of Tomita-Takesaki theory was established; the author’s participation in various research activities which include several short and long visits to the University of Marseille-Aix-Luminy;

several short visits to RIMS of Kyoto University; one year participation in the Mathematical Physics Project of 1975–1976 at ZiF, University of Bielefeld; a full year participation in the operator algebra project of MSRI for 1984–1985; a one year visit to IHES, 1988–1989; two one month long participations in the one year project (1988–1989) on operator algebras at the Mittag-Leffler Institute; several visits to the University of New South Wales; and several month long visits to the Mathematics Institute of University of Warwick. The author would like to express here his sincere gratitude to these institutions and to the mathematicians who hosted him warmly and worked with him. Special thanks are due to Professor Richard V. Kadison with whom the author discussed the philosophy of the subject at length so many times, and to Professor Daniel Kastler who encouraged him in many ways and provided the opportunity to work with him and others including Alain Connes. Throughout the period of the preparation of the book, the author has been continuously supported by the National Science Foundation. Here he would like to record his appreciation of that support. The Guggenheim Foundation also gave the author support at a critical period of his career, for which the author is very grateful. The author also would like to express his gratitude to Professor Masahiro Nakamura who has constantly given his moral support to the author, to Professor Takashi Turumaru whose beautiful lectures inspired the author to be a functional analyst and to the late Professor Yoshinao Misonou under whose leadership the author started his career as a functional analyst. At the final stage of the preparation of the manuscript, Dr. Un Kit Hui and Dr. Toshihiko Masuda took pains to help the author to edit the manuscript. Although any misprints and mistakes are the author's responsibility, the author would like to thank them here.

Guidance to the Reader

Each chapter has its own introduction so that one can quickly get an overview of the content of the chapter. Theorems, Propositions, Lemmas and Definitions are numbered in one sequence, whilst formulas and equations are numbered in each section separately without reference to the section. Formulas (respectively, equations) are referred to by the formula number (respectively, equation number) alone if it is quoted in the same section, and by the section number followed by the formula number if it is quoted in a different section but in the same chapter, and finally by the chapter number, the section number and the formula number (respectively, equation number) if it is quoted in a different chapter. Some exercises are selected to help the reader to get information and techniques not covered in the main text, so they can be viewed as a supplement to the text. Those exercises taken directly literatures are marked by a †-sign, and the references are cited there.

To keep the book within a reasonable size, this three volume book does not include the materials related to the following important areas of operator algebras: K-theory for C^*-algebras, geometric theory of operator algebras such as cyclic cohomology, the classification theory of nuclear C^*-algebras, free probability theory and the advanced theory of subfactors. The interested readers are referred to the forthcoming books in this operator algebra series of encyclopedia.

Chapter IX
Non-Commutative Integration

§ 0 Introduction

The theories of weights, traces and states are often referred as non commutative integration. If the von Neumann algebra in question is abelian, then our theory is precisely the theory of measures and integration. In fact, the weight value of a self-adjoint element is given precisely by the integration of the corresponding function on the spectrum relative to the measure corresponding to the weight. As there are many non-commuting self-adjoint elements in the algebra, we have to consider various spectral measures even if we fix one weight and we can not represent non-commuting self-adjoint elements as functions on the same space. The striking difference between the commutative case and the non-commutative case is the appearance of one parameter automorphism group which is determined by the weight. Namely, weights and/or states determine the dynamics of the system which does not have the commutative counter part. We have explored the relationship between weights and the modular automorphism groups so far. We now further investigate how the dynamics, i.e. the modular automorphism groups, of the algebra relate the different spaces associated with the algebras. First, we study the underlying Hilbert space of the algebra and find the intrinsic pointed convex cone there, which is called the natural cone, in the first section. The theory developed there allows us to view the standard Hilbert space as the square root of the predual of the algebra as well as to represent the automorphism group $\mathrm{Aut}(\mathcal{M})$, of a von Neumann algebra $\mathcal{M}$, as the group of unitaries which leaves the natural cone globally invariant. In §2, we consider the special case that the weight is a trace and see the very special character of a trace, which allows us to realize various spaces associated with the algebra $\mathcal{M}$ as the space of unbounded operators affiliated with $\mathcal{M}$ satisfying certain regulating condition which are called measurable operators. We will see that in the case of a finite von Neumann algebra $\mathcal{M}$ every closed operator affiliated with $\mathcal{M}$ is measurable. Measurable operators form an involutive algebra as we will see in this section. It is remarkable that the trace regulates the behavior of unbounded operator so strongly that closed symmetric measurable operators are automatically self-adjoint. Section 3 relates a von Neumann algebra $\mathcal{M}$ of operators on a Hilbert space $\mathfrak{H}$ to its commutant $\mathcal{M}'$ in very strong way. It will be shown that there uniquely corresponds a non-singular self-adjoint operator, called the spatial derivative, to any pair of a faithful semi-finite normal weights φ on $\mathcal{M}$ and ψ on $\mathcal{M}'$. The spatial derivative behaves very naturally even though they are unbounded operators. Also, we consider right actions, i.e. an anti-representation, of a von Neumann algebra on a Hilbert space and view it a right module over the von Neumann algebra. We then view a Hilbert space equipped with the usual action of a von Neumann algebra as a left module over the von Neumann algebra. Naturally, we consider bimodules over a pair of von Neumann algebras, as a Hilbert space equipped with commuting the left action of one algebra and the right action of the other. This view allows us to introduce the concept of relative tensor product of a pair of a right module and a left module over a fixed von Neumann algebra. It should be noticed that this corresponds to the relative tensor products in algebra but not straightforward way. There are subtle difference here and the situation in pure algebra. For example, the relative tensor product of an arbitrary pair of vectors does not make sense unless we are in the very special case of atomic von Neumann algebras. The last section, §4, discuss conditional expectations and unbounded operator valued weights as a generalization of conditional expectations which allows us to view our theory in a more balanced way. It allows us to factor the usual trace, Tr, on a Hilbert space $\mathfrak{H}$ through a faithful semi-finite normal trace τ on a semi-finite von Neumann algebra $\mathcal{M}$ and the corresponding operator valued weight to $\mathcal{M}$. This in a sense justifies to say that a factor is indeed a *factor* of $\mathcal{L}(\mathfrak{H})$. One should interpret an operator valued weight as a partial integral relative to the related non-commutative measure, which is not visible unless the algebra is abelian.

§ 1 Standard Form of a von Neumann Algebra

We begin our discussion with an example. Let $\mathcal{M} = \mathcal{L}(\mathfrak{H})$ with a Hilbert space $\mathfrak{H}$. We fix the usual trace Tr on $\mathcal{M}$ and denote it by τ, then consider the semi-cyclic representation $\{\pi_\tau, \mathfrak{H}_\tau, \eta_\tau\}$. The Hilbert space $\mathfrak{H}_\tau$ is then identified with the Hilbert space of operators of the Hilbert-Schmidt class, and the representation π_τ is given by the left multiplication. If φ is a faithful normal positive linear functional on $\mathcal{M}$, then φ is of the form:

$$\varphi(x) = \tau(xh) = \left(xh^{\frac{1}{2}} \mid h^{\frac{1}{2}}\right)_{\mathrm{HS}} \tag{1}$$

with h a non-singular nuclear positive operator on $\mathfrak{H}$, where $(\ \mid\ )_{\mathrm{HS}}$ denotes the inner product in $\mathfrak{H}_\tau$. Therefore, $h^{1/2}$ is the cyclic and separating vector associated with the functional φ. Hence we have

$$S_\varphi x = \left(xh^{-\frac{1}{2}}\right)^* h^{\frac{1}{2}} = h^{-\frac{1}{2}} x^* h^{\frac{1}{2}}, \tag{2}$$

for those $x \in \mathfrak{H}_\tau$ such that $xh^{-1/2}$ is bounded. We proceed our discussion formally without worrying about the domain questions and other analytical details, in order to figure out the picture of the entire mechanism. The adjoint involution F_φ of S_φ is then given by

$$F_\varphi x = h^{\frac{1}{2}}\left(h^{-\frac{1}{2}}x\right)^* = h^{\frac{1}{2}} x^* h^{-\frac{1}{2}}, \quad x \in \mathfrak{D}^\flat. \tag{3}$$

Hence we get

$$\Delta_\varphi x = h x h^{-1}, \qquad x \in \mathfrak{D}(\Delta_\varphi); \tag{4}$$

$$\Delta_\varphi^{\mathrm{i}t} x = h^{\mathrm{i}t} x h^{-\mathrm{i}t}, \quad x \in \mathfrak{H}_\tau. \tag{5}$$

We now observe that

$$\left[\Delta_\varphi^{\frac{1}{4}} \mathcal{M}_+ h^{\frac{1}{2}}\right] = \left[h^{\frac{1}{4}} \mathcal{M}_+ h^{\frac{1}{4}}\right] = (\mathfrak{H}_\tau)_+. \tag{6}$$

Thus, the positive cone $(\mathfrak{H}_\tau)_+$ of $\mathfrak{H}_\tau$ is recovered from $\mathcal{M}_+ h^{1/2}$ by multiplying $\Delta_\varphi^{1/4}$. The point here is that the positive cone $(\mathfrak{H}_\tau)_+$ does not depend on the choice of φ; it is intrinsic to the algebra $\mathcal{M} = \mathcal{L}(\mathfrak{H})$. We are going to see that this fact is not special to $\mathcal{L}(\mathfrak{H})$, but true for general von Neumann algebras.

Now, we fix a von Neumann algebra $\mathcal{M}$ and a faithful semi-finite normal weight φ on it; thus also the semi-cyclic representation $\{\pi_\varphi, \mathfrak{H}_\varphi, \eta_\varphi\}$. In the Hilbert space $\mathfrak{H}_\varphi$, we shall consider various pointed convex cones. Let $\mathfrak{A}$ be the full left Hilbert algebra associated with φ and $\mathfrak{A}_0$ the associated Tomita algebra. We identify $\mathcal{M}$ with $\mathcal{R}_\ell(\mathfrak{A})$. We put

$$\left.\begin{aligned} \mathfrak{P}_\varphi^\sharp &= \left\{\xi\xi^\sharp : \xi \in \mathfrak{A}\right\}^-, \\ \mathfrak{P}_\varphi^\flat &= \left\{\xi\xi^\flat : \xi \in \mathfrak{A}'\right\}^-, \\ \mathfrak{P}_\varphi &= \left\{\xi\xi^* : \xi \in \mathfrak{A}_0\right\}^-. \end{aligned}\right\} \tag{7}$$

Here $\xi^* = J\xi,\ \xi \in \mathfrak{H}_\varphi$ and the bar means the closure.

Definition 1.1. For a convex cone $\mathfrak{P}$ in a Hilbert space $\mathfrak{H}$, the *dual cone* $\mathfrak{P}^\circ$ is defined to be the set of all those vectors $\eta \in \mathfrak{H}$ such that $(\xi \mid \eta) \geq 0$ for every $\xi \in \mathfrak{P}$:

$$\mathfrak{P}^\circ = \left\{\eta \in \mathfrak{H} : (\xi \mid \eta) \geq 0,\ \xi \in \mathfrak{P}\right\}. \tag{8}$$

If $\mathfrak{P} = \mathfrak{P}^\circ$, then $\mathfrak{P}$ is called *self-dual*.

To motivate discussion, we state the next result concerning the convex cones defined in (7).

Theorem 1.2. *Under the above notations and assumptions, we have the following:*

(i) *The sets $\mathfrak{P}_\varphi^\sharp$ and $\mathfrak{P}_\varphi^\flat$ are mutually dual pointed convex cones;*

(ii) *$\mathfrak{P}_\varphi$ is a self-dual convex cone and every element of $\mathfrak{H}_\varphi$ is represented as a linear combination of four vectors of $\mathfrak{P}_\varphi$. Furthermore, each self-adjoint vector $\xi \in \mathfrak{H}_\varphi$, i.e. $\xi = \xi^*$, is uniquely represented as difference of two orthogonal vectors of $\mathfrak{P}_\varphi$;*

(iii) *$aJaJ\mathfrak{P}_\varphi \subset \mathfrak{P}_\varphi$ for every $a \in \mathcal{M}$;*

(iv) *To each $\omega \in \mathcal{M}_*^+$, there corresponds a unique $\xi \in \mathfrak{P}_\varphi$ with $\omega = \omega_\xi$. Furthermore, we have*

$$\|\xi - \eta\|^2 \leq \|\omega_\xi - \omega_\eta\| \leq \|\xi - \eta\|\,\|\xi + \eta\|, \quad \xi, \eta \in \mathfrak{P}_\varphi. \tag{9}$$

PROOF OF THEOREM 1.2.(i): Let $\xi \in \mathfrak{A}$ and $\eta \in \mathfrak{A}'$. We then have

$$\left(\xi^\sharp\xi \mid \eta\eta^\flat\right) = \left(\pi_\ell(\xi)^*\xi \mid \pi_r(\eta)^*\eta\right) = \left\|\pi_\ell(\xi)\eta\right\|^2 \geq 0.$$

Suppose that $\xi \in \mathfrak{H}_\varphi$ satisfies $\left(\xi \mid \eta\eta^\flat\right) \geq 0$ for every $\eta \in \mathfrak{A}'$. By the polarization identity, $\mathfrak{A}'^2$ is linearly spanned by $\left\{\eta\eta^\flat : \eta \in \mathfrak{A}'\right\}$, so that Lemma VI.1.13 implies that ξ belongs to $\mathfrak{D}^\sharp$, and $\pi_\ell(\xi)$ is a symmetric operator affiliated with $\mathcal{M}$. The positivity assumption ξ entails that of $\pi_\ell(\xi)$. We denote the Friedricks extension of $\pi_\ell(\xi)$ by a. With its spectral decomposition $a = \int_0^\infty \lambda \, \mathrm{d}e(\lambda)$, set $\xi_n = e(n)\xi$. Then we get $\xi_n \in \mathfrak{A}$ and $\|\xi_n - \xi\| \to 0$ as $n \to \infty$. Furthermore, $\left(\xi_n \mid \eta\eta^\flat\right) = \left(ae(n)\eta \mid \eta\right) \geq 0,\ \eta \in \mathfrak{A}'$. Thus, to prove $\xi \in \mathfrak{P}_\varphi^\sharp$ it suffices to show that $\xi_n \in \mathfrak{P}_\varphi^\sharp$. This means that we are asked to prove $\xi \in \mathfrak{P}_\varphi^\sharp$ if $\left(\xi \mid \eta\eta^\flat\right) \geq 0,\ \eta \in \mathfrak{A}'$, and ξ is left bounded. As observed already, ξ belongs to $\mathfrak{A}$, so $a = \pi_\ell(\xi) \geq 0$ and $\varphi(a^2) < +\infty$. Since $\lambda\left(1 - e(\lambda)\right) \leq a,\ \lambda > 0$, we have $\varphi\left(1 - e(\lambda)\right) < +\infty$. Setting $b_n = \left(1 - e(1/n)\right)a^{1/2}$, we get $\varphi(b_n^2) < +\infty$ and $b_n^2 = \left(1 - e(1/n)\right)a$. Then with $\zeta_n = \eta_\varphi(b_n)$, we have $\zeta_n \in \mathfrak{A}$ and

$$\zeta_n^\sharp\zeta_n = \zeta_n^2 = \left(1 - e(1/n)\right)\xi.$$

Since $\xi \in \left[a\mathfrak{H}_\varphi\right]$, we get $\left\|\xi - \left(1 - e(1/n)\right)\xi\right\| \to 0$ as $n \to \infty$, concluding that $\xi \in \mathfrak{P}_\varphi^\sharp$. Thus $\mathfrak{P}_\varphi^\sharp = \left(\mathfrak{P}_\varphi^\flat\right)^0$. By symmetry, we have $\mathfrak{P}_\varphi^\flat = \left(\mathfrak{P}_\varphi^\sharp\right)^0$. Q.E.D.

Lemma 1.3. $\mathfrak{P}_\varphi = \left(\Delta^{1/4}\mathfrak{P}_\varphi^\sharp\right)^- = \left(\Delta^{-1/4}\mathfrak{P}_\varphi^\flat\right)^-$. *In particular,* $\mathfrak{P}_\varphi$ *is a closed convex cone.*

PROOF: First, we observe that $\mathfrak{P}_\varphi^\sharp \subset \mathfrak{D}^\sharp = \mathfrak{D}\left(\Delta^{1/2}\right)$ and $\mathfrak{P}_\varphi^\flat \subset \mathfrak{D}^\flat = \mathfrak{D}\left(\Delta^{-1/2}\right)$. Hence $\Delta^{1/4}\mathfrak{P}_\varphi^\sharp$ and $\Delta^{-1/4}\mathfrak{P}_\varphi^\flat$ make sense. Furthermore, we have

$$\Delta^{\frac{1}{2}}\mathfrak{P}_\varphi^\sharp = JS\mathfrak{P}_\varphi^\sharp = J\mathfrak{P}_\varphi^\sharp = \mathfrak{P}_\varphi^\flat,$$

so that $\Delta^{1/4}\mathfrak{P}_\varphi^\sharp = \Delta^{-1/4}\mathfrak{P}_\varphi^\flat$. Thus, it only remains to be proven that $\mathfrak{P}_\varphi = \left(\Delta^{1/4}\mathfrak{P}_\varphi^\sharp\right)^-$. Now, if $\xi \in \mathfrak{A}_0$, then

$$\Delta^{\frac{1}{4}}\left(\xi\xi^\sharp\right) = \left(\Delta^{\frac{1}{4}}\xi\right)\left(\Delta^{\frac{1}{4}}\xi^\sharp\right) = \left(\Delta^{\frac{1}{4}}\xi\right)\left(\Delta^{\frac{1}{4}}\xi\right)^*,$$

so that $\Delta^{1/4}\left\{\xi\xi^\sharp : \xi \in \mathfrak{A}_0\right\} = \left\{\xi\xi^* : \xi \in \mathfrak{A}_0\right\}$. Hence

$$\mathfrak{P}_0 = \left\{\xi\xi^* : \xi \in \mathfrak{A}_0\right\} \subset \Delta^{\frac{1}{4}}\mathfrak{P}_\varphi^\sharp.$$

If $\xi \in \mathfrak{A}$, set

$$\xi_r = \sqrt{\frac{r}{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-rt^2}\Delta^{\mathrm{i}t}\xi \, \mathrm{d}t, \quad r > 0. \tag{10}$$

We then have, as $r \to \infty$,

$$\begin{cases} \|\xi - \xi_r\| \to 0, & \left\|\xi^\sharp - \xi_r^\sharp\right\| \to 0 \\ \left\|\pi_\ell(\xi_r)\right\| \le \left\|\pi_\ell(\xi)\right\|, & \pi_\ell(\xi_r) \to \pi_\ell(\xi)\ \sigma\text{-strongly}^*, \end{cases}$$

and that $\xi_r \in \mathfrak{A}_0$. Therefore we conclude

$$\mathfrak{P}_\varphi^\sharp = \left\{\xi\xi^\sharp : \xi \in \mathfrak{A}_0\right\}^-, \qquad \mathfrak{P}_\varphi = \left\{\xi\xi^* : \xi \in \mathfrak{A}_0\right\}^-, \qquad \mathfrak{P}_\varphi^\flat = \left\{\xi\xi^\flat : \xi \in \mathfrak{A}_0\right\}^-.$$

Thus, we obtain the inclusion: $\mathfrak{P}_\varphi \subset \left(\Delta^{1/4}\mathfrak{P}_\varphi^\sharp\right)^-$. To show the reversed inclusion, let $\xi \in \mathfrak{P}_\varphi^\sharp$ and choose $\xi_n \in \mathfrak{P}_0^\sharp = \left\{\eta\eta^\sharp : \eta \in \mathfrak{A}_0\right\}$ so that $\xi_n \to \xi$. Then we have $S\xi = \xi,\ S\xi_n = \xi_n$, so that $\Delta^{1/2}\xi_n = J\xi_n \to J\xi = \Delta^{1/2}\xi$, which yields the convergence:

$$\left\|\Delta^{\frac{1}{4}}\xi - \Delta^{\frac{1}{4}}\xi_n\right\|^2 = \left(\Delta^{\frac{1}{2}}(\xi - \xi_n) \mid \xi - \xi_n\right) \to 0.$$

Since $\Delta^{1/4}\xi_n \in \mathfrak{P}_0,\ \Delta^{1/4}\xi \in \mathfrak{P}_\varphi$. Thus $\Delta^{1/4}\mathfrak{P}_\varphi^\sharp \subset \mathfrak{P}_\varphi$. Q.E.D.

Lemma 1.4. *$\mathfrak{P}_\varphi$ is a self-dual cone and $\Delta^{\mathrm{i}t}\mathfrak{P}_\varphi = \mathfrak{P}_\varphi$.*

PROOF: For any $\xi \in \mathfrak{P}_\varphi^\sharp$ and $\eta \in \mathfrak{P}_\varphi^\flat$,

$$\left(\Delta^{\frac{1}{4}}\xi \mid \Delta^{-\frac{1}{4}}\eta\right) = (\xi \mid \eta) \ge 0$$

by Theorem 1.2.(i). Hence $(\xi \mid \eta) \ge 0$ for every pair $\xi, \eta \in \mathfrak{P}_\varphi$.

Now suppose that $\xi \in \mathfrak{H}_\varphi$ satisfies the inequality

$$(\xi \mid \eta) \ge 0 \quad \text{for every } \eta \in \mathfrak{P}_\varphi.$$

For each $r > 0$, define ξ_r by (10). Then for each $\eta \in \mathfrak{A}_0$, we have

$$\begin{aligned}\left(\xi_r \mid \eta\eta^*\right) &= \sqrt{\frac{r}{\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-rt^2}\left(\Delta^{\mathrm{i}t}\xi \mid \eta\eta^*\right)\,\mathrm{d}t \\ &= \sqrt{\frac{r}{\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-rt^2}\left(\xi \,\middle|\, \left(\Delta^{-\mathrm{i}t}\eta\right)\left(\Delta^{-\mathrm{i}t}\eta\right)^*\right)\,\mathrm{d}t \ge 0.\end{aligned}$$

Now, we get

$$\left(\Delta^{-\frac{1}{4}}\xi_r \mid \eta\eta^\flat\right) = \left(\xi_r \mid \Delta^{-\frac{1}{4}}(\eta\eta^\flat)\right) = \left(\xi_r \,\middle|\, \left(\Delta^{-\frac{1}{4}}\eta\right)\left(\Delta^{-\frac{1}{4}}\eta\right)^*\right) \ge 0.$$

Hence $\zeta_r = \Delta^{-1/4}\xi_r \in \mathfrak{P}_\varphi^\sharp$ because $\mathfrak{P}_0^\flat = \left\{\eta\eta^\flat : \eta \in \mathfrak{A}_0\right\}$ is dense in $\mathfrak{P}_\varphi^\flat$, so that $\xi_r = \Delta^{1/4}\zeta_r \in \mathfrak{P}_\varphi$. Therefore ξ belongs to $\mathfrak{P}_\varphi$ because $\|\xi - \xi_r\| \to 0$. Thus $\mathfrak{P}_\varphi$ is self-dual.

The invariance: $\Delta^{\mathrm{i}t}\mathfrak{P}_\varphi = \mathfrak{P}_\varphi$ follows from the fact that $\Delta^{\mathrm{i}t}(\xi\xi^*) = \left(\Delta^{\mathrm{i}t}\xi\right)\left(\Delta^{\mathrm{i}t}\xi\right)^*$, $\xi \in \mathfrak{A}_0$. Q.E.D.

Lemma 1.5. *If ψ is another faithful semi-finite normal weight on $\mathcal{M}$, then there exists uniquely a unitary $U_{\varphi,\psi}$ from $\mathfrak{H}_\psi$ onto $\mathfrak{H}_\varphi$ such that*

(i) $U_{\varphi,\psi}\pi_\psi(x)U^*_{\varphi,\psi} = \pi_\varphi(x)\,,\quad x \in \mathcal{M}\,;$

(ii) $U_{\varphi,\psi}\mathfrak{P}_\psi = \mathfrak{P}_\varphi\,.$

PROOF: As in the previous chapter, we consider $\mathcal{N} = \mathcal{M} \otimes M(2;\mathbf{C})$ and the balanced weight $\rho = \varphi \oplus \psi$ given by (VIII.3.7). Then define $S_\varphi,\ S_{\varphi,\psi},\ S_{\psi,\varphi}$ and S_ψ by (VIII.3.13) and consider their polar decompositions:

$$\begin{aligned} S_\varphi &= J_\varphi \Delta_\varphi^{\frac{1}{2}}, & S_{\varphi,\psi} &= J_{\varphi,\psi}\Delta_{\varphi,\psi}^{\frac{1}{2}},\\ S_{\psi,\varphi} &= J_{\psi,\varphi}\Delta_{\psi,\varphi}^{\frac{1}{2}}, & S_\psi &= J_\psi \Delta_\psi^{\frac{1}{2}}. \end{aligned} \tag{11}$$

As in Theorem VIII.3.2, the operator

$$U_{\varphi,\psi} = J_\varphi J_{\varphi,\psi} \tag{12}$$

implements the unitary equivalence of π_φ and π_ψ. We recall the fact that $U_{\varphi,\psi}$ is also given by:

$$U_{\varphi,\psi} = J_{\varphi,\psi}J_\psi\,. \tag{12$'$}$$

Since $\mathfrak{P}_\varphi$ is the closure of $\mathfrak{P}_0$, we get

$$\mathfrak{P}_\varphi = \left\{\pi_\varphi(x)J_\varphi\eta_\varphi(x) : x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*\right\}^-,$$

$$\mathfrak{P}_\psi = \left\{\pi_\psi(y)J_\psi\eta_\psi(y) : y \in \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*\right\}^-.$$

If $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ and $y \in \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*$, then we get

$$\begin{aligned} &\big(\pi_\varphi(x)J_\varphi\eta_\varphi(x)\ \big|\ U_{\varphi,\psi}\pi_\psi(y)J_\psi\eta_\psi(y)\big)\\ &\quad= \big(J_\varphi\eta_\varphi(x)\ \big|\ U_{\varphi,\psi}\pi_\psi(x^*)\pi_\psi(y)J_\psi\eta_\psi(y)\big)\\ &\quad= \big(J_\varphi\eta_\varphi(x)\ \big|\ U_{\varphi,\psi}\pi_\psi(x^*)J_\psi\pi_\psi(y)J_\psi\eta_\psi(y)\big) \quad {}^{1}\\ &\quad= \big(J_\varphi\eta_\varphi(x)\ \big|\ U_{\varphi,\psi}J_\psi\pi_\psi(y)J_\psi\pi_\psi(x^*)\eta_\psi(y)\big)\\ &\quad= \big(J_\varphi\eta_\varphi(x)\ \big|\ J_\varphi U_{\varphi,\psi}\pi_\psi(y)J_\psi\eta_\psi(x^*y)\big)\\ &\quad= \big(\pi_\varphi(y)U_{\varphi,\psi}J_\psi\eta_\psi(x^*y)\ \big|\ \eta_\varphi(x)\big)\\ &\quad= \big(J_{\varphi,\psi}\eta_\psi(x^*y)\ \big|\ \eta_\varphi(y^*x)\big) = \Big(\Delta_{\varphi,\psi}^{\frac{1}{2}}\eta_\varphi(y^*x)\ \Big|\ \eta_\varphi(y^*x)\Big) \geq 0, \end{aligned}$$

where we used the fact that $y^*x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\psi$. Hence, the self-duality of $\mathfrak{P}_\varphi$ implies that $U_{\varphi,\psi}\mathfrak{P}_\psi \subset \mathfrak{P}_\varphi$. The converse inclusion relation follows from the fact that $U^*_{\varphi,\psi} = U_{\psi,\varphi}$.

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[1] $\pi_\psi(y)J_\psi\eta_\psi(y) = J_\psi\pi_\psi(y)J_\psi\eta_\psi(y)$.

Now, suppose that U is a unitary from $\mathfrak{H}_\psi$ onto $\mathfrak{H}_\varphi$ satisfying (i) and (ii). To prove $U = U_{\varphi,\psi}$, set $V = U_{\varphi,\psi}U^*$. Then $V \in \pi_\varphi(\mathcal{M})'$ and $V\mathfrak{P}_\varphi = \mathfrak{P}_\varphi$. Hence for any $\xi \in \mathfrak{P}_\varphi$, we have $J_\varphi V\xi = V\xi = J_\varphi V J_\varphi \xi$. By the polarization identity, $\mathfrak{A}_0^2$ is spanned by $\left\{ \xi\xi^* : \xi \in \mathfrak{A}_0 \right\}$ and $\mathfrak{A}_0^2$ is dense in $\mathfrak{H}_\varphi$. Thus, we obtain $V = J_\varphi V J_\varphi$. Therefore, $V = V^* \in \pi_\varphi(\mathcal{M}) \cap \pi_\varphi(\mathcal{M})'$. So we write $V = E - F$ as a difference of orthogonal projections E, F (in the center of $\pi_\varphi(\mathcal{M})$ with $E + F = 1$). Since $F = F J_\varphi F J_\varphi$, for each $\xi \in \mathfrak{A}_0$ we have

$$F(\xi\xi^*) = (F\xi)(F\xi)^* \in \mathfrak{P}_\varphi,$$

so that $F\mathfrak{P}_\varphi \subset \mathfrak{P}_\varphi$. If $0 \neq F\xi = \xi,\ \xi \in \mathfrak{P}_\varphi$, then

$$\left(V\xi \;\middle|\; \xi\right) = \left((E - F)\xi \;\middle|\; \xi\right) = -\|F\xi\|^2 < 0,$$

which is a contradiction. Hence $F\mathfrak{P}_\varphi = \{0\}$. Since $\left[\mathfrak{P}_\varphi\right] = \mathfrak{H}_\varphi$ as we just observed, $F = 0$. Thus $V = 1$, which means that $U = U_{\varphi,\psi}$. Q.E.D.

Lemma 1.6. *For each $\omega \in \mathcal{M}_*^+$, there exists $\xi \in \mathfrak{P}_\varphi$ such that*

$$\omega(x) = \left(\pi_\varphi(x)\xi \;\middle|\; \xi\right).$$

PROOF: We first prove the case where ω and φ commute and $\omega \leq \varphi$. As in §VII.2, there exists $h_\omega \in \pi_\varphi(\mathcal{M})'_+$ and $\eta_\omega \in \mathfrak{H}_\varphi$ such that

$$h_\omega^{\frac{1}{2}}\eta_\varphi(x) = \pi_\varphi(x)\eta_\omega, \qquad \omega(x) = \left(\pi_\varphi(x)\eta_\omega \;\middle|\; \eta_\omega\right), \qquad x \in \mathcal{M}.$$

It follows easily that $\eta_\omega \in \mathfrak{P}_\varphi^\flat$. By assumption, ω and φ commute, so that $\Delta_\varphi^{it}\eta_\omega = \eta_\omega$. Hence $\eta_\omega \in \Delta_\varphi^{-1/4}\mathfrak{P}_\varphi^\flat \subset \mathfrak{P}_\varphi$.

We next prove the case that ω is faithful. We then consider the cyclic representation $\left\{\pi_\omega, \mathfrak{H}_\omega, \xi_\omega\right\}$ and set $\eta_\omega = U_{\varphi,\omega}\xi_\omega$. Since $\xi_\omega \in \mathfrak{P}_\omega$, Lemma 1.5 implies that $\eta_\omega \in \mathfrak{P}_\varphi$ and $\omega(x) = \left(\pi_\varphi(x)\eta_\omega \;\middle|\; \eta_\omega\right)$.

Finally we prove the general case. If ω is not faithful, then we choose a semifinite normal weight ψ' on $\mathcal{M}$ such that $s(\psi') = 1 - s(\omega)$ and set $\psi = \psi' + \omega$. Then ω and ψ commute and $\omega \leq \psi$, so that there exists $\eta_\omega \in \mathfrak{P}_\psi$ such that $\omega(x) = \left(\pi_\psi(x)\eta_\omega \;\middle|\; \eta_\omega\right)$. Now, we put $\xi = U_{\varphi,\psi}\eta_\omega$, which is the required vector by the previous lemma. Q.E.D.

In the above discussion, we used the notations π_φ and π_ψ in order to avoid confusion. But we now identify $\pi_\varphi(x)$ and x, so we write x alone instead of $\pi_\varphi(x)$. Thus $\mathfrak{H}$ means $\mathfrak{H}_\varphi$.

Lemma 1.7. *If $J\xi = \xi,\ \ \xi \in \mathfrak{H}$, then ξ is uniquely written as a difference $\xi = \xi_+ - \xi_-$ of two orthogonal vectors $\xi_+, \xi_- \in \mathfrak{P}_\varphi$.*

PROOF: Since $\mathfrak{P}_\varphi$ is a closed convex cone, the distance d from ξ to $\mathfrak{P}_\varphi$ is given by a vector $\xi_+ \in \mathfrak{P}_\varphi$, so that $d = \|\xi - \xi_+\|$ and ξ_+ is orthogonal to $\xi - \xi_+$. If $\eta \in \mathfrak{P}_\varphi$, $\eta \neq 0$, then

$$\left\|\xi - \xi_+\right\|^2 = d^2 < \left\|\xi - (\xi_+ + \lambda\eta)\right\|^2, \quad \lambda > 0,$$

so

$$\lambda^2\|\eta\|^2 - 2\lambda\big(\xi - \xi_+ \mid \eta\big) > 0, \quad \lambda > 0,$$

which is possible only if $\big(\xi - \xi_+ \mid \eta\big) \leq 0$. Thus $\xi_- = \xi_+ - \xi$ belongs to $\mathfrak{P}_\varphi$ and $\xi = \xi_+ - \xi_-$.

Suppose that $\xi = \xi'_+ - \xi'_-$, $\xi'_+ \perp \xi'_-$ and $\xi'_+, \xi'_- \in \mathfrak{P}_\varphi$. For each $\eta \in \mathfrak{P}_\varphi$, we have

$$\begin{aligned}\|\xi - \eta\|^2 &= \left\|\xi'_+ - \eta\right\|^2 + \|\xi'_-\|^2 - 2\big(\xi'_+ - \eta \mid \xi'_-\big)\\ &= \left\|\xi'_+ - \eta\right\|^2 + \|\xi'_-\|^2 + 2\big(\eta \mid \xi'_-\big) \geq \|\xi'_-\|^2.\end{aligned}$$

The last inequality becomes an equality only when $\eta = \xi'_+$. Hence ξ'_+ is the vector in $\mathfrak{P}_\varphi$ which gives the distance from ξ to $\mathfrak{P}_\varphi$. Thus $\xi'_+ = \xi_+$ and so $\xi'_- = \xi_-$. Q.E.D.

For each $\xi \in \mathfrak{P}_\varphi$, we consider $\omega = \omega_\xi \in \mathcal{M}_*^+$ and set $e = s(\omega)$. Then ω can be viewed as a faithful positive normal functional on the reduced algebra $\mathcal{M}_e$. Viewing $\{\pi_\omega, \mathfrak{H}_\omega, \xi_\omega\}$ as the cyclic representation of $\mathcal{M}_e$, we consider naturally $\{J_\omega, \Delta_\omega, \mathfrak{P}_\omega, \mathfrak{P}^\sharp_\omega, \mathfrak{P}^\flat_\omega\}$. Now put

$$U_{\varphi,\omega}\pi_\omega(x)\xi_\omega = x\xi, \quad x \in \mathcal{M}_e. \tag{13}$$

Lemma 1.8. *$U_{\varphi,\omega}$ is an isometry from $\mathfrak{H}_\omega$ onto $[\mathcal{M}\xi] \cap [\mathcal{M}'\xi]$ which enjoys the following properties:*

(i) $U_{\varphi,\omega}\pi_\omega(x) = xU_{\varphi,\omega}$, $x \in \mathcal{M}_e$;

(ii) $U_{\varphi,\omega}\mathfrak{P}_\omega = \mathfrak{P}_\varphi \cap [\mathcal{M}\xi] \cap [\mathcal{M}'\xi]$;

(iii) *$U_{\varphi,\omega}$ is unique, subject to conditions* (i) *and* (ii).

PROOF: We note that $e = [\mathcal{M}'\xi]$. If e' is the projection to $[\mathcal{M}\xi]$, then $e' = JeJ$ and $f = ee'$ commutes with J. It is clear that $U_{\varphi,\omega}$ is an isometry of $\mathfrak{H}_\omega$ onto $f\mathfrak{H}$ and satisfies (i).

We put $J_0 = U^*_{\varphi,\omega}JU_{\varphi,\omega}$. Then it follows that:

$$\left.\begin{aligned}&J_0\xi_\omega = \xi_\omega;\\ &J_0\pi_\omega(\mathcal{M}_e)J_0 = \pi_\omega(\mathcal{M}_e)';\\ &\big(J_0\pi_\omega(x)J_0\pi_\omega(x)\xi_\omega \mid \xi_\omega\big) \geq 0, \quad x \in \mathcal{M}_e.\end{aligned}\right\} \tag{14}$$

From this, we want to conclude that $J_0 = J_\omega$. To this end, put $H = J_0S_\omega$ on $\mathfrak{H}_\omega$, where S_ω means of course the $\sharp$-operation in $\mathfrak{H}_\omega$ determined by ω. For each $x \in \mathcal{M}_e$, we have

$$\bigl(H\pi_\omega(x)\xi_\omega \bigm| \pi_\omega(x)\xi_\omega\bigr) = \bigl(J_0\pi_\omega(x^*)\xi_\omega \bigm| \pi_\omega(x)\xi_\omega\bigr) = \bigl(\pi_\omega(x^*)J_0\pi_\omega(x^*)\xi_\omega \bigm| \xi_\omega\bigr)$$
$$= \bigl(J_0\pi_\omega(x^*)J_0\pi_\omega(x^*)\xi_\omega \bigm| \xi_\omega\bigr) \geq 0.$$

Since $\pi_\omega(\mathcal{M}_e)\xi_\omega$ is a core for both H and $\Delta_\omega^{1/2}$, we get $H = \Delta_\omega^{1/2}$ and $J_0 = J_\omega$ from the uniqueness of polar decomposition.

We have seen already

$$\mathfrak{P}_\omega = \overline{\Bigl\{ J_\omega\pi_\omega(x)J_\omega\pi_\omega(x)\xi_\omega : x \in \mathcal{M}_e \Bigr\}},$$

and also

$$U_{\varphi,\omega}J_\omega\pi_\omega(x)J_\omega\pi_\omega(x)\xi_\omega = JxJx\xi \in \mathfrak{P}_\varphi, \quad x \in \mathcal{M}_e,$$

so that $U_{\varphi,\omega}\mathfrak{P}_\omega \subset \mathfrak{P}_\varphi$. By the self-duality of $\mathfrak{P}_\omega$, we have

$$U_{\varphi,\omega}^*\bigl(\mathfrak{P}_\varphi \cap [\mathcal{M}\xi] \cap [\mathcal{M}'\xi]\bigr) \subset \mathfrak{P}_\omega.$$

Thus we obtain assertion (ii).

The uniqueness of $U_{\varphi,\omega}$ follows from the arguments similar to that for $U_{\varphi,\psi}$ in Lemma 1.5. We leave the detail to the reader. Q.E.D.

Remark 1.9. We note here that the property (14) characterizes the modular conjugation J_ω for a faithful $\omega \in \mathcal{M}_*^+$.

We shall use the following notations:

$$\Delta_\xi = U_{\varphi,\omega}\Delta_\omega U_{\varphi,\omega}^*, \qquad S_\xi = U_{\varphi,\omega}S_\omega U_{\varphi,\omega}^*, \qquad \mathfrak{P}_\xi = U_{\varphi,\omega}\mathfrak{P}_\omega. \tag{15}$$

Corollary 1.10. *For each $\omega \in \mathcal{M}_*^+$, the vector $\xi \in \mathfrak{P}_\varphi$ with $\omega = \omega_\xi$ is unique.*

PROOF: Suppose that $\omega = \omega_\xi = \omega_\eta$ for some $\xi, \eta \in \mathfrak{P}_\varphi$. Then there exists a partial isometry u of $\mathcal{M}'$ such that $u\xi = \eta$ and $u^*\eta = \xi$. Hence $[\mathcal{M}'\xi] = [\mathcal{M}'\eta]$. It then follows that

$$\bigl[\mathcal{M}\xi\bigr] = J\bigl[\mathcal{M}'\xi\bigr] = J\bigl[\mathcal{M}'\eta\bigr] = \bigl[\mathcal{M}\eta\bigr].$$

Therefore, if we define $V_{\varphi,\omega}$ by (13) replacing ξ with η, then $V_{\varphi,\omega}$ enjoys the exactly same property as $U_{\varphi,\omega}$ and $V_{\varphi,\omega}$ has the same range as $U_{\varphi,\omega}$. Now $V_{\varphi,\omega}^*U_{\varphi,\omega}$ leaves $\mathfrak{P}_\omega$ invariant. Hence, we have $V_{\varphi,\omega}^*U_{\varphi,\omega} = 1$ by Lemma 1.5. Thus $\xi = \eta$. Q.E.D.

This result allows us to identify $\bigl\{\mathfrak{H}_\omega, \xi_\omega\bigr\}$ with $\bigl\{[\mathcal{M}\xi] \cap [\mathcal{M}'\xi], \xi\bigr\}$. Hence we shall view ξ_ω as a vector in $\mathfrak{P}_\varphi$ for any $\omega \in \mathcal{M}_*^+$, so that $\mathfrak{P}_\omega = \mathfrak{P}_\xi$.

Lemma 1.11. *If $\xi \in \mathfrak{P}_\varphi$, then*

$$\mathfrak{P}_\xi = \Bigl\{\bigl(\mathbf{R}_+\xi - \mathfrak{P}_\varphi\bigr) \cap \mathfrak{P}_\varphi\Bigr\}^-.$$

PROOF: Putting $e = s(\omega_\xi)$, we have

$$\mathfrak{P}_\xi = \left\{ \Delta_\xi^{\frac{1}{4}} x\xi : x \in \mathcal{M}_e^+ \right\}^- .$$

Now, from the identity:

$$\|x\|\xi = \Delta_\xi^{\frac{1}{4}} x\xi + \Delta_\xi^{\frac{1}{4}} \big(\|x\| - x\big)\xi, \quad x \in \mathcal{M}_e^+,$$

it follows that $\Delta_\xi^{1/4} x\xi \in \big(\mathbf{R}_+\xi - \mathfrak{P}_\varphi\big) \cap \mathfrak{P}_\varphi$. Hence we get

$$\mathfrak{P}_\xi \subset \left\{ \big(\mathbf{R}_+\xi - \mathfrak{P}_\varphi\big) \cap \mathfrak{P}_\varphi \right\}^- .$$

To prove the converse inclusion, suppose that $\xi, \eta \in \mathfrak{P}_\varphi$ and $\zeta = \xi - \eta \in \mathfrak{P}_\varphi$. We choose a semi-finite normal weight ψ' with $s(\psi') = 1 - s(\omega_\xi)$ and put $\psi = \omega_\xi + \psi'$. Transferring all the structure in $\mathfrak{H}_\psi$ into $\mathfrak{H}$ by $U_{\varphi,\psi}$, we can replace φ by ψ, so that we may assume that ω and φ commute, which means that Δ leaves ξ invariant. For each $r > 0$, we consider ξ_r, η_r and ζ_r given by (10). But $\xi = \xi_r$ since $\xi = \Delta^{\mathrm{i}t}\xi$, so that

$$\xi = \eta_r + \zeta_r, \quad \eta_r, \zeta_r \in \mathfrak{P}_\varphi, \quad r > 0.$$

Now, we have

$$\xi = \Delta^{-\frac{1}{4}}\eta_r + \Delta^{-\frac{1}{4}}\zeta_r \in \mathfrak{P}_\varphi^\sharp; \quad \Delta^{-\frac{1}{4}}\eta_r \in \mathfrak{P}_\varphi^\sharp, \quad \Delta^{-\frac{1}{4}}\zeta_r \in \mathfrak{P}_\varphi^\sharp.$$

Since ξ is left and right bounded, $\Delta^{1/4}\eta_r$ and $\Delta^{-1/4}\zeta_r$ are both left bounded, and satisfy the inequality:

$$\pi_\ell\big(\Delta^{-\frac{1}{4}}\eta_r\big) \le \pi_\ell(\xi) \le s(\omega_\xi), \qquad \pi_\ell\big(\Delta^{-\frac{1}{4}}\zeta_r\big) \le \pi_\ell(\xi).$$

Setting $e = s(\omega_\xi)$, we have $e\Delta^{-1/4}\eta_r = \Delta^{-1/4}\eta_r$ and $e\Delta^{-1/4}\zeta_r = \zeta_r$. Since e and Δ commute, $\eta_r = e\eta_r$ and $\zeta_r = e\zeta_r$. As $r \to \infty$, we get $\eta_r \to \eta$ and $\zeta_r \to \zeta$, so that $\eta = e\eta = JeJ\eta$ and $\zeta = e\zeta = JeJ\zeta$. Thus, η and ζ both belong to $\mathfrak{P}_\xi$. Therefore, we get $\big(\mathbf{R}_+\xi - \mathfrak{P}_\varphi\big) \cap \mathfrak{P}_\varphi \subset \mathfrak{P}_\xi$. Q.E.D.

Lemma 1.12. *For each pair $\xi_1, \xi_2 \in \mathfrak{P}_\varphi$, the following three conditions are equivalent:*

(i) $\xi_1 \perp \xi_2$;

(ii) $\mathfrak{P}_{\xi_1} \perp \mathfrak{P}_{\xi_2}$;

(iii) $s(\omega_{\xi_1}) \perp s(\omega_{\xi_2})$.

PROOF: The implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are obvious. In general $[\mathfrak{P}_\xi] = [\mathcal{M}\xi] \cap [\mathcal{M}'\xi]$ for $\xi \in \mathfrak{P}_\varphi$, so that (ii) $\Longrightarrow$ (iii) follows.

(i) $\Longrightarrow$ (ii): By the previous lemma, it is enough to show that

$$\big(\mathbf{R}_+\xi_1 - \mathfrak{P}_\varphi\big) \cap \mathfrak{P}_\varphi \perp \big(\mathbf{R}_+\xi_2 - \mathfrak{P}_\varphi\big) \cap \mathfrak{P}_\varphi.$$

If $\xi_1, \eta_2 \in \mathfrak{P}_\varphi$ and $\lambda\xi_1 - \eta_1 \in \mathfrak{P}_\varphi$, $\lambda > 0$, and $\mu\xi_2 - \eta_2 \in \mathfrak{P}_\varphi$, $\mu > 0$, then we have

$$0 \le (\eta_1 \mid \eta_2) \le \lambda\,(\xi_1 \mid \eta_2) \le \lambda\mu\,(\xi_1 \mid \xi_2) = 0,$$

so that $\eta_1 \perp \eta_2$. Q.E.D.

End of Proof of Theorem 1.2: We have already proved almost all of Theorem 1.2 except inequality (9). The first half of the inequality follows from the general fact:

$$\omega_\xi - \omega_\eta = \frac{1}{2}\bigl(\omega_{\xi+\eta,\,\xi-\eta} + \omega_{\xi-\eta,\,\xi+\eta}\bigr).$$

For a given pair $\xi, \eta \in \mathfrak{P}_\varphi$, choose orthogonal $\zeta_+, \zeta_- \in \mathfrak{P}_\varphi$ such that $\xi - \eta = \zeta_+ - \zeta_-$ by Lemma 1.7. By Lemma 1.12, $e = s(\omega_{\zeta_+}) \perp f = s(\omega_{\zeta_-})$. Putting $a = e - f$, we have

$$\begin{aligned}
\bigl\|\omega_\zeta - \omega_\eta\bigr\| &\ge \bigl|\omega_\xi(a) - \omega_\eta(a)\bigr| \qquad (\text{since } \|a\| = 1)\\
&= \frac{1}{2}\Bigl|\bigl(a\,(\xi+\eta) \bigm| \xi-\eta\bigr) + \bigl(a\,(\xi-\eta) \bigm| \xi+\eta\bigr)\Bigr|\\
&= \frac{1}{2}\Bigl|\bigl(a\,(\xi+\eta) \bigm| \zeta_+-\zeta_-\bigr) + \bigl(a\,(\zeta_+-\zeta_-) \bigm| \xi+\eta\bigr)\Bigr|\\
&= \frac{1}{2}\Bigl|\bigl(\xi+\eta \bigm| \zeta_++\zeta_-\bigr) + \bigl(\zeta_++\zeta_- \bigm| \xi+\eta\bigr)\Bigr| \quad {}^{2}\\
&= \bigl(\xi+\eta \bigm| \zeta_++\zeta_-\bigr)\\
&\ge \bigl(\xi \mid \zeta_+\bigr) - \bigl(\xi \mid \zeta_-\bigr) - \bigl(\xi \mid \zeta_-\bigr) - \bigl(\eta \mid \zeta_+\bigr) + \bigl(\eta \mid \zeta_-\bigr) = \|\xi-\eta\|^2.
\end{aligned}$$

This completes the proof for the first part of inequality (9). The second part of (9) is a general fact. Q.E.D.

Based on the theorem just proven, we make the following definition:

Definition 1.13. Given a von Neumann algebra $\{\mathcal{M}, \mathfrak{H}\}$, a quadruple $\{\mathcal{M}, \mathfrak{H}, J, \mathfrak{P}\}$ of a unitary involution[3] J, called *modular conjugation*, and a self-dual cone $\mathfrak{P}$ in $\mathfrak{H}$ is said to be a *standard form* of $\mathcal{M}$ if the following requirements are satisfied:

(i) $J\mathcal{M}J = \mathcal{M}'$;

(ii) $JaJ = a^*$, $a \in \mathcal{C}_\mathcal{M}$ $(= \mathcal{M} \cap \mathcal{M}')$;

(iii) $J\xi = \xi$, $\xi \in \mathfrak{P}$;

(iv) $aJaJ\mathfrak{P} \subset \mathfrak{P}$, $a \in \mathcal{M}$.

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2 $a(\zeta_+ - \zeta_-) = \zeta_+ + \zeta_-$.

3 Anti linear isometry J with $J^2 = 1$.

Theorem 1.14. *Suppose that* $\{\mathcal{M}_1, \mathfrak{H}_1, J_1, \mathfrak{P}_1\}$ *and* $\{\mathcal{M}_2, \mathfrak{H}_2, J_2, \mathfrak{P}_2\}$ *are standard forms. If* π *is an isomorphism of* $\mathcal{M}_1$ *onto* $\mathcal{M}_2$, *then there exists uniquely a unitary* u *from* $\mathfrak{H}_1$ *onto* $\mathfrak{H}_2$ *such that*

(i) $\pi(x) = uxu^*$, $x \in \mathcal{M}_1$;

(ii) $J_2 = uJ_1u^*$;

(iii) $\mathfrak{P}_2 = u\mathfrak{P}_1$.

PROOF: Let $\{\mathcal{M}, \mathfrak{H}, J, \mathfrak{P}\}$ be a standard form. We shall prove that

$$\{\mathcal{M}, \mathfrak{H}, J, \mathfrak{P}\} \cong \{\pi_\varphi(\mathcal{M}), \mathfrak{H}_\varphi, J_\varphi, \mathfrak{P}_\varphi\}$$

with φ a faithful weight on $\mathcal{M}$, which in turn shows the existence of u. The uniqueness of u then follows from that of $U_{\varphi,\psi}$ in Lemma 1.5.

For each $\xi \in \mathfrak{P}$, set $\mathfrak{H}(\xi) = [\mathcal{M}\xi] \cap [\mathcal{M}'\xi]$ and $e = s(\omega_\xi)$. We define S_ξ naturally and set $H = JS_\xi$. The arguments that follow (14) show $\Delta_\xi^{1/2} = H$ and $J_\xi = J$. Since $\mathfrak{P}_\xi = \{xJx\xi : x \in \mathcal{M}_e\}^-$, we have $\mathfrak{P}_\xi \subset \mathfrak{P}$. The self-duality of $\mathfrak{P}_\xi$ in $\mathfrak{H}(\xi)$ then implies $\mathfrak{P}_\xi = \mathfrak{P} \cap \mathfrak{H}(\xi)$.

The σ-finite case: Assume the σ-finiteness for $\mathcal{M}$. Take a maximal orthogonal family $\{\xi_i : i \in I\}$ in $\mathfrak{P}$. Since $s(\omega_{\xi_i}) \perp s(\omega_{\xi_j})$, $i \neq j$, I is countable. Adjusting norms, we may assume that $\sum\|\xi_i\|^2 < +\infty$. Put $\xi = \sum_{i\in I}\xi_i$. The maximality of $\{\xi_i\}$ implies, by Lemma 1.12, that ω_ξ is faithful. Hence we have $\mathfrak{H} = \mathfrak{H}(\xi)$, so that $\mathfrak{P} = \mathfrak{P}_\xi$.

The general case: For each $e \in \mathrm{Proj}(\mathcal{M})$, we set $\mathfrak{H}(e) = e\mathfrak{H} \cap Je\mathfrak{H}$. If e is σ-finite, then $\mathfrak{P} \cap \mathfrak{H}(e) = \mathfrak{P}_\xi$ for some $\xi \in \mathfrak{P}$ and there exists uniquely a unitary U_e from $\mathfrak{H}(e)$ onto $\pi_\varphi(e)J_\varphi\pi_\varphi(e)\mathfrak{H}_\varphi$ such that $U_e x = \pi_\varphi(x)U_e$, $x \in \mathcal{M}_e$, and $U_e\mathfrak{P}_\xi = \mathfrak{P}_\varphi \cap U_e\mathfrak{H}(e)$. The uniqueness of U_e means that U_e extends U_f if $f \leq e$. Since the family of σ-finite projections is upward directed and has supremum 1, there exists a common extension U of all possible U_e's. Since $\mathfrak{P} = \bigcup \mathfrak{P}_\xi$, this U enjoys the required properties. Q.E.D.

We shall close this section with an application of the above theorem to the automorphism group. Given a von Neumann algebra $\mathcal{M}$, we denote by $\mathrm{Aut}(\mathcal{M})$ the group of all automorphisms of $\mathcal{M}$. The group of all inner automorphism groups will be denoted by $\mathrm{Int}(\mathcal{M})$. It is easy to see that $\mathrm{Int}(\mathcal{M})$ is a normal subgroup, so we can form the quotient group $\mathrm{Out}(\mathcal{M})$ of $\mathrm{Aut}(\mathcal{M})$ by $\mathrm{Int}(\mathcal{M})$. To each finite subset $\{\omega_1, \omega_2, \dots, \omega_n\}$ of $\mathcal{M}_*$, and each $\alpha \in \mathrm{Aut}(\mathcal{M})$, we set

$$U(\alpha; \omega_1, \dots, \omega_n) = \left\{\beta \in \mathrm{Aut}(\mathcal{M}) : \begin{matrix} \|\omega_i \circ \alpha - \omega_i \circ \beta\| < 1; \\ \|\omega_i \circ \alpha^{-1} - \omega_i \circ \beta^{-1}\| < 1, \end{matrix} \quad i = 1, \dots, n\right\}. \tag{16}$$

The family $\{U(\alpha; \omega_1, \dots, \omega_n) : \omega_1, \dots, \omega_n \in \mathcal{M}_*\}$ gives rise to a topology in $\mathrm{Aut}(\mathcal{M})$ which makes $\mathrm{Aut}(\mathcal{M})$ a topological group. It is not difficult to show that $\mathrm{Aut}(\mathcal{M})$ is complete with respect to this uniform structure.

Theorem 1.15. *If* $\{\mathcal{M}, \mathfrak{H}, J, \mathfrak{P}\}$ *is a standard form, then the group* $\mathcal{U}$ *of all unitaries* u *satisfying*

$$u\mathcal{M}u^* = \mathcal{M}, \qquad uJu^* = J, \qquad u\mathfrak{P} = \mathfrak{P}$$

is isomorphic to $\mathrm{Aut}(\mathcal{M})$ *under the map:* $u \in \mathcal{U} \mapsto \alpha_u \in \mathrm{Aut}(\mathcal{M})$*, where*

$$\alpha_u(x) = uxu^*, \quad x \in \mathcal{M}.$$

Furthermore, this map is a homeomorphism of $\mathcal{U}$ *equipped with the strong operator topology onto* $\mathrm{Aut}(\mathcal{M})$*.*

PROOF: The map: $u \in \mathcal{U} \mapsto \alpha_u \in \mathrm{Aut}(\mathcal{M})$ is clearly a homomorphism. By Theorem 1.14, it is surjective. The uniqueness in the theorem yields the injectivity. By inequality (9), the map: $\xi \in \mathfrak{P} \mapsto \omega_\xi \in \mathcal{M}_*^+$ is a homeomorphism, which in turn means that the map: $u \mapsto \alpha_u$ is a homeomorphism. Q.E.D.

Definition 1.16. The inverse map of the above map: $u \mapsto \alpha_u \in \mathrm{Aut}(\mathcal{M})$ is called the *standard implementation* and denoted by $U(\theta)$ for each $\theta \in \mathrm{Aut}(\mathcal{M})$.

The standard implementation $U(\theta)$ of $\theta \in \mathrm{Aut}(\mathcal{M})$ is characterized by the following:

$$\begin{aligned} \theta(x) &= U(\theta)xU(\theta)^*, \quad x \in \mathcal{M}, \\ U(\theta)\mathfrak{P} &= \mathfrak{P}. \end{aligned} \tag{17}$$

If we fix a standard form $\{\mathcal{M}, \mathfrak{H}, J, \mathfrak{P}\}$, then for any faithful weight φ, there exists uniquely a unitary U_φ of $\mathfrak{H}_\varphi$ onto $\mathfrak{H}$ such that

$$U_\varphi \pi_\varphi(x) U_\varphi^* = x, \quad x \in \mathcal{M}, \qquad U_\varphi \mathfrak{P} = \mathfrak{P}.$$

With this U_φ, we identify $\mathfrak{H}_\varphi$ and $\mathfrak{H}$. We then realize $\{\mathfrak{H}_\varphi, \eta_\varphi, J_\varphi, \mathfrak{P}_\varphi\}$ in $\mathfrak{H}$ so that

$$\mathfrak{H}_\varphi = \mathfrak{H}, \qquad J_\varphi = J, \qquad \mathfrak{P}_\varphi = \mathfrak{P}$$

and η_φ is a map from $\mathfrak{n}_\varphi$ into $\mathfrak{H}$ such that

$$\Delta_\varphi^{\frac{1}{4}} \eta_\varphi(x^*x) \in \mathfrak{P}, \quad x \in \mathfrak{n}_\varphi.$$

Proposition 1.17. *In the above setting, the standard implementation* $U(\theta)$ *of* $\theta \in \mathrm{Aut}(\mathcal{M})$ *has the property:*

$$U(\theta)\eta_\varphi(x) = \eta_{\varphi\circ\theta^{-1}}\big(\theta(x)\big), \quad x \in \mathfrak{n}_\varphi. \tag{18}$$

In particular, we have

$$U(\theta)\xi_\omega = \xi_{\omega\circ\theta^{-1}}, \quad \omega \in \mathcal{M}_*^+, \tag{19}$$

where ξ_ω *is the vector in* $\mathfrak{P}$ *such that* $\omega = \omega_{\xi_\omega}$*.*

PROOF: Since $\mathfrak{n}_{\varphi\circ\theta^{-1}} = \theta(\mathfrak{n}_\varphi)$, the right hand side of (18) makes sense. We define V to be the operator given by the right hand side of (18). It follows that V can be extended to a unitary on $\mathfrak{H}$ denoted by V again, which implements θ on $\mathcal{M}$ because for each $a \in \mathcal{M}$ and $x \in \mathfrak{n}_\varphi$

$$\begin{aligned} VaV^*\eta_\varphi(x) &= Va\eta_{\varphi\circ\theta}\big(\theta^{-1}(x)\big) = V\eta_{\varphi\circ\theta}\big(a\theta^{-1}(x)\big) \\ &= \eta_\varphi\big(\theta(a)x\big) = \theta(a)\eta_\varphi(x). \end{aligned}$$

To show $V = U(\theta)$, we have only to prove $V\mathfrak{P} = \mathfrak{P}$. For each $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$, we have

$$\begin{aligned} VS_\varphi\eta_\varphi(x) &= V\eta_\varphi(x^*) = \eta_{\varphi\circ\theta^{-1}}\big(\theta(x^*)\big) \\ &= S_{\varphi\circ\theta^{-1}}\eta_{\varphi\circ\theta^{-1}}\big(\theta(x)\big) = S_{\varphi\circ\theta^{-1}}V\eta_\varphi(x). \end{aligned}$$

Since $\eta_\varphi(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ is a core for S_φ and $\eta_{\varphi\circ\theta^{-1}}\big(\theta(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)\big)$ is also a core for $S_{\varphi\circ\theta^{-1}}$, we have $VS_\varphi = S_{\varphi\circ\theta^{-1}}V$. Hence the uniqueness of the polar decomposition implies that

$$VJ = JV \qquad \text{and} \qquad V\Delta_\varphi V^* = \Delta_{\varphi\circ\theta^{-1}}.$$

Recalling that $\mathfrak{P} = \big\{ xJ\eta_\varphi(x) : x \in \mathfrak{n}_\varphi \big\}^-$, we compute:

$$\begin{aligned} V\mathfrak{P} &= \big\{ VxJ\eta_\varphi(x) : x \in \mathfrak{n}_\varphi \big\}^- = \big\{ \theta(x)JV\eta_\varphi(x) : x \in \mathfrak{n}_\varphi \big\}^- \\ &= \Big\{ \theta(x)J\eta_{\varphi\circ\theta^{-1}}\big(\theta(x)\big) : x \in \mathfrak{n}_\varphi \Big\}^- \\ &= \big\{ yJ\eta_{\varphi\circ\theta^{-1}}(y) : y \in \mathfrak{n}_{\varphi\circ\theta^{-1}} \big\}^- = \mathfrak{P}_{\varphi\circ\theta^{-1}} = \mathfrak{P}. \end{aligned}$$

Q.E.D.

Let $\mathfrak{W}(\mathcal{M})$ be the set of semi-finite normal weights on $\mathcal{M}$ and $\mathfrak{W}_0(\mathcal{M})$ be the set of faithful semi-finite normal weights on $\mathcal{M}$. To each pair $(\psi, \varphi) \in \mathfrak{W}_0(\mathcal{M}) \times \mathfrak{W}_0(\mathcal{M})$, there corresponds canonically a unitary $U_{\psi,\varphi}\colon \mathfrak{H}_\varphi \mapsto \mathfrak{H}_\psi$ which carries the natural cone $\mathfrak{P}_\varphi$ of $\mathfrak{H}_\varphi$ onto $\mathfrak{P}_\psi$ of $\mathfrak{H}_\psi$ and J_φ onto J_ψ, and intertwines π_φ and π_ψ. We now define the canonical Hilbert space $L^2(\mathcal{M})$ associated with a given von Neumann algebra $\mathcal{M}$ as the Hilbert space:

$$\Big\{ \zeta = \{\zeta_\varphi\} \in \prod_{\varphi \in \mathfrak{W}_0(\mathcal{M})} \mathfrak{H}_\varphi \ :\ \zeta_\psi = U_{\psi,\varphi}\zeta_\varphi \ \text{ for every } (\psi, \varphi) \in \mathfrak{W}_0(\mathcal{M}) \times \mathfrak{W}_0(\mathcal{M}) \Big\} \tag{20}$$

equipped with the inner product defined by:

$$\big(\{\zeta_{1,\varphi}\} \,\big|\, \{\zeta_{2,\varphi}\}\big) = \big(\zeta_{1,\varphi_0} \,\big|\, \zeta_{2,\varphi_0}\big)$$

with any fixed $\varphi_0 \in \mathfrak{W}_0(\mathcal{M})$. The positive cone $L^2(\mathcal{M})_+$ of $L^2(\mathcal{M})$ is then defined as the set of those $\zeta = \{\zeta_\varphi\} \in L^2(\mathcal{M})$ such that $\zeta_\varphi \in \mathfrak{P}_\varphi,\ \varphi \in \mathfrak{W}_0(\mathcal{M})$. The action of $\mathcal{M}$ on $L^2(\mathcal{M})$ is then given by

$$x\{\zeta_\varphi\} = \big\{\pi_\varphi(x)\zeta_\varphi\big\}, \quad x \in \mathcal{M}.$$

By virtue of Theorem 1.14, for any fixed $\varphi_0 \in \mathfrak{W}_0(\mathcal{M})$ the map $U\colon \xi \in \mathfrak{H}_{\varphi_0} \mapsto \big\{U_{\psi,\varphi_0}\xi\big\} \in L^2(\mathcal{M})$ gives rise to the unitary equivalence of

$$\big\{\pi_{\varphi_0}(\mathcal{M}),\ \mathfrak{H}_{\varphi_0},\ \mathfrak{P}_{\varphi_0},\ J_{\varphi_0}\big\}$$

and the canonical $\big\{\mathcal{M},\ L^2(\mathcal{M}),\ L^2(\mathcal{M})_+,\ J\big\}$.

Definition 1.18. We call $\{\mathcal{M},\ L^2(\mathcal{M}),\ L^2(\mathcal{M})_+,\ J\}$ *the* standard form of $\mathcal{M}$ or the *canonical* standard form.

For any $\varphi \in \mathfrak{W}_0(\mathcal{M})$, the projection $U_\varphi\colon \{\xi_\psi\} \mapsto \xi_\varphi \in \mathfrak{H}_\varphi$ gives unitary equivalence between the canonical standard form $\{\mathcal{M},\ L^2(\mathcal{M}),\ L^2(\mathcal{M})_+,\ J\}$ and the one $\{\pi_{\varphi_0}(\mathcal{M}),\ \mathfrak{H}_{\varphi_0},\ \mathfrak{P}_{\varphi_0},\ J_{\varphi_0}\}$ associated with φ_0. The map: $x \in \mathfrak{n}_\varphi \mapsto U_\varphi^*\eta_\varphi(x) \in L^2(\mathcal{M})$ gives the realization of the semi-cyclic representation π_φ on $L^2(\mathcal{M})$. We will identify this map with η_φ and therefore the semi-cyclic representation π_φ with $\{L^2(\mathcal{M}), \eta_\varphi\}$ equipped with the canonical action of $\mathcal{M}$. In the case that we want to emphasize the dependence on φ, we will write $L^2(\mathcal{M}, \varphi)$. To cover a non-faithful weight, we use the following trick. Let φ be a faithful semi-finite normal weight on $\mathcal{M}$ and $e \in \mathrm{Proj}(\mathcal{M}_\varphi)$. For the semi-finite normal weight ψ defined by $\psi(x) = \varphi(exe),\ x \in \mathcal{M}_+$, we set:

$$\eta_\psi(x) = \eta_\varphi(xe), \quad x \in \mathfrak{n}_\psi. \tag{21}$$

It follows immediately that the action of $\mathcal{M}$ on $JeJL^2(\mathcal{M})$ together with the map η_ψ is the realization of the semi-cyclic representation $\{\pi_\psi,\ \mathfrak{H}_\psi,\ \eta_\psi\}$ inside the standard form.

Theorem 1.19. *Let $\mathcal{M}$ be a σ-finite von Neumann algebra in the standard form $\{\mathcal{M},\ \mathfrak{H},\ \mathfrak{H}_+,\ J\}$. Suppose that a sequence $\{\varphi_n\}$ of faithful normal positive functionals on $\mathcal{M}$ converges to a faithful φ in norm. Then for any fixed faithful semi-finite normal weight ψ we have the following convergence:*

(i) *the sequence $\{\Delta_{\varphi_n,\psi}\}$ of relative modular operators converges to $\Delta_{\varphi,\psi}$ in the strong resolvent sense.*

(ii) *The sequence $\{(\mathrm{D}\varphi_n:\mathrm{D}\psi)_t\}$ of the cocycle derivatives converges to $(\mathrm{D}\varphi:\mathrm{D}\psi)_t$ σ-strongly and uniformly in t on any bounded interval.*

(iii) *The sequence $\{\sigma^{\varphi_n}\}$ of modular automorphism groups converges to the modular automorphism group σ^φ of φ in the following sense:*

$$\lim_{n\to\infty} \left\|\sigma_t^{\varphi_n}(x)\eta - \sigma_t^{\varphi}(x)\eta\right\| = 0, \quad \eta \in \mathfrak{H}, \quad x \in \mathcal{M},$$

and the convergence is uniform on any bounded interval of t.

PROOF:

(i) Let $\mathfrak{a}_\psi = \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*$ and $\mathfrak{A}_\psi = \eta_\psi(\mathfrak{a}_\psi) \subset \mathfrak{H}$, a full left Hilbert algebra. Then with ξ_φ the representing vector of $\varphi \in \mathcal{M}_*$ in the cone $\mathfrak{H}_+$, we have $J\Delta_{\varphi,\psi}^{1/2}\eta_\psi(x) = x^*\xi_\varphi$. By inequality (9), we have

$$\lim_{n\to\infty}\left\|\xi_{\varphi_n} - \xi_\varphi\right\| = 0.$$

Hence we get, for any $x \in \mathfrak{a}_\psi$,

$$\begin{aligned}\lim_{n\to\infty}\left\|\left(\Delta_{\varphi_n,\psi} - \Delta_{\varphi,\psi}\right)\eta_\psi(x)\right\| &= \lim_{n\to\infty}\left\|\left(J\Delta_{\varphi_n,\psi} - J\Delta_{\varphi,\psi}\right)\eta_\psi(x)\right\| \\ &= \lim_{n\to\infty}\left\|x^*\xi_{\varphi_n} - x^*\xi_\varphi\right\| = 0.\end{aligned}$$

Since $\mathfrak{A}_\psi = \eta_\psi(\mathfrak{a}_\psi)$ is a core for $\Delta_{\varphi,\psi}^{1/2}$ and contained in $\bigcap_{n=1}^\infty \mathfrak{D}\big(\Delta_{\varphi_n,\psi}^{1/2}\big)$, the sequence $\{\Delta_{\varphi_n,\psi}\}$ of relative modular operators converges to $\Delta_{\varphi,\psi}$ in the strong resolvent sense by Theorem A.6.(iii).

(ii) and (iii): We have $(\mathrm{D}\varphi_n : \mathrm{D}\psi)_t = \Delta_{\varphi_n,\psi}^{\mathrm{i}t}\Delta_\psi^{-\mathrm{i}t}$, $t \in \mathbf{R}$. The strong resolvent convergence of $\{\Delta_{\varphi_n,\psi}^{\mathrm{i}t}\}$ to $\Delta_{\varphi,\psi}$ implies the strong convergence of the sequence $\{\Delta_{\varphi_n,\psi}^{\mathrm{i}t}\}$ of one parameter unitary groups to the one parameter unitary group $\{\Delta_{\varphi,\psi}^{\mathrm{i}t}\}$ uniformly on any bounded interval of t by Theorem A.6. Thus our assertion follows.

Q.E.D.

Given a standard form $\{\mathcal{M}, \mathfrak{H}, J, \mathfrak{P}\}$ of a von Neumann algebra $\mathcal{M}$, we define an action of $\mathcal{M}$ on $\mathfrak{H}$ from the right as follows:

$$\xi x = Jx^*J\xi, \quad x \in \mathcal{M}, \quad \xi \in \mathfrak{H}. \tag{22}$$

Since $J\mathcal{M}J = \mathcal{M}'$, $\mathfrak{H}$ becomes a two sided $\mathcal{M}$-module and the commutant of $\mathcal{M}$ is precisely given by the right action of $\mathcal{M}$. Writing $J\xi = \xi^*$ as before, we have

$$(x\xi)^* = \xi x^*, \quad x \in \mathcal{M}, \quad \xi \in \mathfrak{H}. \tag{23}$$

We must note however that when $\{\mathcal{M}, \mathfrak{H}, J, \mathfrak{P}\}$ is given by a Tomita algebra $\mathfrak{A}$, for $\xi, \eta \in \mathfrak{A}$, we have

$$\xi\pi_\ell(\eta) = J\pi_\ell(\eta)^*J\xi = J(\eta^\sharp J\xi) = \xi\Delta^{\frac{1}{2}}\eta = \pi_r\big(\Delta^{\frac{1}{2}}\eta\big)\xi.$$

Thus, the right action of $x = \pi_\ell(\eta)$ is not $\pi_r(\eta)$ but $\pi_r\big(\Delta^{1/2}\eta\big)$.

For each $\varphi \in \mathcal{M}_*^+$, we denote the representing vector of φ in $\mathfrak{P}$ by ξ_φ, i.e. $\varphi(x) = \big(x\xi_\varphi \mid \xi_\varphi\big)$, $x \in \mathcal{M}$. It is easy to see that $x\xi_\varphi = \xi_\varphi x \iff x\varphi = \varphi x$. For each pair $x \in \mathcal{M}$ and $\xi \in \mathfrak{H}$ we write

$$[x, \xi] = x\xi - \xi x.$$

Proposition 1.20. *If h and k are self-adjoint elements of $\mathcal{M}$, then for each $\varphi \in \mathcal{M}_*^+$, there exists a Radon measure μ on $\mathbf{R}\times\mathbf{R}$ such that*

$$\big\| f(h)\xi_\varphi - \xi_\varphi g(k)\big\|^2 = \iint_{\mathbf{R}\times\mathbf{R}} \big| f(x) - g(y)\big|\, \mathrm{d}\mu(x, y) \tag{24}$$

for every bounded Borel functions f and g on $\mathbf{R}$.

PROOF: Let π (resp. π') be the left (resp. right) representation of $\mathcal{M}$ on $\mathfrak{H}$, i.e. $\pi(x)\xi = x\xi$ and $\pi'(x)\xi = \xi x$. Let A be the C^*-algebra generated by $\pi(h)$ and $\pi'(k)$ and 1. Since $\pi(h)$ and $\pi'(k)$ commute, A is abelian. Hence $A \cong C\big(\mathrm{Sp}(A)\big)$. For each $\omega \in \mathrm{Sp}(A)$ put

$$x(\omega) = \omega\big(\pi(h)\big), \qquad y(\omega) = \omega\big(\pi'(k)\big).$$

Then we have $\big(x(\omega), y(\omega)\big) \in \mathbf{R}^2$. Since A is generated by $\pi(h)$ and $\pi'(k)$ and 1, the map: $\omega \in \mathrm{Sp}(A) \mapsto \big(x(\omega), y(\omega)\big) \in \mathbf{R}^2$ is injective and continuous. Hence under this map, $\mathrm{Sp}(A)$ is identified with a compact subset of $\mathbf{R}^2$, called the joint spectrum of $\pi(h)$ and $\pi'(k)$. The operators $\pi(h)$ and $\pi'(k)$ are respectively identified with the projection function from $\mathrm{Sp}(A)$ to the first (resp. second) coordinate of $\omega \in \mathrm{Sp}(A)$. Let μ be the spectral measure corresponding to the vector ξ_φ, i.e.

$$\|a\xi_\varphi\|^2 = \int_{\mathrm{Sp}(A)} |a(\omega)| \,\mathrm{d}\mu(\omega), \quad a \in A.$$

Viewing the measure μ as a measure on $\mathbf{R}^2$ supported by $\mathrm{Sp}(A)$, we have

$$\left\| f\big(\pi(h), \pi'(k)\big)\xi_\varphi \right\|^2 = \iint_{\mathbf{R}^2} |f(x, y)|^2 \,\mathrm{d}\mu(x, y)$$

for every bounded Borel function f on $\mathbf{R}^2$. In particular, we have (24). Q.E.D.

Proposition 1.21. *Writing*

$$[x, \varphi] = x\varphi - \varphi x, \quad x \in \mathcal{M}, \quad \varphi \in \mathcal{M}_*,$$

we have

$$\big\|[x, \varphi]\big\| \leq 2\|\varphi\|^{\frac{1}{2}} \big\|[x, \xi_\varphi]\big\|, \quad \varphi \in \mathcal{M}_*^+, \quad x \in \mathcal{M}. \tag{25}$$

PROOF: For each $y \in \mathcal{M}$, we have

$$\begin{aligned}
\big|\langle y, [x, \varphi]\rangle\big| &= |\langle yx - xy, \varphi\rangle| = \big|(yx\xi_\varphi - xy\xi_\varphi \mid \xi_\varphi)\big| \\
&\leq \big|(y(x\xi_\varphi - \xi_\varphi x) \mid \xi_\varphi)\big| + \big|(y\xi_\varphi x - xy\xi_\varphi \mid \xi_\varphi)\big| \\
&\leq \|y\| \big\|[x, \xi_\varphi]\big\| \|\xi_\varphi\| + \big|(y\xi_\varphi \mid \xi_\varphi x^* - x^*\xi_\varphi)\big| \\
&\leq \|y\| \|\xi_\varphi\| \big\|[x, \xi_\varphi]\big\| + \|y\| \|\xi_\varphi\| \big\|[x^*, \xi_\varphi]\big\|.
\end{aligned}$$

But we have

$$\big\|[x^*, \xi_\varphi]\big\| = \big\|J[x^*, \xi_\varphi]\big\| = \big\|[J\xi_\varphi, x]\big\| = \big\|[\xi_\varphi, x]\big\|. \qquad \text{Q.E.D.}$$

For each $a \in \mathbf{R}_+$ let E_a denote the characteristic function of the half line $[a, \infty[$. For each $x \in \mathcal{M}$ let $x = u(x)|x|$ be the polar decomposition, and define

$$u_a(x) = u(x)E_a\big(|x|\big), \quad x \in \mathcal{M}. \tag{26}$$

Proposition 1.22. *For any $x \in \mathcal{M}$ and $\varphi \in \mathcal{M}_*^+$ we have*

(i)
$$\int_0^\infty \varphi\Big(E_{\sqrt{a}}\big(|x|\big)\Big)\,\mathrm{d}a = \varphi(x^*x), \tag{27}$$

(ii)
$$\int_0^\infty \Big\|\big[u_{\sqrt{a}}(x), \xi_\varphi\big]\Big\|^2\,\mathrm{d}a \le 4\big\|[x, \xi_\varphi]\big\|\varphi(x^*x + xx^*)^{\frac{1}{2}}. \tag{28}$$

PROOF:

(i) We have $E_{\sqrt{a}}\big(|x|\big) = E_a(x^*x)$ and

$$x^*x = \int_0^\infty E_a(x^*x)\,\mathrm{d}a\,.$$

This implies (i).

(ii) First, we assume $x = x^*$. Put $F_a(t) = \operatorname{sign} t\,E_a\big(|t|\big)$ for $a \in \mathbf{R}_+$ and $t \in \mathbf{R}$. Let μ be the measure on $\mathbf{R}^2$ determined by $h = k = x$ in Proposition 1.20. Then we have

$$\begin{aligned}\int_0^\infty \Big\|\big[u_{\sqrt{a}}(x), \xi_\varphi\big]\Big\|^2\,\mathrm{d}a &= \int_0^\infty \iint_{\mathbf{R}^2} \big|F_{\sqrt{a}}(\alpha) - F_{\sqrt{a}}(\beta)\big|^2\,\mathrm{d}\mu(\alpha,\beta)\,\mathrm{d}a \\ &= \iint_{\mathbf{R}^2}\bigg(\int_0^\infty \big|F_{\sqrt{a}}(\alpha) - F_{\sqrt{a}}(\beta)\big|^2\,\mathrm{d}a\bigg)\,\mathrm{d}\mu(\alpha,\beta)\,.\end{aligned}$$

If $\operatorname{sign}(\alpha) = \operatorname{sign}(\beta)$, then

$$\begin{aligned}\int_0^\infty \big|F_{\sqrt{a}}(\alpha) - F_{\sqrt{a}}(\beta)\big|^2\,\mathrm{d}a &= \int_0^\infty \big|E_{\sqrt{a}}(\alpha) - E_{\sqrt{a}}(\beta)\big|^2\,\mathrm{d}a \\ &= \int_0^\infty \big|E_a(\alpha^2) - E_a(\beta^2)\big|^2\,\mathrm{d}a = \big|\alpha^2 - \beta^2\big| = |\alpha - \beta|\big(|\alpha| + |\beta|\big).\end{aligned}$$

If $\operatorname{sign}(\alpha) = -\operatorname{sign}(\beta)$, then we have

$$\big|F_{\sqrt{a}}(\alpha) - F_{\sqrt{a}}(\beta)\big|^2 \le 2\big(E_a(\alpha^2) + E_a(\beta^2)\big),$$

so that

$$\begin{aligned}\int_0^\infty \big|F_{\sqrt{a}}(\alpha) - F_{\sqrt{a}}(\beta)\big|^2\,\mathrm{d}a &\le 2\big(\alpha^2 + \beta^2\big) \\ &\le 4(\alpha - \beta)^2 = 4|\alpha - \beta|\big(|\alpha| + |\beta|\big).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\int_0^\infty \left\| \left[u_{\sqrt{a}}(x), \xi_\varphi\right] \right\|^2 \mathrm{d}a &\leq 4 \iint_{\mathbf{R}^2} |\alpha - \beta| \big(|\alpha| + |\beta|\big) \, \mathrm{d}\mu(\alpha, \beta) \\
&\leq 4 \Bigg(\iint_{\mathbf{R}^2} |\alpha - \beta|^2 \, \mathrm{d}\mu(\alpha, \beta) \Bigg)^{\frac{1}{2}} \\
&\qquad \times \Bigg(\Bigg(\iint_{\mathbf{R}^2} |\alpha|^2 \, \mathrm{d}\mu(\alpha, \beta) \Bigg)^{\frac{1}{2}} + \Bigg(\iint_{\mathbf{R}^2} |\beta|^2 \, \mathrm{d}\mu(\alpha, \beta) \Bigg)^{\frac{1}{2}} \Bigg)^{\frac{1}{2}} \\
&= 4 \big\| [x, \xi_\varphi] \big\| \big(\|x\xi_\varphi\| + \|\xi_\varphi x\| \big) = 4 \big\| [x, \xi_\varphi] \big\| \big(\|x\xi_\varphi\| + \|x\xi_\varphi\| \big)^{\frac{1}{2}} \\
&= 4\sqrt{2} \big\| [x, \xi_\varphi] \big\| \varphi(x^2)^{\frac{1}{2}}.
\end{aligned}$$

This proves (28) in this self-adjoint case.

For the case $x \neq x^*$, we consider the 2×2-matrix algebra $\mathcal{M}_2 = M_2(\mathbf{C}) \otimes \mathcal{M}$. The standard form for $\mathcal{M}_2$ is given by 2×2-matrices over $\mathfrak{H}$. So let $\widetilde{\mathfrak{H}} = M_2(\mathbf{C}) \otimes \mathfrak{H}$ be the vector space of all 2×2-matrices with entries from $\mathfrak{H}$. Thus each element $\xi \in \widetilde{\mathfrak{H}}$ is of the form:

$$\xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}, \quad \xi_{i,j} \in \mathfrak{H}, \quad i, j = 1, 2;$$

$$\xi^* = \widetilde{J}\xi = \begin{pmatrix} J\xi_{11} & J\xi_{21} \\ J\xi_{12} & J\xi_{22} \end{pmatrix}.$$

Each element of $\mathcal{M}_2$ acts on $\widetilde{\mathfrak{H}}$ from the left by the matrix multiplication. The commutant $\mathcal{M}_2'$ of $\mathcal{M}_2$ is given by the right multiplication of $\mathcal{M}_2$. The natural positive cone $\widetilde{\mathfrak{P}}$ is then the closure of the set of matrices:

$$\begin{pmatrix} x_{11}\xi x_{11}^* + x_{12}\eta x_{12}^* & x_{11}\xi x_{21}^* + x_{12}\eta x_{22}^* \\ x_{21}\xi x_{11}^* + x_{22}\eta x_{12}^* & x_{21}\xi x_{21}^* + x_{22}\eta x_{22}^* \end{pmatrix}, \quad \xi, \eta \in \mathfrak{P}, \quad x = [x_{i,j}] \in \mathcal{M}_2.$$

Now with this setting, we consider

$$\widetilde{x} = \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}, \qquad \xi_{\widetilde{\varphi}} = \begin{pmatrix} \xi_\varphi & 0 \\ 0 & \xi_\varphi \end{pmatrix}.$$

Then we get

$$\widetilde{x}^2 = \begin{pmatrix} x^*x & 0 \\ 0 & xx^* \end{pmatrix}, \qquad |\widetilde{x}| = \begin{pmatrix} |x| & 0 \\ 0 & |x^*| \end{pmatrix}, \qquad u(\widetilde{x}) = \begin{pmatrix} 0 & u(x)^* \\ u(x) & 0 \end{pmatrix},$$

$$E_a\big(|\widetilde{x}|\big) = \begin{pmatrix} E_a\big(|x|\big) & 0 \\ 0 & E_a\big(|x^*|\big) \end{pmatrix}, \qquad u_a(\widetilde{x}) = \begin{pmatrix} 0 & u_a(x)^* \\ u_a(x) & 0 \end{pmatrix}.$$

Furthermore, we have

$$u_a(\widetilde{x})\xi_{\widetilde{\varphi}} = \begin{pmatrix} 0 & u_a(x)^*\xi_\varphi \\ u_a(x)\xi_\varphi & 0 \end{pmatrix}, \qquad \xi_{\widetilde{\varphi}}u_a(\widetilde{x}) = \begin{pmatrix} 0 & \xi_\varphi u_a(x)^* \\ \xi_\varphi u_a(x) & 0 \end{pmatrix},$$

so that

$$\left\|\left[u_a(\widetilde{x}), \xi_{\widetilde{\varphi}}\right]\right\|^2 = \left\|\left[u_a(x), \xi_\varphi\right]\right\|^2 + \left\|\left[u_a(x)^*, \xi_\varphi\right]\right\|^2 = 2\left\|\left[u_a(x), \xi_\varphi\right]\right\|^2.$$

Hence we get

$$\begin{aligned} 2\int_0^\infty \left\|\left[u_{\sqrt{a}}(x), \xi_\varphi\right]\right\|^2 \mathrm{d}a &= \int_0^\infty \left\|\left[u_{\sqrt{a}}(\widetilde{x}), \xi_{\widetilde{\varphi}}\right]\right\|^2 \mathrm{d}a \\ &\le 4\sqrt{2}\left\|[\widetilde{x}, \xi_{\widetilde{\varphi}}]\right\|\widetilde{\varphi}(\widetilde{x}^2)^{\frac{1}{2}} = 8\left\|[x, \xi_\varphi]\right\|\varphi(x^*x + xx^*)^{\frac{1}{2}}. \end{aligned}$$

Q.E.D.

Corollary 1.23. *If $x \in \mathcal{M}$, $x \neq 0$, and $\varphi \in \mathcal{M}_*^+$ satisfy the inequality for $\varepsilon > 0$:*

$$\left\|[x, \xi_\varphi]\right\| \le \varepsilon\varphi(x^*x + xx^*)^{\frac{1}{2}},$$

then there exists $a > 0$ such that

$$\left\|\left[u_{\sqrt{a}}(x), \xi_\varphi\right]\right\| \le 2\varepsilon\varphi\Big(u_{\sqrt{a}}(x)^*u_{\sqrt{a}}(x) + u_{\sqrt{a}}(x)u_{\sqrt{a}}(x)^*\Big)^{\frac{1}{2}}, \quad u_a(x) \neq 0.$$

PROOF: By (27) and (28), we have

$$\begin{aligned} \int_0^\infty \left\|\left[u_{\sqrt{a}}(x), \xi_\varphi\right]\right\|^2 \mathrm{d}a &\le 4\left\|[x, \xi_\varphi]\right\|\varphi(x^*x + xx^*)^{\frac{1}{2}} \\ &= \left\|[x, \xi_\varphi]\right\|\left(\int_0^\infty \varphi\Big(E_{\sqrt{a}}\big(|x|\big) + E_{\sqrt{a}}\big(|x^*|\big)\Big)\,\mathrm{d}a\right)^{\frac{1}{2}} \\ &= 4\left\|[x, \xi_\varphi]\right\|\left(\int_0^\infty \varphi\Big(u_{\sqrt{a}}(x)^*u_{\sqrt{a}}(x) + u_{\sqrt{a}}(x)u_{\sqrt{a}}(x)^*\Big)\right)^{\frac{1}{2}} \\ &\le 4\varepsilon\int_0^\infty \varphi\Big(u_{\sqrt{a}}(x)^*u_{\sqrt{a}}(x) + u_{\sqrt{a}}(x)u_{\sqrt{a}}(x)^*\Big)\,\mathrm{d}a\,. \end{aligned}$$

Thus, we have for some $a > 0$

$$\left\|\left[u_{\sqrt{a}}(x), \xi_\varphi\right]\right\|^2 \leq 4\varepsilon\varphi\Big(u_{\sqrt{a}}(x)^* u_{\sqrt{a}}(x) + u_{\sqrt{a}}(x) u_{\sqrt{a}}(x)^*\Big) \neq 0.$$

Hence with this $a > 0$ we obtain:

$$\left\|\left[u_{\sqrt{a}}(x), \xi_\varphi\right]\right\| \leq 2\sqrt{\varepsilon}\varphi\Big(u_{\sqrt{a}}(x)^* u_{\sqrt{a}}(x) + u_{\sqrt{a}}(x) u_{\sqrt{a}}(x)^*\Big)^{\frac{1}{2}}. \qquad \text{Q.E.D.}$$

The following lemma will be used later, in Chapter XVIII.

Lemma 1.24. *There exists a universal constant $C > 0$ such that*

$$\left\|\left(\Delta_\varphi^{\mathrm{i}t} - 1\right)x\xi_\varphi\right\| \leq C\big(1 + |t|\big)\left\|[x, \xi_\varphi]\right\|, \quad x \in \mathcal{M}, \tag{29}$$

for any faithful $\varphi \in \mathcal{M}_*^+$.

PROOF: Since $\left\|[x, \xi_\varphi]\right\| = \left\|\left(1 - \Delta_\varphi^{1/2}\right)x\xi\right\|$, we simply estimate a bound of the function: $(\lambda, t) \in \mathbf{R}_+^* \times \mathbf{R} \mapsto \left(\lambda^{\mathrm{i}t} - 1\right)\left(\lambda^{1/2} - 1\right)^{-1} \in \mathbf{C}$. Thus we consider the function:

$$f(\lambda, t) = \frac{\mathrm{e}^{\mathrm{i}t} - 1}{\big(1 + |t|\big)\big(\mathrm{e}^{\frac{1}{2}\lambda} - 1\big)}, \quad (\lambda, t) \in \mathbf{R}^2,$$

and observe that this function is indeed bounded on $\mathbf{R}^2$. Q.E.D.

Exercise IX.1

1) Let $\{\mathcal{M}, \mathfrak{H}, J, \mathfrak{P}\}$ be a von Neumann algebra in a standard form. In $\mathfrak{H}$, define an order relation $\xi \geq \eta$ by $\xi - \eta \in \mathfrak{P}$. For each $\omega \in \mathcal{M}_*^+$, let $\xi(\omega)$ be the vector in $\mathfrak{P}$ such that $\omega = \omega_{\xi(\omega)}$. Show that the map: $\omega \in \mathcal{M}_*^+ \mapsto \xi(\omega) \in \mathfrak{P}$ enjoys the following properties:

$$\xi(\lambda\omega) = \sqrt{\lambda}\xi(\omega), \quad \lambda \geq 0, \quad \omega \in \mathcal{M}_*^+;$$

$$\omega_1 \geq \omega_2 \Longrightarrow \xi(\omega_1) \geq \xi(\omega_2), \quad \omega_1, \omega_2 \in \mathcal{M}_*^+.$$

(Hint: Use Exercise VIII.3.8.)

$$\xi\big(\lambda\omega_1 + (1-\lambda)\omega_2\big) \geq \lambda\xi(\omega_1) + (1-\lambda)\xi(w_2), \quad 0 \leq \lambda \leq 1,$$

(Hint: $\lambda\omega_{\xi_1} + (1-\lambda)\omega_{\xi_2} \geq \omega_{\lambda\xi_1 + (1-\lambda)\xi_2}$ in general.)

2) With $\{\mathcal{M}, \mathfrak{H}, J, \mathfrak{P}\}$ as above, show that for every $\xi \in \mathfrak{H}$ there exist uniquely a partial isometry $u \in \mathcal{M}$ and $|\xi| \in \mathfrak{P}$ such that $\xi = u|\xi|$, u^*u is the cyclic projection to $\left[\mathcal{M}'|\xi|\right]$ and uu^* is the cyclic projection to $[\mathcal{M}'\xi]$. The vector $|\xi|$ is called the *absolute value* of ξ and u the *phase* of ξ. (Hint: Consider $\omega = \omega_\xi \in (\mathcal{M}'_*)^+$ and $|\xi| = \xi(\omega_\xi)$ relative to $\mathcal{M}'$.)

3) Show that if $\xi = u|\xi|$ is the polar decomposition, then $|\xi^*| = u|\xi|u^*$ and $\xi^* = u^*|\xi^*|$ is the polar decomposition of ξ^*, and $u|\xi| = |\xi^*|u$.

4) When we consider non faithful elements of $\mathcal{M}_*^+$, their modular automorphism groups and related objects mean the ones in the reduced algebra by their support. Let φ and ψ be elements in $\mathcal{M}_*^+$. Show that if $\varphi \leq M\psi$ for some M, then

$$\xi(\varphi) = (\mathrm{D}\varphi : \mathrm{D}\psi)_{-\frac{i}{2}}\xi(\psi) = (\mathrm{D}\varphi : \mathrm{D}\psi)_{-\frac{i}{4}}\xi(\psi)(\mathrm{D}\varphi : \mathrm{D}\psi)^*_{-\frac{i}{4}}.$$

(Hint: For the first identity, compare $(\mathrm{D}\varphi : \mathrm{D}\psi)_z\xi(\psi)$ and $\Delta^{iz}_{\varphi,\psi}\xi(\psi)$. Then observe that they agree on **R** and conclude that

$$\xi(\varphi) = \Delta^{\frac{1}{2}}_{\varphi,\psi}\xi(\psi) = (\mathrm{D}\varphi : \mathrm{D}\psi)_{-\frac{i}{2}}\xi(\psi).$$

For the second identity, show that

$$a\xi(\psi) = \xi(\psi)\sigma^{\psi}_{\frac{i}{2}}(a), \quad a \in \mathcal{D}\big(\sigma^{\psi}_{\frac{i}{2}}\big).$$

5) Assume that $\mathcal{M}$ is σ-finite and ψ is faithful. Prove that $\Theta_\psi : x \in \mathcal{M} \mapsto \Delta_\psi^{1/4}x\xi(\psi)$ is an order isomorphism of $\mathcal{M}$ onto $\Delta_\psi^{1/4}\mathcal{M}\xi(\psi)$, which is the linear span of the face $\bigcup_{\lambda>0}\big[0, \lambda\xi(\psi)\big]$ of $\mathfrak{P}$ generated by $\xi(\psi)$, where $[0,\xi] = \big\{\, \eta \in \mathfrak{H} : 0 \leq \eta \leq \xi \,\big\}$ in the ordered Hilbert space $\mathfrak{H}$ given by $\mathfrak{P}$.

6) Let $\big\{\mathcal{M}_i,\ \mathfrak{H}_i,\ J_i,\ \mathfrak{P}_i\big\}$, $i = 1, 2$, be the standard form of two σ-finite von Neumann algebras. Following the steps described below, prove that U is an isometry of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ with $U(\mathfrak{P}_1) = \mathfrak{P}_2$, then U gives rise to an order isomorphism θ, i.e. a Jordan *-isomorphism, of $\mathcal{M}_1$ onto $\mathcal{M}_2$ such that

$$\big(\theta(x)U\xi \;\big|\; U\xi\big) = \big(x\xi \;\big|\; \xi\big), \quad x \in \mathcal{M}_1, \quad \xi \in \mathfrak{H}_1.$$

In other words, the order structure of the standard form determines the order (Jordan) structure of a von Neumann algebra. [459]

Before going to the next step, recall that an order isomorphism between C^*-algebras is precisely a Jordan isomorphism by Exercise IV.1.2.

(**a**) Prove that U maps the order interval $\big[0, \xi(\psi_1)\big]$ given by a faithful $\psi_1 \in \mathcal{M}_*^+$ onto the order interval $\big[0, U\xi(\psi_1)\big]$ which generates a dense face of $\mathfrak{P}_2$, so that $\psi_2 = \omega_{U\xi(\psi_1)}$ is faithful.

(**b**) Prove that $\theta = \Theta_{\psi_2}^{-1} \circ U \circ \Theta_{\psi_1}$ is the required order isomorphism of $\mathcal{M}_1$ onto $\mathcal{M}_2$.

7) Show that if θ is a Jordan isomorphism of a von Neumann algebra $\mathcal{M}_1$ onto another $\mathcal{M}_2$, then there exists a unitary U_θ which maps the standard form $\big\{\mathcal{M}_1,\ \mathfrak{H}_1,\ J_1,\ \mathfrak{P}_1\big\}$ of $\mathcal{M}_1$ onto that $\big\{\mathcal{M}_2,\ \mathfrak{H}_2,\ J_2,\ \mathfrak{P}_2\big\}$ of $\mathcal{M}_2$ in such a way that

$$\big(\theta(x)\xi_2 \;\big|\; \xi_2\big) = \big(xU_\theta^{-1}\xi_2 \;\big|\; U_\theta^{-1}\xi_2\big), \quad \xi_2 \in \mathfrak{P}_2, \quad x \in \mathcal{M}_1.$$

(Hint: By Exercise IV.1.1 and 2, there exists a central projection $z \in \mathcal{M}_1$ such that θ is an isomorphism of $\mathcal{M}_1 z_1$ and an anti-isomorphism of $\mathcal{M}_1(1 - z_1)$. Use the isomorphism $\bar{\theta}(x) = \theta(xz_1) + J\theta\big(x(1-z_1)\big)^* J,\ x \in \mathcal{M}_1$, of $\mathcal{M}_1$, onto $\mathcal{M}_2\theta(z_1) \oplus \mathcal{M}_2'\big(1 - \theta(z_1)\big)$ to construct the required unitary U_θ.)

8) Keep the notations and the σ-finiteness assumption of Problem 5. Following the steps described below, prove that if $T \in \mathcal{L}(\mathfrak{H})$ is positive and invertible such that $T(\mathfrak{P}_1) = \mathfrak{P}_2$, then there exists a unique invertible $h \in \mathcal{M}_+$ such that $T = hJhJ$. [459]

(**a**) Show that if Θ is a linear map of $\mathcal{M}$ into $\mathfrak{H}$ such that $\Theta(\mathcal{M}_+)$ is a dense face of $\mathfrak{P}$, then there exist a unitary $U \in \mathcal{L}(\mathfrak{H})$ with $U(\mathfrak{P}) = \mathfrak{P}$ and a faithful $\psi \in \mathcal{M}_*^+$ such that

$$\Theta = U \circ \Theta_\psi .$$

(Hint: As $\Theta(\mathcal{M}_+)$ is dense in $\mathfrak{P}$, $\big[0, \Theta(1)\big]$ is total. So $\varphi = \omega_{\Theta(1)}$ is faithful. As Θ_φ is an order isomorphism from $\mathcal{M}$ onto the face of $\mathfrak{P}$ generated by $\xi(\varphi) = \Theta(1)$, $\theta = \Theta_\varphi^{-1} \circ \Theta$ is an order isomorphism of $\mathcal{M}$ onto itself such that $\theta(1) = 1$. Let $U = U_\theta$ be the unitary given by Problem 7, and $\psi = \omega_{U^{-1}\xi(\varphi)} = \varphi \circ \theta$. Then $U\Theta_\psi(x) = \Theta_\varphi \circ \Theta(x) = \Theta(x)$.)

(**b**) Let $\varphi \in \mathcal{M}_*^+$ be faithful. Show that there exists a faithful $\psi \in \mathcal{M}_*^+$ such that the absolute value of the bounded invertible operator: $\Delta_\varphi^{1/4} x\xi(\varphi) \mapsto \Delta_\psi^{1/4} x\xi(\psi)$ is exactly T. (Hint: Let $\Theta = T \circ \Theta_\varphi$ and apply (a) to Θ to find a unitary U and ψ such that

$$T\Delta_\varphi^{\frac14} x\xi(\varphi) = U\Delta_\psi^{\frac14} x\xi(\psi), \quad x \in \mathcal{M}.$$

Hence $U^* T\Delta_\varphi^{1/4} x\xi(\varphi) = \Delta_\psi^{1/4} x\xi(\psi),\ x \in \mathcal{M}$.)

(**c**) Show the equivalence of the following statements:

(i) $\big\|\Delta_\varphi^{1/4} x\xi(\varphi)\big\| \geq \big\|\Delta_\psi^{1/4} x\xi(\psi)\big\|,\ x \in \mathcal{M}$;

(ii) The function: $t \in \mathbf{R} \mapsto u_t = (\mathrm{D}\psi : \mathrm{D}\varphi)_t \in \mathcal{M}$ has an analytic extension to a member $\{u_z\}$ of $\mathcal{A}_{\mathcal{M}}\big(\mathbf{D}_{1/4}\big)$ such that $\big\|u_{-\mathrm{i}/4}\big\| \leq 1$.

Then show that if the above conditions hold, then

$$\Delta_\psi^{\frac14} x\xi(\psi) = u_{-\frac{\mathrm{i}}{4}}\Big(\Delta_\varphi^{\frac14} x\xi(\varphi)\Big) u_{-\frac{\mathrm{i}}{4}}^* .$$

(Hint: Making use of the 2×2-matrix technique, justify the following formal computations:

$$J\,\Delta_{\psi,\varphi}^{\frac12}\,\Delta_\varphi^{-\frac12}\,J = \Delta_\psi^{-\frac12}\Delta_{\psi,\varphi}^{\frac12}$$

as seen below:

$$J\Delta_{\psi,\varphi}^{\frac12}\Delta_\varphi^{-\frac12} Jx\xi(\varphi) = J\Delta_{\psi,\varphi}^{\frac12} x^*\xi(\varphi) = x\xi(\psi) = \Delta_\psi^{-\frac12} Jx^*\xi(\varphi)$$
$$= \Delta_\psi^{-\frac12}\Delta_{\psi,\varphi}^{\frac12} x\xi(\varphi);$$

$$\begin{aligned}
\Delta_\psi^{\frac14} x\xi(\psi) &= \Delta_\psi^{\frac14} J \Delta_{\psi,\varphi}^{\frac12} x^*\xi(\varphi) \;=\; \Delta_\psi^{\frac14} J \Delta_{\psi,\varphi}^{\frac12} \Delta_\varphi^{-\frac12} J x\xi(\varphi)\\
&= \Delta_\psi^{\frac14} \Delta_\psi^{-\frac12} \Delta_{\psi,\varphi}^{\frac12} x\xi(\varphi) \;=\; \Delta_\psi^{-\frac14} \Delta_{\psi,\varphi}^{\frac12} \Delta_\varphi^{-\frac14} \Delta_\varphi^{\frac14} x\xi(\varphi)\\
&= \Delta_\psi^{-\frac14} \Delta_{\psi,\varphi}^{\frac14} \Delta_{\psi,\varphi}^{\frac14} \Delta_\varphi^{-\frac14} \Delta_\varphi^{\frac14} x\xi(\varphi)\\
&= \Big(J \Delta_{\psi,\varphi}^{\frac14} \Delta_\varphi^{-\frac14} J\Big)\Big(\Delta_{\psi,\varphi}^{\frac14} \Delta_\varphi^{-\frac14}\Big) \Delta_\varphi^{\frac14} x\xi(\varphi)\\
&= u_{-\frac{\mathrm{i}}{4}} \Big(\Delta_\varphi^{\frac14} x\xi(\varphi)\Big) u^*_{-\frac{\mathrm{i}}{4}}\,.
\end{aligned}$$

(**d**) In view of (c), T is the absolute value of the operator $aJaJ$ for some $a \in \mathcal{M}$. But the absolute value $|aJaJ|$ is given by $T = |aJaJ| = |a|J|a|J$.

(**e**) If $aJaJ = 1,\ a \in \mathcal{M}$, then a is invertible and $a^{-1} = JaJ \in \mathcal{M} \cap \mathcal{M}' = \mathfrak{Z}$, so that a is central. If $h, k \in \mathcal{M}_+$ are invertible and if $hJhJ = kJkJ$, then $h = ak$ with some $a \in \mathfrak{Z}_+$ such that $1 = aJaJ = a^2$, so that $a = 1$. Thus the uniqueness of h with $T = hJhJ$ follows.

9) Keep the notations and the assumptions in the previous problem. Assume the innerness of a derivation on a von Neumann algebra which will be proven later, Theorem XI.3.5. Following the arguments presented below, prove that if $\delta \in \mathcal{L}(\mathfrak{H})$ generates a one parameter group $\exp(t\delta)$ of invertible operators such that $\exp(t\delta)(\mathfrak{P}) = \mathfrak{P},\ t \in \mathbf{R}$, then $\delta = x + JxJ$ for some $x \in \mathcal{M}$.

(**a**) Based on the formula, called the *Lie-Trotter formula*:

$$\exp\big(t(\delta + \delta^*)\big) = \lim_{n\to\infty} \big(\exp(t\delta/n)\exp(t\delta/n)^*\big)^n\,;$$

$$\exp\big(t(\delta - \delta^*)\big) = \lim_{n\to\infty} \big(\exp(t\delta/n)\exp(-t\delta/n)^*\big)^n,$$

observe that it suffices to prove the claim for a self-adjoint δ and a skew-adjoint δ separately.

(**b**) If $\delta = \delta^*$, then $T_t = \exp(t\delta)$ is of the form $T_t = H_t J H_t J,\ H_t \in \mathcal{M}_+$, by Problem 8. By the uniqueness of H_t, we have $H_{s+t} = H_s H_t$. Hence with $h = \lim_{t\to 0}(H_t - 1)/t$, we obtain $\delta = h + JhJ$.

(**c**) Assume $\delta = -\delta^*$. Then $U(t) = \exp(t\delta)$ is a one parameter unitary group such that $U(t)(\mathfrak{P}) = \mathfrak{P}$. By Problem 6, $U(t)$ gives rise to a one parameter group $\{\theta_t\}$ of Jordan *-automorphisms of $\mathcal{M}$ such that $U(t) = U_{\theta_t}$. As $\theta_t = (\theta_{t/2})^2$, each θ_t is the square of a Jordan *-automorphism which is an automorphism. Hence $\{\theta_t\}$ is a one parameter automorphism group of $\mathcal{M}$ which is given by $\theta_t(x) = U(t)xU(t)^*$, $t \in \mathbf{R},\ x \in \mathcal{M}$.

(**d**) Differentiating the last equation, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta_t(x)\bigg|_{t=0} = \delta x - x\delta = [\delta, x] \in \mathcal{M}, \quad x \in \mathcal{M}\,,$$

so that $\mathrm{ad}(\delta) = [\delta,\ \cdot\ ]$ is a derivation of $\mathcal{M}$.

(**e**) By Theorem XI.3.5, there exists $h \in \mathcal{M}_h$ such that $\delta x - x\delta = \mathrm{i}\,[h, x],\ x \in \mathcal{M}$. Hence $\delta - \mathrm{i}h = \mathrm{i}k \in \mathcal{M}'$. As $U(t)$ preserves $\mathfrak{P}$, $JU(t)J = U(t)$, so that $J\delta J = \delta$. Hence

$$\delta = \mathrm{i}h + \mathrm{i}k = J\big(\mathrm{i}h + \mathrm{i}h + \mathrm{i}k\big)J = -\mathrm{i}\,J(h+k)J.$$

Thus $a = h + JkJ = -(JhJ + k) \in \mathcal{M} \cap \mathcal{M}' = \mathfrak{Z}$, so that

$$a = a^* = JaJ = JhJ + k = -a.$$

Therefore $a = 0$, which means that

$$\delta = \mathrm{i}h - \mathrm{i}JhJ = \mathrm{i}h + J(\mathrm{i}h)J.$$

10) Keep the above notations and the assumptions. Observe that the set $\mathfrak{g} = \mathfrak{g}(\mathfrak{P})$ of all $\delta \in \mathcal{L}(\mathfrak{H})$ with $\exp(t\delta)(\mathfrak{P}) = \mathfrak{P},\ t \in \mathbf{R}$, is a Lie algebra under the Lie bracket operation: $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$. By the previous problem, the map $x \in \mathcal{M} \mapsto x + JxJ \in \mathfrak{g}$ is a real linear surjective Lie algebra homomorphism from the Lie algebra $\mathcal{M}$ onto $\mathfrak{g}$ when $\mathcal{M}$ is viewed as a Lie algebra under the Lie product $[x, y] = xy - yx,\ x, y \in \mathcal{M}$.

(**a**) Prove that the kernel of the above Lie homomorphism is precisely $\big\{ a \in \mathfrak{Z} : a^* = -a \big\}$ and that the center $\mathfrak{c}$ of $\mathfrak{g}$ is $\big\{ a + JaJ : a \in \mathfrak{Z},\ a = a^* \big\}$.

(**b**) Consider the quotient Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g}/\mathfrak{c}$, and denote the coset $\delta + \mathfrak{c} \in \hat{\mathfrak{g}}$ of δ by $\hat{\delta}$. Prove that the homomorphism: $x \in \mathcal{M} \mapsto x + JxJ \in \mathfrak{g}$ gives rise to an isomorphism j of the quotient Lie algebra $\mathcal{M}/\mathfrak{Z} = \hat{\mathcal{M}}$, i.e. the Lie algebra of derivations of $\mathcal{M}$, onto $\hat{\mathfrak{g}}$.

(**c**) Prove that if $a = [x, y],\ x, y \in \mathcal{L}(\mathfrak{H})$, commutes with both x and y, then $\mathrm{Sp}(a) = \{0\}$. Hence if $x, y \in \mathcal{M}$ and $a \in \mathfrak{Z}$, then $a = 0$. (Hint: As a and x commute, $[\mathrm{e}^{tx}, y] = ta\mathrm{e}^{tx}$, so that $\mathrm{e}^{tx}y\mathrm{e}^{-tx} = y + ta$. If $\lambda \in \mathrm{Sp}(a)$, then $t\lambda + \mathrm{Sp}(y) \subset \mathrm{Sp}(\mathrm{e}^{tx}y\mathrm{e}^{-tx}) = \mathrm{Sp}(y)$. Hence the boundedness of $\mathrm{Sp}(y)$ implies $\lambda = 0$.) Therefore, $\hat{\mathcal{M}}$ has no center other than $\{0\}$.

(**d**) Define $I_{\mathcal{M}} : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ by

$$I_{\mathcal{M}}(\hat{\delta}) = j\big(\mathrm{i}\,j^{-1}(\hat{\delta})\big).$$

Prove that $I = I_{\mathcal{M}}$ enjoys the following property:

$$\left.\begin{aligned} &I^2 = -\,\mathrm{id}, \qquad I(\hat{\delta}^*) = -I(\hat{\delta})^*; \\ &\big[\pm\delta_1, \delta_2\big] = \big[\delta_1, I\delta_2\big] = I\big[\delta_1, \delta_2\big]. \end{aligned}\right\} \tag{$*$}$$

(**e**) Call a real linear map I of $\hat{\mathfrak{g}}$ onto itself satisfying the above condition $(*)$ an *orientation* of $\mathfrak{P}$. Prove that if I_1 and I_2 are two orientations of $\mathfrak{P}$, then there exists a central projection $e \in \mathcal{M}$ such that $I_1 = I_2$ on $\hat{\mathfrak{g}}(e\mathfrak{P})$ and $I_1 = -I_2$ on $\hat{\mathfrak{g}}\big((1-e)\mathfrak{P}\big)$.

(Hint: Set $\mathcal{E} = I_2 \circ I_1^{-1}$. Prove that $\mathcal{E}^2(\hat{\delta}) - \hat{\delta}$ belongs to the center of $\hat{\mathfrak{g}}$ which is $\{0\}$ by (c). Thus $\hat{\mathfrak{g}}$ is the direct sum of two ideals

$$\hat{\mathfrak{g}}_1 = \big\{ \hat{\delta} \in \hat{\mathfrak{g}} : \mathcal{E}(\hat{\delta}) = \hat{\delta} \big\} \quad \text{and} \quad \hat{\mathfrak{g}}_{-1} = \big\{ \hat{\delta} \in \hat{\mathfrak{g}} : \mathcal{E}(\hat{\delta}) = -\hat{\delta} \big\}.$$

Set

$$\mathcal{M}_1 = \left\{ x \in \mathcal{M} : j \circ \mathrm{ad}(x) \in \hat{\mathfrak{g}}_1 \right\} \quad \text{and} \quad \mathcal{M}_{-1} = \left\{ x \in \mathcal{M} : j \circ \mathrm{ad}(x) \in \hat{\mathfrak{g}}_{-1} \right\}.$$

If $x \in \mathcal{M}_1$ and $y \in \mathcal{M}_{-1}$, then $j \circ \mathrm{ad}[x, y] = 0$, so $[x, y] \in \mathfrak{Z}$; hence $[x, y] = 0$ by (c). As $\mathcal{E}(\hat{\delta}^*) = \mathcal{E}(\hat{\delta})^*$, $\hat{\mathfrak{g}}_1$, $\hat{\mathfrak{g}}_{-1}$, $\mathcal{M}_1$ and $\mathcal{M}_1$ are all closed under the *-operation. If $x \in \mathcal{M}_1' \cap \mathcal{M}$, then $j \circ \mathrm{ad}(x)$ commutes with $\hat{\mathfrak{g}}_1$, hence it belongs to $\hat{\mathfrak{g}}_{-1}$. Hence $\mathcal{M}_{-1} = \mathcal{M}_1' \cap \mathcal{M}$. Similarly, $\mathcal{M}_1 = \mathcal{M}_{-1}' \cap \mathcal{M}$. Thus $\mathcal{M}_1$ and $\mathcal{M}_{-1}$ are both von Neumann subalgebras of $\mathcal{M}$, which contain $\mathfrak{Z}$. As $j \circ \mathrm{ad}(x) \in \hat{\mathfrak{g}}_1 + \hat{\mathfrak{g}}_{-1}$ for every $x \in \mathcal{M}$, we have $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_{-1}$. If $x = x_1 + x_{-1}$ and $y = y_1 + y_{-1}$ with $x_1, y_1 \in \mathcal{M}_1$ and $x_{-1}, y_{-1} \in \mathcal{M}_{-1}$, then $[x, y] = [x_1, y_1] + [x_{-1}, y_{-1}]$; hence $[x_1, y] = [x_1, y_1] \in \mathcal{M}_1$. If $z = z_1 + z_{-1}$ with $z_1 \in \mathcal{M}_1$ and $z_{-1} \in \mathcal{M}_{-1}$, then

$$[x_1, y_1]z = [x_1, y_1]z_1 + x_1 y_1 z_{-1} - y_1 z_{-1} x_1 = [x_1, y_1]z_1 + [x_1, y_1 z_{-1}] \in \mathcal{M}_1.$$

Hence if $e = \bigvee \left\{ us_\ell\big([x, y]\big)u^* : x, y \in \mathcal{M}_1,\ u \in \mathcal{U}(\mathcal{M}_1) \right\}$, then e is a central projection of $\mathcal{M}_1$ such that $\mathcal{M}_{1,e}$ is an ideal of $\mathcal{M}$. As $\mathcal{M}_1 \cap \mathcal{M}_1' = \mathfrak{Z}$, $e \in \mathfrak{Z}$ and $\mathcal{M}e = \mathcal{M}_1 e$. Since $\mathcal{M}_1(1-e) = \mathfrak{Z}(1-e) = \mathfrak{Z}(1-e)$, we have $\mathcal{M}_1 = \mathcal{M}(1-e) + \mathfrak{Z}$. Now we have seen that $\mathcal{E} = \mathrm{id}$ on $\hat{\mathfrak{g}}(e\mathfrak{P})$ and $\mathcal{E} = -\,\mathrm{id}$ on $\hat{\mathfrak{g}}((1-e)\mathfrak{P})$.)

(**f**) Prove that if with $e \in \mathrm{Proj}(\mathfrak{Z})$ $I_1 = I_{\mathcal{M}}$ on $\hat{\mathfrak{g}}(e\mathfrak{P})$ and $I_1 = -I_{\mathcal{M}}$ on $\hat{\mathfrak{g}}\big((1-e)(\mathfrak{P})\big)$, then $I = I_{\mathcal{N}}$ where $\mathcal{N} = \mathcal{M}e + \mathcal{M}'(1-e)$.

(**g**) Prove, with $I = I_{\mathcal{M}}$, that

$$\mathcal{M} = \left\{ \delta_1 - I\delta_2 : \hat{\delta}_2 = I(\hat{\delta}_1),\ \delta_1, \delta_2 \in \mathfrak{g}(\mathfrak{P}) \right\};$$

$$\mathcal{M}' = \left\{ \delta_1 + I\delta_2 : \hat{\delta_2} = I(\hat{\delta}_1),\ \delta_2 \in \mathfrak{g}(\mathfrak{P}) \right\}.$$

Therefore, the natural positive cone $\mathfrak{P}$ together with the orientation I determines the von Neumann algebra. [459].

11) Drop the σ-finiteness assumption from Problems (6), (8), (9) and (10).

12) Consider the standard form $\{\widetilde{\mathcal{M}}, \tilde{\mathfrak{H}}, \tilde{J}, \tilde{\mathfrak{P}}\}$ of $\widetilde{\mathcal{M}} = \mathcal{M} \otimes M_2(\mathbf{C})$, viewing $\tilde{\mathfrak{H}}$ as the Hilbert space of 2×2-matrices with entries from $\mathfrak{H}$ and letting $\tilde{\mathcal{M}}$ acts by the left multiplication, where the inner product in $\tilde{\mathfrak{H}}$ is defined as follows:

$$\big([\xi_{i,j}] \mid [\eta_{k,\ell}]\big) = \sum_{i,j=1}^{2} \big(\xi_{ij} \mid \eta_{i,j}\big).$$

(**a**) Observe that

$$(\xi^*)_{i,j} = (\xi_{j,i})^*, \qquad \xi = [\xi_{i,j}] \in \widetilde{\mathfrak{H}}.$$

(**b**) Show that $\begin{pmatrix} a & b^* \\ b & c \end{pmatrix} \in \widetilde{\mathcal{M}}$ is positive if and only if a, $b \geq 0$ and

$$\big|(b\xi \mid \eta)\big| \leq \big(a\xi \mid \xi\big)^{\frac{1}{2}} \big(c\eta \mid \eta\big)^{\frac{1}{2}}, \quad \xi, \eta \in \mathfrak{H};$$

if and only if there exists a unique $s \in \mathcal{M}$ with $\|s\| \leq 1$ such that $b = c^{1/2} s a^{1/2}$.

(c) Show that $\begin{pmatrix} \varphi & \rho^* \\ \rho & \psi \end{pmatrix} \in \widetilde{\mathcal{M}}_*$ is positive when the pairing with $\widetilde{\mathcal{M}}$ is given by

$$\left\langle \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \right\rangle = \sum_{i,j=1}^{2} \omega_{i,j}(x_{i,j}),$$

if and only if $\varphi, \psi \geq 0$ and

$$\left|\rho(y^* x)\right| \leq \varphi(x^* x)^{\frac{1}{2}} \psi(y^* y)^{\frac{1}{2}}, \quad x, y \in \mathcal{M}.$$

(d) Suppose that $\varphi \in \mathcal{M}_*^+$ is faithful. Show that $\begin{pmatrix} \xi & \zeta^* \\ \zeta & \eta \end{pmatrix} \in \widetilde{\mathfrak{H}}$ belongs to $\widetilde{\mathfrak{P}}$ if and only if $\xi, \eta \in \mathfrak{P}$ and

$$\left|\left(x^* \zeta y \mid \xi(\varphi)\right)\right| \leq \left(x^* \xi x \mid \xi(\varphi)\right)^{\frac{1}{2}} \left(y^* \eta y \mid \xi(\varphi)\right)^{\frac{1}{2}}, \quad x, y \in \mathcal{M}.$$

13[†]) Let $\{\mathcal{M}, \mathfrak{H}, J, \mathfrak{P}\}$ be a standard von Neumann algebra. If $\xi_0 \in \mathfrak{P}$ is cyclic, hence separating, then $\mathfrak{P}$ is precisely the set of all vectors of the form

$$\xi = TJTJ\xi_0$$

where T is a densely defined closed operator affiliated with $\mathcal{M}$ such that $\xi_0 \in \mathfrak{D}(T)$ and $JTJ\xi_0 = JT\xi_0 = JT\xi_0 \in \mathfrak{D}(T)$. [601, 602]

§ 2 Measurable Operators and Integral for a Trace

We fix a faithful semi-finite normal trace τ on a semi-finite von Neumann algebra $\mathcal{M}$.

Definition 2.1. The *measure topology* of $\mathcal{M}$ with respect to τ (or simply τ-*measure topology*) is the uniform topology given by a neighborhood system $\{x + N(\varepsilon, \delta) : \varepsilon, \delta > 0\}$, $x \in \mathcal{M}$, where $N(\varepsilon, \delta)$ is the set of all operators $a \in \mathcal{M}$ such that

$$\|ap\| < \varepsilon \quad \text{and} \quad \tau(p^\perp) < \delta$$

for some $p \in \mathrm{Proj}(\mathcal{M})$. The convergence with respect to this topology is called τ-*measure* convergence. The completion of $\mathcal{M}$ with respect to this topology is denoted by $\mathfrak{M}(\mathcal{M})$.

When $\mathcal{M}$ acts on $\mathfrak{H}$, we define the $(\mathcal{M}, \tau)$-*measure* (or simply *measure*) *topology* of $\mathfrak{H}$ as the uniform topology given by a neighborhood system $\{\xi + O(\varepsilon, \delta) : \varepsilon > 0,\ \delta > 0\}$, $\xi \in \mathfrak{H}$, where $O(\varepsilon, \delta)$ is the set of all $\eta \in \mathfrak{H}$ such that

$$\|p\eta\| < \varepsilon \quad \text{and} \quad \tau(p^\perp) < \delta$$

for some $p \in \mathrm{Proj}(\mathcal{M})$. The completion of $\mathfrak{H}$ is denoted by $\mathfrak{M}(\mathfrak{H})$. We naturally define the boundedness for subsets of $\mathcal{M}$ and $\mathfrak{H}$ with respect to the measure topology.

Theorem 2.2. *Consider the following*

$$a \in \mathcal{M} \mapsto a^* \in \mathcal{M}; \tag{1}$$

$$(a, b) \in \mathcal{M} \times \mathcal{M} \mapsto a + b \in \mathcal{M}; \tag{2}$$

$$(a, b) \in \mathcal{M} \times \mathcal{M} \mapsto ab \in \mathcal{M}; \tag{3}$$

$$(\xi, \eta) \in \mathfrak{H} \times \mathfrak{H} \mapsto \xi + \eta \in \mathfrak{H}; \tag{4}$$

$$(a, \xi) \in \mathcal{M} \times \mathfrak{H} \mapsto a\xi \in \mathfrak{H}. \tag{5}$$

These maps can respectively be extended to maps: $\mathfrak{M}(\mathcal{M}) \to \mathfrak{M}(\mathcal{M})$, $\mathfrak{M}(\mathcal{M}) \times \mathfrak{M}(\mathcal{M}) \to \mathfrak{M}(\mathcal{M})$, $\mathfrak{M}(\mathcal{M}) \times \mathfrak{M}(\mathcal{M}) \to \mathfrak{M}(\mathcal{M})$, $\mathfrak{M}(\mathfrak{H}) \times \mathfrak{M}(\mathfrak{H}) \to \mathfrak{M}(\mathfrak{H})$ *and* $\mathfrak{M}(\mathcal{M}) \times \mathfrak{M}(\mathfrak{H}) \to \mathfrak{M}(\mathfrak{H})$. *These extensions are unique, and maps* (1), (2) *and* (4) *together with their extensions are uniformly continuous. Maps* (3) *and* (5) *together with their extensions are uniformly continuous on the product set of bounded subsets. Thus,* $\mathfrak{M}(\mathcal{M})$ *is a topological involutive algebra with a continuous representation on a topological vector space* $\mathfrak{M}(\mathfrak{H})$.

PROOF: To prove the theorem, we shall show the following inclusions:

$$N(\varepsilon, \delta)^* \subset N(\varepsilon, 2\delta) \tag{1'}$$

$$N(\varepsilon_1, \delta_1) + N(\varepsilon_2, \delta_2) \subset N(\varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2) \tag{2'}$$

$$N(\varepsilon_1, \delta_1)N(\varepsilon_2, \delta_2) \subset N(\varepsilon_1\varepsilon_2, \delta_1 + \delta_2) \tag{3'}$$

$$O(\varepsilon_1, \delta_1) + O(\varepsilon_2, \delta_2) \subset O(\varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2) \tag{4'}$$

$$N(\varepsilon_1, \delta_1)O(\varepsilon_2, \delta_2) \subset O(\varepsilon_1\varepsilon_2, \delta_1 + \delta_2) \tag{5'}$$

Assume these inclusions for a moment. It then follows that the maps (1), (2), (4) are uniformly continuous. Suppose S_1 and S_2 are bounded subsets of $\mathcal{M}$. We want to show that the map (3) is uniformly continuous on $S_1 \times S_2$. To this end, for any $\varepsilon, \delta > 0$ we want to find $\varepsilon_1, \delta_1, \varepsilon_2, \delta_2 > 0$ so that

$$\big(a + N(\varepsilon_1, \delta_1)\big)\big(b + N(\varepsilon_2, \delta_2)\big) \subset ab + N(\varepsilon, \delta) \tag{6}$$

for every $a \in S_1$ and $b \in S_2$.

The boundedness of S_1 and S_2 means that for any $\alpha, \beta > 0$ there exists $\gamma > 0$ such that $\gamma S_1 \subset N(\alpha, \beta)$ and $\gamma S_2 \subset N(\alpha, \beta)$, which is equivalent to $S_1 \subset N(\alpha/\gamma, \beta)$ and $S_2 \subset N(\alpha/\gamma, \beta)$. Thus, for each $\alpha_1, \alpha_2 > 0$, we can find $\eta_1, \eta_2 > 0$ such that $S_1 \subset N(\eta_1, \alpha_1)$ and $S_2 \subset N(\eta_2, \alpha_2)$. By (2′) and (3′), we have, if $a \in S_1$ and $b \in S_2$,

$$\begin{aligned}
&\big(a + N(\varepsilon_1, \delta_1)\big)\big(b + N(\varepsilon_2, \delta_2)\big) - ab \\
&\quad \subset N(\eta_1, \alpha_1)N(\varepsilon_2, \delta_2) + N(\varepsilon_1, \delta_1)N(\eta_2, \alpha_2) + N(\varepsilon_1, \delta_1)N(\varepsilon_2, \delta_2) \\
&\quad \subset N\big(\eta_1\varepsilon_2, \alpha_1 + \delta_2\big) + N\big(\varepsilon_1\eta_2, \delta_1 + \alpha_2\big) + N\big(\varepsilon_1\varepsilon_2, \delta_1 + \delta_2\big) \\
&\quad \subset N\big(\eta_1\varepsilon_2 + \varepsilon_1\eta_2 + \varepsilon_1\varepsilon_2, \alpha_1 + \alpha_2 + 2\delta_1 + 2\delta_2\big).
\end{aligned}$$

Thus, we choose $\alpha_1, \alpha_2, \delta_1, \delta_2 > 0$ in such a way that $\alpha_1 + \alpha_2 + 2\delta_1 + 2\delta_2 \leq \delta$, and then choose $\varepsilon_1, \varepsilon_2 > 0$ further so that $\eta_1\varepsilon_2 + \eta_2\varepsilon_1 + \varepsilon_1\varepsilon_2 \leq \varepsilon$, to conclude (6). In a similar way, the uniform continuity of map (5) on a bounded set can be shown. Now, we shall show (1′), (2′) and (3′). The proof for (4′) and (5′) is similar, so we leave it to the reader.

(1′) If $a \in N(\varepsilon, \delta)$, then $\|ap\| < \varepsilon$ and $\tau(p^\perp) < \delta$ for some $p \in \mathrm{Proj}(\mathcal{M})$. We put

$$p_a = \bigvee \{ q \in \mathrm{Proj}(\mathcal{M}) : aq = paq \}. \tag{7}$$

It then follows that

$$p_a^\perp = \bigwedge \{ q \in \mathrm{Proj}(\mathcal{M}) : qa^* p^\perp = a^* p^\perp \},$$

so that $p_a^\perp$ is the range projection of $a^* p^\perp$. Hence

$$p_a^\perp \precsim p^\perp. \tag{8}$$

Applying (8) to p_{a^*}, we get

$$\tau\big((p \wedge p_{a^*})^\perp\big) = \tau\big(p^\perp \vee p_{a^*}^\perp\big) \leq \tau\big(p^\perp\big) + \tau\big(p_{a^*}^\perp\big) \leq 2\tau\big(p^\perp\big) < 2\delta,$$

and

$$a^*\big(p \wedge p_{a^*}\big) = pa^*\big(p \wedge p_{a^*}\big); \qquad \big\|a^*\big(p \wedge p_{a^*}\big)\big\| \leq \big\|pa^*\big\| = \|ap\| < \varepsilon.$$

Thus, a^* belongs to $N(\varepsilon, 2\delta)$.

(2′) Suppose $a \in N(\varepsilon_1, \delta_1)$ and $b \in N(\varepsilon_2, \delta_2)$. Choose $p, q \in \mathrm{Proj}(\mathcal{M})$ so that

$$\|ap\| < \varepsilon_1, \qquad \|bq\| < \varepsilon_2, \qquad \tau\big(p^\perp\big) < \delta_1, \qquad \tau\big(q^\perp\big) < \delta_2.$$

Since $(a+b)(p \wedge q) = (ap + bq)(p \wedge q)$, we have $\big\|(a+b)(p \wedge q)\big\| < \varepsilon_1 + \varepsilon_2$ and

$$\tau\big((p \wedge q)^\perp\big) = \tau\big(p^\perp \vee q^\perp\big) \leq \tau\big(p^\perp\big) + \tau\big(q^\perp\big) \leq \delta_1 + \delta_2.$$

(3′) We use p_b given by (7) with b in the place of a. Now we have

$$ab(q \wedge p_b) = apb(q \wedge p_b),$$

so

$$\big\|ab(q \wedge p_b)\big\| < \varepsilon_1\varepsilon_2.$$

By (8), we get

$$\tau\big((q \wedge p_b)^\perp\big) = \tau\big(q^\perp \vee p_b^\perp\big) \leq \tau(q^\perp) + (p_b^\perp) \leq \tau(q^\perp) + \tau(p^\perp) < \delta_1 + \delta_2,$$

thus $ab \in N(\varepsilon_1\varepsilon_2, \delta_1 + \delta_2)$. Q.E.D.

Lemma 2.3.

(i) *The measure topologies of* $\mathcal{M}$ *and* $\mathfrak{H}$ *both satisfy the Hausdorff separation axiom. Thus* $\mathcal{M}$ *and* $\mathfrak{H}$ *can be viewed as subsets of their completion* $\mathfrak{M}(\mathcal{M})$ *and* $\mathfrak{M}(\mathfrak{H})$ *respectively.*

(ii) *For each* $a \in \mathfrak{M}(\mathcal{M})$ *and* $\varepsilon > 0$, *there exists* $p \in \mathrm{Proj}(\mathcal{M})$ *such that* $ap \in \mathcal{M}$ *and* $\tau(p^{\perp}) < \varepsilon$.

PROOF:

(i) Since they are both uniform topologies, it suffices to prove the T_1-separation axiom. Suppose $\xi \in \bigcap\left\{ O(\varepsilon, \delta) : \varepsilon > 0,\ \delta > 0\right\}$. For each $n = 1, 2, \dots$, choose $q_n \in \mathrm{Proj}(\mathcal{M})$ so that $\|q_n\xi\| < 2^{-n}$ and $\tau(q_n^{\perp}) < 2^{-n}$. We set $p_n = \bigwedge_{k=n}^{\infty} q_k$. Then $\tau(p_n^{\perp}) < 2^{-n+1}$ and $p_1 \le p_2 \le \cdots$, so that $p_n^{\perp} \searrow 0$ which means $p_n \nearrow 1$. But $\|p_n\xi\| \le \|q_k\xi\| < 2^{-k}$ for every $k \ge n$, so that $p_n\xi = 0$. Thus $\xi = 0$.

If $a \ne 0$, then $a\xi \ne 0$ for some $\xi \in \mathfrak{H}$. If $a \in \bigcap N(\varepsilon, \delta)$, then $a\xi \in \bigcap O(\varepsilon, \delta)$ by Theorem 2.2, which is impossible as just shown. Thus $\mathcal{M}$ satisfies the T_1-axiom.

(ii) If $a \in \mathfrak{M}(\mathcal{M})$, then a is the measure convergence limit of a sequence $\{a_n\}$ of $\mathcal{M}$. By selecting a subsequence, we may assume that

$$a = a_0 + \sum_{k=1}^{\infty}(a_{k+1} - a_k), \qquad a_{k+1} - a_k \in N(2^{-k}, 2^{-k}).$$

Put $b_k = a_{k+1} - a_k,\ k = 1, 2, \dots$, and choose $q_k \in \mathrm{Proj}(\mathcal{M})$ so that $\|b_k q_k\| < 2^{-k}$ and $\tau(q_k^{\perp}) < 2^{-k}$. We then set $p_n = \bigwedge_{k \ge n} q_k$. It follows that $\{p_n\}$ is increasing and $\tau(p_n^{\perp}) < 2^{-n+1} \to 0$, so that $p_n \nearrow 1$. By Theorem 2.2, we get

$$ap_n = a_0 p_n + \sum_{k=1}^{\infty} b_k p_n = a_0 p_n + \sum_{k=1}^{n-1} b_k p_n + \sum_{k=n}^{\infty} b_k q_k p_n.$$

The last summation converges in norm, so that $ap_n \in \mathcal{M}$. Q.E.D.

Definition 2.4. For each $a \in \mathfrak{M}(\mathcal{M})$, we set

$$\mathfrak{D}\big(M(a)\big) = \left\{ \xi \in \mathfrak{H} : a\xi \in \mathfrak{H} \right\}, \qquad M(a)\xi = a\xi, \quad \xi \in \mathfrak{D}\big(M(a)\big). \tag{9}$$

Theorem 2.5.

(i) *For each* $a \in \mathfrak{M}(\mathcal{M})$, $M(a)$ *is a densely defined closed operator affiliated with* $\mathcal{M}$ *and maximal in the sense that* $M(a)$ *has no proper closed extension affiliated with* $\mathcal{M}$.

(ii) *For each pair* $a, b \in \mathfrak{M}(\mathcal{M})$, *we have*

$$M(a^*) = M(a)^*,$$

$$M(a + b) = \overline{M(a) + M(b)},$$

$$M(ab) = \overline{M(a)M(b)},$$

where the bar on the right hand side means the closure.

(iii) *If a sequence* $\{p_n\}$ *in* $\mathrm{Proj}(\mathcal{M})$ *is increasing and* $\tau(p_n^\perp) \searrow 0$ *and if a linear operator* A *defined on* $\mathfrak{D} = \bigcup p_n\mathfrak{H}$ *has the property that* $Ap_n \in \mathcal{M}$, *then* A *is preclosed and there exists uniquely an element* $a \in \mathfrak{M}(\mathcal{M})$ *with* $M(a) = \bar{A}$.

We need a few lemmas.

Lemma 2.6.

(i) *If* $p \wedge q = 0$, $p, q \in \mathrm{Proj}(\mathcal{M})$, *then* $p \precsim q^\perp$ *and* $q \precsim p^\perp$.

(ii) *Suppose that* e *and* f *are projections in* $\mathcal{M}$. *If for any* $\varepsilon > 0$ *there exists* $p \in \mathrm{Proj}(\mathcal{M})$ *with*

$$e \wedge p = f \wedge p \quad \textit{and} \quad \tau(p^\perp) < \varepsilon,$$

then $e = f$.

PROOF:
(i) If $p \wedge q = 0$, then $p^\perp \vee q^\perp = 1$, so

$$p = p^\perp \vee q^\perp - p^\perp \sim q^\perp - q^\perp \wedge p^\perp \le q^\perp,$$
$$q = p^\perp \vee q^\perp - q^\perp \sim p^\perp - q^\perp \wedge p^\perp \le p^\perp.$$

(ii) We have

$$(e - e \wedge f) \wedge p \le e \wedge p - (e \wedge f) \wedge p = f \wedge p - f \wedge p = 0,$$

so (i) implies that $e - e \wedge f \precsim p^\perp$. Hence we get $\tau(e - e \wedge f) < \varepsilon$ for any $\varepsilon > 0$, so $e = e \wedge f$. Similarly, we get $f = e \wedge f$. Q.E.D.

Lemma 2.7. *Given two densely defined closed operators A and B affiliated with* $\mathcal{M}$, *if for any* $\varepsilon > 0$, *there exists* $p \in \mathrm{Proj}(\mathcal{M})$ *with* $\tau(p^\perp) < \varepsilon$ *such that*

$$p\mathfrak{H} \cap \mathfrak{D}(A) \cap A^{-1}(p\mathfrak{H}) = p\mathfrak{H} \cap \mathfrak{D}(B) \cap B^{-1}(p\mathfrak{H}) = \mathfrak{D},$$
$$A\xi = B\xi \quad \textit{for every } \xi \in \mathfrak{D},$$

then $A = B$. *In particular, if* $p\mathfrak{H} \subset \mathfrak{D}(A) \cap \mathfrak{D}(B)$ *and* $Ap = Bp$ *for some* $p \in \mathrm{Proj}(\mathcal{M})$ *with* $\tau(p^\perp) < \varepsilon$, *then* $A = B$.

PROOF: Consider $\tilde{\mathcal{M}} = \mathcal{M} \otimes M_2(\mathbf{C})$ on $\tilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}$ and the trace $\tilde{\tau}$ defined by

$$\tilde{\tau}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \tau(a) + \tau(b).$$

It follows that $\tilde{\tau}$ is a faithful semi-finite normal trace on $\tilde{\mathcal{M}}$. Let $g(A)$ and $g(B)$ be the projections of $\tilde{\mathfrak{H}}$ onto the graphs of A and B respectively. The direct computation shows that

$$g(A) = \begin{pmatrix} (1 + A^*A)^{-1} & A^*(1 + AA^*)^{-1} \\ A(1 + A^*A)^{-1} & AA^*(1 + AA^*)^{-1} \end{pmatrix}. \tag{10}$$

Hence $g(A)$ and $g(B)$ both belong to $\tilde{M}$. Now, set

$$\tilde{p} = p \otimes 1 = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \in \tilde{\mathcal{M}}.$$

Then we have $\tilde{\tau}(\tilde{p}^{\perp}) = 2\tau(p^{\perp}) < 2\varepsilon$. The assumption on A and B means here that $g(A) \wedge \tilde{p} = g(B) \wedge \tilde{p}$. The previous lemma then shows that $g(A) = g(B)$, hence the conclusion. Q.E.D.

We are now ready to complete the proof of Theorem 2.5.

PROOF OF THEOREM 2.5:

(i) If $a \in \mathfrak{M}(\mathcal{M})$, then we can find a sequence $\{a_n\}$ in $\mathcal{M}$ which converges to a in the measure convergence. By Theorem 2.2, $\{a_n\xi\}$ converges to $a\xi \in \mathfrak{M}(\mathfrak{H})$ in measure. If $\xi \in \mathfrak{D}(M(a))$, then $\{a_n\xi\}$ converges to $a\xi = M(a)\xi \in \mathfrak{H}$ in measure. If $u \in \mathcal{U}(\mathcal{M}')$, then $a_n u\xi \to au\xi$ in measure and $a_n u\xi = ua_n\xi$. Since operators in $\mathcal{M}'$ are continuous in the measure topology, we get the measure convergence: $ua_n\xi \to uM(a)\xi$. Thus we have $u\xi \in \mathfrak{D}(M(a))$ and $M(a)u\xi = uM(a)\xi$, so that $M(a)$ is affiliated with $\mathcal{M}$. Suppose now that $\{\xi_n\} \subset \mathfrak{D}(M(a))$, $\xi = \lim \xi_n$ and $\eta = \lim M(a)\xi_n$. Since $\xi_n \to \xi$ in measure, $a\xi_n \to a\xi \in \mathfrak{M}(\mathfrak{H})$ in measure. The convergence of $M(a)\xi_n$ to η implies the measure convergence. The separation property of the measure topology in $\mathfrak{M}(\mathfrak{H})$ yields that $\eta = a\xi$. Thus $\xi \in \mathfrak{D}(M(a))$ and $\eta = M(a)\xi$, which means that $M(a)$ is closed. Finally, if $ap \in \mathcal{M}$, $p \in \mathrm{Proj}(\mathcal{M})$, then $p\mathfrak{H} \subset \mathfrak{D}(M(a))$, so that Lemma 2.3.(ii) guarantees the density of $\mathfrak{D}(M(a))$ in $\mathfrak{H}$.

Suppose that A is a closed operator affiliated with $\mathcal{M}$ and $A \supset M(a)$. If $ap \in \mathcal{M}$, $p \in \mathrm{Proj}(\mathcal{M})$, then $Ap = ap = M(a)p \in \mathcal{M}$. By Lemma 2.7, we have $A = M(a)$. Thus $M(a)$ is maximal.

(ii) Let $\varepsilon > 0$. Choose $p \in \mathrm{Proj}(\mathcal{M})$ by Lemma 2.3.(ii) so that $a^*p \in \mathcal{M}$ and $\tau(p^{\perp}) < \varepsilon$. Then $p\mathfrak{H} \subset \mathfrak{D}(M(a^*))$ and $a^*p = M(a^*)p$. From the spectral analysis of the absolute value $|M(a)|$ of $M(a)$, it follows that there exists an increasing sequence $\{q_n\}$ in $\mathrm{Proj}(\mathcal{M})$ with $1 = \lim q_n$ such that $M(a)q_n = aq_n \in \mathcal{M}$ and $\xi \in \mathfrak{H}$ belongs to $\mathfrak{D}(M(a))$ if and only if $M(a)q_n\xi$ converges, and $M(a)\xi = \lim M(a)q_n\xi$. Now if $\xi \in \mathfrak{D}(M(a))$ and $\eta \in p\mathfrak{H}$, then we have

$$\begin{aligned}\big(M(a)\xi \mid \eta\big) &= \lim\big(aq_n\xi \mid \eta\big) = \lim\big(\xi \mid (q_n a)^* p\eta\big) = \lim\big(\xi \mid q_n(a^*p)\eta\big)\\ &= \lim\big(q_n\xi \mid a^*p\eta\big) = \big(\xi \mid a^*\eta\big) = \big(\xi \mid M(a^*)\eta\big).\end{aligned}$$

Hence $p\mathfrak{H} \subset \mathfrak{D}(M(a)^*)$ and $M(a)^*p = M(a^*)p$. Lemma 2.7 now implies that $M(a)^* = M(a^*)$.

Trivially, we have

$$M(a) + M(b) \subset M(a+b) \qquad \text{and} \qquad M(a)M(b) \subset M(ab).$$

Thus, $M(a) + M(b)$ and $M(a)M(b)$ are both preclosed. Choose $p, q \in \mathrm{Proj}(\mathcal{M})$ with $\tau(p^{\perp}) < \varepsilon$ and $\tau(q^{\perp}) < \varepsilon$ such that $ap \in \mathcal{M}$ and $bq \in \mathcal{M}$. Put $r = p \wedge q$. Then we get $\tau(r^{\perp}) < 2\varepsilon$ and

$$r\mathfrak{H} \subset \mathfrak{D}\big(M(a)\big) \cap \mathfrak{D}\big(M(b)\big),$$

$$\big[M(a) + M(b)\big]r = ar + br = M(a+b)r.$$

Thus, Lemma 2.7 implies that $M(a+b) = \overline{M(a) + M(b)}$. With p_b given by applying (7) to b, we put $s = q \wedge p_b$. By (8), we have $\tau(s^{\perp}) < 2\varepsilon$, and furthermore

$$s\mathfrak{H} \subset \mathfrak{D}\big(M(a)M(b)\big), \qquad M(a)M(b)s = apbs = M(ab)s.$$

Thus, we obtain $\overline{M(a)M(b)} = M(ab)$ by Lemma 2.7.

(iii) Put $a_n = Ap_n \in \mathcal{M}$. Then $\{a_n\}$ is a Cauchy sequence in measure, so that it converges to $a \in \mathfrak{M}(\mathcal{M})$ in measure. If $\xi \in \mathfrak{D}$, then for a sufficiently large n we have $a_n\xi = a_{n+1}\xi = \cdots = A\xi$, so that $\{a_n\xi\}$ converges to $A\xi$ strongly. On the other hand, $\{a_n\xi\}$ also converges to $a\xi$ in measure. Hence $a\xi = A\xi \in \mathfrak{H}$ and $\xi \in \mathfrak{D}\big(M(a)\big)$, so $A\xi = M(a)\xi$. Therefore, we get $M(a) \supset A$. By Lemma 2.7, we conclude $M(a) = \bar{A}$. Q.E.D.

Definition 2.8. When $\{\mathcal{M}, \mathfrak{H}\}$ and τ are given as before, a closed operator A of the form $A = M(a),\ a \in \mathfrak{M}(\mathcal{M})$, is said to be *$\tau$-measurable*.

By Theorems 2.2 and 2.5, the measurable operators form a *-algebra.

Corollary 2.9. *Let A be a densely defined closed operator affiliated with $\mathcal{M}$ and*

$$A = uH, \quad H = \int_0^\infty \lambda \, \mathrm{d}e(\lambda)$$

be its polar decomposition and the spectral decomposition of the absolute value. A necessary and sufficient condition for A to be τ-measurable is that

$$\lim_{\lambda\to\infty} \tau\big(e(\lambda)^{\perp}\big) = 0. \tag{11}$$

This condition is equivalent to saying that

$$\tau\big(e(\lambda)^{\perp}\big) < +\infty \quad \textit{for large } \lambda > 0. \tag{12}$$

The proof is straightforward. We leave it to the reader.

Corollary 2.10. *In the algebra $\mathfrak{M}(\mathcal{M})$, let*

$$\mathfrak{M}(\mathcal{M})_+ = \big\{ a^*a : a \in \mathfrak{M}(\mathcal{M}) \big\}. \tag{13}$$

Then $\mathfrak{M}(\mathcal{M})_+$ is a pointed convex cone and each element a of $\mathfrak{M}(\mathcal{M})_+$ has a unique square root in $\mathfrak{M}(\mathcal{M})_+$ denoted by $a^{1/2}$. Every $a \in \mathfrak{M}(\mathcal{M})$ has the polar decomposition

$$a = u|a|, \quad \textit{with} \quad |a| = (a^*a)^{\frac{1}{2}}.$$

The proof may be carried out by considering $\{ M(a) : a \in \mathfrak{M}(\mathcal{M}) \}$. We call $|a| = (a^*a)^{1/2}$ the *absolute value* of $a \in \mathfrak{M}(\mathcal{M})$.

Definition 2.11. The algebra $\mathfrak{M}(\mathcal{M})$ is called the *τ-measurable operator algebra*. If $\mathcal{M}$ acts on $\mathfrak{H}$, then $\{ M(a) : a \in \mathfrak{M}(\mathcal{M}) \}$ is also called the *τ-measurable operator algebra*. When we need to distinguish them, we call the former the *abstract* τ-measurable operator algebra and the latter the *concrete* τ-measurable operator algebra.

For the sum and the product of two measurable operators A and B, we use the closures of the algebraic sum and product. When we emphasize this fact, we write them as

$$A \dotplus B, \qquad A \cdot B$$

and call them the *strong* sum and the *strong* product respectively.

We now extend the trace τ on $\mathcal{M}_+$ to $\mathfrak{M}(\mathcal{M}_+)$ as follows:

$$\tau(h) = \lim_{\varepsilon \to 0} \tau\big(h\,(1 + \varepsilon h)^{-1}\big), \quad h \in \mathfrak{M}(\mathcal{M})_+. \tag{14}$$

For each $x \in \mathfrak{M}(\mathcal{M})$, we define

$$x_\varepsilon = x\big(1 + \varepsilon |x|\big)^{-1} \in \mathcal{M}, \quad \varepsilon > 0. \tag{15}$$

As before, we consider $\mathfrak{m}_\tau$ and $\mathfrak{n}_\tau$. The trace property of τ, $\tau(x^*x) = \tau(xx^*)$, implies that $\mathfrak{m}_\tau$ and $\mathfrak{n}_\tau$ are both ideals of $\mathcal{M}$. By the polarization identity, we have

$$\tau(xy) = \tau(yx), \quad x, y \in \mathfrak{n}_\tau. \tag{16}$$

By the polar decomposition, we know that every $a \in \mathfrak{m}_\tau$ is of the form $a = xy$, $x, y \in \mathfrak{n}_\tau$, so that for every $b \in \mathcal{M}$, we have

$$\tau(ab) = \tau(xyb) = \tau\big(x(yb)\big) = \tau\big((yb)x\big) = \tau\big(y(bx)\big) = \tau(bxy) = \tau(ba).$$

so we get

$$\tau(ab) = \tau(ba), \quad a \in \mathfrak{m}_\tau, \quad b \in \mathcal{M}. \tag{16$'$}$$

For $x \in \mathfrak{m}_\tau$ and $y \in \mathcal{M}$, let $x = u|x|$, $y = v|y|$ and $xy = w|xy|$ be the polar decompositions. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\big|\tau(xy)\big| &= \Big|\tau\big(u\,|x|^{\frac12}\,|x|^{\frac12}\,v\,|y|^{\frac12}\,|y|^{\frac12}\big)\Big| = \Big|\tau\big(|y|^{\frac12}\,u\,|x|^{\frac12}\,|x|^{\frac12}\,v\,|y|^{\frac12}\big)\Big| \\
&\le \tau\big(|y|^{\frac12}\,u\,|x|\,u^*|y|^{\frac12}\big)^{\frac12}\;\tau\big(|y|^{\frac12}\,v^*|x|\,v\,|y|^{\frac12}\big)^{\frac12} \\
&= \tau\big(|x^*||y|\big)^{\frac12}\;\tau\big(|x||y^*|\big)^{\frac12}.
\end{aligned}$$

Namely, we have the inequality:

$$\big|\tau(xy)\big|^2 \le \tau\big(|x^*||y|\big)\tau\big(|x||y^*|\big), \quad x \in \mathfrak{m}_\tau, \quad y \in \mathcal{M}. \tag{17}$$

The right hand side is further bounded by the following:

$$\leq \|y\|\tau\big(|x^*|\big)\|y\|\tau\big(|x|\big) = \|y\|^2\tau\big(|x|\big)^2;$$

thus we get

$$\big|\tau(xy)\big| \leq \|y\|\tau\big(|x|\big), \quad x \in \mathfrak{m}_\tau, \quad y \in \mathcal{M}. \tag{18}$$

Hence each $x \in \mathfrak{m}_\tau$ gives rise to an element τ_x of $\mathcal{M}_*$: $y \in \mathcal{M} \mapsto \tau_x(y) = \tau(xy) \in \mathbf{C}$.

Lemma 2.12.

(i) *Each $x \in \mathfrak{m}_\tau$ gives rise to $\tau_x \in \mathcal{M}_*$ by the formula:*

$$\tau_x(y) = \tau(xy), \quad x \in \mathfrak{m}_\tau, \quad y \in \mathcal{M}, \tag{19}$$

whose norm is given by:

$$\|\tau_x\| = \tau\big(|x|\big), \quad x \in \mathfrak{m}_\tau. \tag{20}$$

(ii) *The set*

$$L^1(\mathcal{M}, \tau) = \Big\{ x \in \mathfrak{M}(\mathcal{M}) : \tau\big(|x|\big) < +\infty \Big\}$$

is a two sided $\mathcal{M}$-submodule of $\mathfrak{M}(\mathcal{M})$ and the function:

$$\|x\|_1 = \tau\big(|x|\big), \quad x \in L^1(\mathcal{M}, \tau)$$

is a complete norm of $L^1(\mathcal{M}, \tau)$ with respect to which $\mathfrak{m}_\tau$ is a dense subspace.

(iii) *We can extend τ on $\mathfrak{m}_\tau$ to $L^1(\mathcal{M}, \tau)$ continuously to a linear functional and the bilinear form:*

$$(x, y) \in \mathcal{M} \times L^1(\mathcal{M}, \tau) \mapsto \tau(xy) \in \mathbf{C}$$

identifies $L^1(\mathcal{M}, \tau)$ with the predual $\mathcal{M}_*$.

PROOF:

(i) We have proved $\|\tau_x\| \leq \tau\big(|x|\big)$ for $x \in \mathfrak{m}_\tau$. By the polar decomposition $x = u|x|$, we have

$$\tau\big(|x|\big) = \tau(u^*x) = \tau(xu^*) = \tau_x(u^*),$$

so that we have $\|\tau_x\| \geq \tau\big(|x|\big)$. Hence $\|x\|_1 = \tau\big(|x|\big)$ gives the norm of $\tau_x \in \mathcal{M}_*$ for $x \in \mathfrak{m}_\tau$.

(ii) and (iii): Fix an $x \in \mathfrak{M}(\mathcal{M})$ and let $|x| = \int_0^\infty \lambda \, \mathrm{d}e(\lambda)$ be the spectral decomposition of $|x|$. Consider the abelian von Neumann subalgebra $\mathcal{A}$ generated by $\big\{e(\lambda)\big\}$. If $x \in L^1(\mathcal{M}, \tau)$, then $\tau\big(|x|_\varepsilon\big) \nearrow \tau\big(|x|\big) < +\infty$ as $\varepsilon \searrow 0$. This means that the increasing function: $\lambda \in \mathbf{R}_+ \mapsto \tau\big(e(\lambda)\big) \in [0, +\infty]$ gives rise to a measure μ on the open half line $\mathbf{R}_+^* =]0, +\infty[$. Hence as $\varepsilon \searrow 0$ we have

$$\tau\big(|x|_\varepsilon\big) = \int_0^\infty \frac{\lambda}{1 + \varepsilon\lambda} \, \mathrm{d}\mu(\lambda) \nearrow \tau\big(|x|\big) = \int_0^\infty \lambda \, \mathrm{d}\mu(\lambda) < +\infty.$$

Therefore, we get

$$\big\||x| - |x|_\varepsilon\big\|_1 \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and

$$\big\|x - x_\varepsilon\big\|_1 = \tau(|x - x_\varepsilon|) = \tau(|x| - |x|_\varepsilon) = \|\,|x| - |x|_\varepsilon\|_1 \to 0.$$

Thus we have

$$\lim_{\varepsilon\to 0}\big\|x - x_\varepsilon\big\|_1 = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\|x_\varepsilon\|_1 = \|x\|_1.$$

Therefore, x is approximated by $x_\varepsilon \in \mathfrak{m}_\tau$. This means that $L^1(\mathcal{M}, \tau)$ can be viewed as a subspace of the completion of the normed space $\{\mathfrak{m}_\tau, \|\cdot\|_1\}$. Hence $\|\cdot\|_1$ is a norm on $L^1(\mathcal{M}, \tau)$ and τ can be extended continuously to $L^1(\mathcal{M}, \tau)$.

By Corollary VIII.3.6, every semi-finite normal weight φ on $\mathcal{M}$ is of the form $\varphi = \tau_h$ with a closed self-adjoint positive operator affiliated with $\mathcal{M}$. If $\varphi \in \mathcal{M}_*^+$, then $\tau(h) = \lim_{\varepsilon\to 0}\tau(h_\varepsilon) = \varphi(1) < +\infty$. With the spectral decomposition $h = \int_0^\infty \lambda \, de(\lambda)$, we have $\lambda e(\lambda)^\perp \le h$, so that $\tau\big(e(\lambda)^\perp\big)$ is finite. Thus h is τ-measurable and belongs to $L^1(\mathcal{M}, \tau)$. So $\mathcal{M}_*^+$ can be identified with $L^1(\mathcal{M}, \tau)_+$ under the correspondence $h \in L^1(\mathcal{M}, \tau)_+ \longleftrightarrow \tau_h \in \mathcal{M}_*^+$.

Consider the polar decomposition $x = u|x| = |x^*|u$ of $x \in \mathfrak{M}(\mathcal{M})$. Then we have $|x^*|_\varepsilon = u|x|_\varepsilon u^*$, so that

$$\tau\big(|x^*|\big) = \lim \tau\big(|x^*|_\varepsilon\big) = \lim \tau\big(|x|_\varepsilon\big) = \tau\big(|x|\big).$$

Hence $L^1(\mathcal{M}, \tau)$ is self-adjoint. For any unitary $a \in \mathcal{M}$ and $x \in L^1(\mathcal{M}, \tau)$, we have $|ax| = |x|$, so that $L^1(\mathcal{M}, \tau)$ is a left $\mathcal{M}$-module. The self-adjointness of $L^1(\mathcal{M}, \tau)$ then yields the two sided module property over $\mathcal{M}$.

Now, the polar decompositions in $\mathcal{M}_*$ and in $L^1(\mathcal{M}, \tau)$ allow us to identify them. Q.E.D.

Theorem 2.13.

(i) *The function* τ *on* $\mathfrak{M}(\mathcal{M})_+$ *enjoys the following properties:*

$$\left.\begin{aligned} \tau(a+b) &= \tau(a) + \tau(b), && a, b \in \mathfrak{M}(\mathcal{M})_+,\\ \tau(\lambda a) &= \lambda\tau(a), && \lambda \ge 0,\\ \tau(x^*x) &= \tau(xx^*), && x \in \mathfrak{M}(\mathcal{M}). \end{aligned}\right\} \tag{21}$$

(ii) *For* $1 \le p < +\infty$, *set*

$$\begin{aligned} \|x\|_p &= \tau\big(|x|^p\big)^{1/p}, \quad x \in \mathfrak{M}(\mathcal{M}),\\ L^p(\mathcal{M}, \tau) &= \big\{\, x \in \mathfrak{M}(\mathcal{M}) : \|x\|_p < +\infty \,\big\}. \end{aligned} \tag{22}$$

Then $L^p(\mathcal{M}, \tau)$ *is a Banach space in which* $\mathcal{M} \cap L^p(\mathcal{M}, \tau)$ *is dense. Furthermore,* $L^p(\mathcal{M}, \tau)$ *is invariant under the multiplications of* $\mathcal{M}$ *from both sides, and*

$$\|ax\|_p \le \|a\|\|x\|_p, \qquad \|xa\|_p \le \|a\|\|x\|_p \tag{23}$$

for each $a \in \mathcal{M}$, $x \in L^p(\mathcal{M}, \tau)$.

(iii) *The extended trace* τ *on* $L^1(\mathcal{M}, \tau)$ *identifies* $\mathcal{M}_*$ *with* $L^1(\mathcal{M}, \tau)$ *by the bilinear form:*

$$(x, y) \in \mathcal{M} \times L^1(\mathcal{M}, \tau) \mapsto \tau(xy) \in \mathbf{C}.$$

(iv) *If* $1/p + 1/q = 1$ *and* $p > 1$ *, then the product of* $L^p(\mathcal{M}, \tau)$ *and* $L^q(\mathcal{M}, \tau)$ *agrees with* $L^1(\mathcal{M}, \tau)$ *and we have the Hölder's inequality:*

$$\left|\tau(xy)\right| \leq \|x\|_p \|y\|_q, \quad x \in L^p(\mathcal{M}, \tau), \quad y \in L^q(\mathcal{M}, \tau). \tag{24}$$

Furthermore, $L^p(\mathcal{M}, \tau)$ *and* $L^q(\mathcal{M}, \tau)$ *are the conjugate space of each other.*

PROOF:

(i) Take a and b from $\mathfrak{M}(\mathcal{M})_+$. If $\tau(a) < +\infty$ and $\tau(b) < +\infty$ then a and b are both in $L^1(\mathcal{M}, \tau)$. The linearity of τ on $L^1(\mathcal{M}, \tau)$ implies that $\tau(a) + \tau(b) = \tau(a + b)$. If $\tau(a + b) < +\infty$, then $a \leq a + b$ and $b \leq a + b$, so $a_\varepsilon \leq (a + b)_\varepsilon$ and $b_\varepsilon \leq (a + b)_\varepsilon$. Thus we get

$$\tau(a) = \lim_{\varepsilon \to 0} \tau(a_\varepsilon) \leq \lim_{\varepsilon \to 0} \tau\big((a + b)_\varepsilon\big) = \tau(a + b) < +\infty,$$

similarly $\tau(b) < +\infty$. Therefore, $\tau(a) + \tau(b) = \tau(a + b)$.

The homogeneity of τ on $\mathfrak{M}(\mathcal{M})_+$ is trivial.

For any $x \in \mathfrak{M}(\mathcal{M})$, let $x = u|x|$ be the polar decomposition. Then we have $xx^* = ux^*xu^*$, so $u(x^*x)_\varepsilon u^* = (xx^*)_\varepsilon$. Hence

$$\tau(xx^*) = \lim_{\varepsilon \to 0} \tau\big((xx^*)_\varepsilon\big) = \lim_{\varepsilon \to 0} \tau\big(u(x^*x)_\varepsilon u^*\big) = \lim_{\varepsilon \to 0} \tau\big((x^*x)_\varepsilon\big) = \tau(x^*x).$$

(ii) Assume $p > 1$, since we have proven the case $p = 1$. Choose q so that $1/p + 1/q = 1$. Take $a, b \in \mathfrak{m}_\tau$ and $c, d \in \mathfrak{m}_\tau^+$. Assume that $c \geq \varepsilon s(c)$ and $d \geq \varepsilon s(d)$ for some $\varepsilon > 0$, where $s(c)$ and $s(d)$ are the support of c and d respectively. In this case, we can define $\mathfrak{m}_\tau$-valued entire functions: $\lambda \in \mathbf{C} \to c^\lambda$ and $\lambda \in \mathbf{C} \to d^\lambda$. For each x and y in $\mathcal{M}$ with $\|x\| \leq 1$ and $\|y\| \leq 1$, the function f:

$$f(\lambda) = \tau\big(xc^\lambda yd^{1-\lambda}\big), \quad \lambda \in \mathbf{C},$$

is entire, to which the Phragmen-Lindelöf theorem applies. Thus, we get

$$\left|\tau\big(xc^\sigma yd^{1-\sigma}\big)\right| \leq \sup_{\operatorname{Re} s=1} \left|\tau\big(xc^s yd^{1-s}\big)\right|^\sigma \sup_{\operatorname{Re} s=0} \left|\tau\big(xc^s yd^{1-c}\big)\right|^{1-\sigma}$$

for $0 \leq \sigma \leq 1$. By (18), the right hand side is bounded by $\|c\|_1^\sigma \|d\|_1^{1-\sigma}$. Now, consider the polar decomposition $a = u|a|$ and $b = v|b|$ and put $c = |a|^p$ and $d = |b|^q$, and further $\sigma = 1/q$. Then under the hypothesis for c and d, we have the Hölder type inequality:

$$\left|\tau(ab)\right| \leq \tau\big(|a|^p\big)^{1/p} \tau\big(|b|^q\big)^{1/q} = \|a\|_p \|b\|_q. \tag{25}$$

By passing to the limit, the above inequality holds for general pair $a, b \in \mathfrak{m}_\tau$.

Let $\mathfrak{m}_0 = \left\{ x \in \mathcal{M} : s_\ell(x) \in \mathfrak{m}_\tau \right\} \subset \mathfrak{m}_\tau$. It follows that $\mathfrak{m}_0$ is an ideal of $\mathcal{M}$ and dense in $L^1(\mathcal{M}, \tau)$. If $a \in \mathfrak{m}_0$, then we put

$$b = \|a\|_p^{-p/q} |a|^{p-1} u^* \in \mathfrak{m}_0,$$

and obtain $\tau(ab) = \|a\|_p$ and $\|b\|_q = 1$. Thus, we obtain

$$\|a\|_p = \sup\left\{ \left|\tau(ab)\right| : b \in \mathfrak{m}_\tau,\ \|b\|_q \le 1 \right\}, \quad a \in \mathfrak{m}_0. \tag{26}$$

For a general $a \in \mathfrak{m}_\tau$, we have

$$\begin{aligned}
\|a\|_p &= \sup\left\{ \|ea\|_p : e \in \mathrm{Proj}(\mathcal{M}) \cap \mathfrak{m}_\tau \right\} \\
&= \sup\left\{ \left|\tau(eab)\right| : b \in \mathfrak{m}_\tau,\ \|b\|_q \le 1,\ e \in \mathrm{Proj}(\mathcal{M}) \cap \mathfrak{m}_\tau \right\} \\
&= \sup\left\{ \left|\tau(ab)\right| : b \in \mathfrak{m}_\tau,\ \|b\|_q \le 1 \right\}.
\end{aligned}$$

Therefore, we have the Minkowski type inequality:

$$\|a + b\|_p \le \|a\|_p + \|b\|_p, \quad a, b \in \mathfrak{m}_\tau.$$

We now want to show that the embedding of $\left\{\mathfrak{m}_\tau, \|\cdot\|_p\right\}$ into $\mathfrak{M}(\mathcal{M})$ is continuous, so that the identity map of $\mathfrak{m}_\tau$ is extendable to a continuous map of the completion E_p of $\left\{\mathfrak{m}_\tau, \|\cdot\|_p\right\}$ into $\mathfrak{M}(\mathcal{M})$. Let $\varepsilon, \delta > 0$. Suppose $\|x\|_p < \varepsilon\delta^{1/p}$, $x \in \mathfrak{m}_\tau$, and let $x = uh$ be the polar decomposition and $h = \int_0^\infty \lambda \, de(\lambda)$ the spectral decomposition. Then we have, since $\lambda e(\lambda)^\perp \le h$,

$$\varepsilon^p \tau\left(e(\varepsilon)^\perp\right) \le \tau(h^p) < \varepsilon^p \delta,$$

so that $\tau\left(e(\varepsilon)^\perp\right) < \delta$; hence $x \in N(2\varepsilon, \delta)$ because $\left\|xe(\varepsilon)\right\| \le \varepsilon$.

We prove next the injectivity of the extended map of E_p into $\mathfrak{M}(\mathcal{M})$. Suppose that a Cauchy sequence $\{x_n\}$ in $\left\{\mathfrak{m}_\tau, \|\cdot\|_p\right\}$ converges to zero in measure. For a fixed $\varepsilon > 0$, choose n_0 so that $\|x_n - x_m\|_p < \varepsilon$ for $n, m \ge n_0$. By (26) extended to $\mathfrak{m}_\tau$, there exists $y \in \mathfrak{m}_\tau$ with $\|y\|_q \le 1$ such that $\left|\tau(x_{n_0} y)\right| \ge \|x_{n_0}\|_p - \varepsilon$. Then we have for $n \ge n_0$

$$\left|\tau(x_n y)\right| \ge \left|\tau(x_{n_0} y)\right| - \left|\tau\left((x_n - x_{n_0})y\right)\right| \ge \|x_{n_0}\|_p - 2\varepsilon.$$

Therefore, if $x = \lim x_n \in E_p$ is not zero, then there exists $y \in \mathfrak{m}_\tau,\ \|y\|_q \le 1$, such that $\left\{\tau(x_n y)\right\}$ is bounded away from zero, i.e. $\left|\tau(x_n y)\right| \ge \delta > 0$ for $n \ge n_0$ and some $\delta > 0$. But the measure convergence of $\{x_n\}$ to zero implies that for any $\varepsilon > 0$ there exists $e \in \mathrm{Proj}(\mathcal{M})$ with $\|x_n e\| < \varepsilon$ and $\tau(e^\perp) < \varepsilon^q$. But

$$\begin{aligned}
\delta &\le \left|\tau(x_n y)\right| \le \left|\tau(x_n e y)\right| + \left|\tau(x_n e^\perp y)\right| \\
&\le \|x_n e\|_\infty \, \|y\|_1 + \|y\|_\infty \, \|x_n\|_p \, \|e^\perp\|_q \le \varepsilon\left(\|y_1\| + \|x_n\|_p \, \|y\|_\infty\right).
\end{aligned}$$

Since $\{\|x_n\|_p\}$ is bounded and y is fixed, we can choose $\varepsilon > 0$ small enough so that the above inequality does not hold. Therefore, $\|x_n\|_p$ must converge to zero, that is, the map of E_p into $\mathfrak{M}(\mathcal{M})$ is injective. So we embed E_p into $\mathfrak{M}(\mathcal{M})$.

Before going further, we observe the following:

$$\|axb\|_p \leq \|a\|\|b\|\|x\|_p, \quad x \in \mathfrak{m}_\tau, \quad a, b \in \mathcal{M}. \tag{27}$$

By the inequalities (26) and (25) extended to $\mathfrak{m}_\tau$, we have

$$\begin{aligned}
\|ax\|_p &= \sup\Big\{ \big|\tau(axy)\big| : y \in \mathfrak{m}_\tau,\ \|y\|_q \leq 1 \Big\} \\
&\leq \|a\| \sup\big\{ \|xy\|_1 : y \in \mathfrak{m}_\tau,\ \|y\|_q \leq 1 \big\} = \|a\|\|x\|_p;
\end{aligned}$$

$$\begin{aligned}
\|xb\|_p &= \sup\Big\{ \big|\tau(xby)\big| : y \in \mathfrak{m}_\tau,\ \|y\|_q \leq 1 \Big\} \\
&= \sup\Big\{ \big|\tau(yxb)\big| : y \in \mathfrak{m}_\tau,\ \|y\|_q \leq 1 \Big\} \\
&\leq \|b\| \sup\big\{ \|yx\|_1 : y \in \mathfrak{m}_\tau,\ \|y\|_q \leq 1 \big\} = \|b\|\|x\|_p.
\end{aligned}$$

Therefore, E_p is a two sided $\mathcal{M}$-module. We define the actions of $\mathcal{M}$ on $(E_q)^*$ by the following:

$$\big\langle x, afb \big\rangle = \big\langle bxa,\ f \big\rangle, \quad x \in E_q, \quad a, b \in \mathcal{M}, \quad f \in E_q^*.$$

Suppose that a bounded net $\{a_i\}$ in $\mathcal{M}$ converges to zero $*$-strongly. If $x = exe \in \mathfrak{m}_0$, $e \in \mathrm{Proj}(\mathcal{M}) \cap \mathfrak{m}_\tau$, then we have

$$\|a_i x\|_p = \tau\big((x^* a_i^* a_i x)^{p/2}\big)^{1/p} = \tau_e\big((x^* a_i^* a_i x)^{p/2}\big)^{1/p} \to 0;$$

$$\|x a_i\|_p = \tau\big((a_i^* x^* x a_i)^{p/2}\big)^{1/p} = \tau_e\big((a_i^* x^* x a_i)^{p/2}\big)^{1/p} \to 0.$$

Since $\mathfrak{m}_0$ is dense in E_p, the actions of $\mathcal{M}$ on E_p are *-strongly continuous on bounded sets.

Fix an $x \in L^p(\mathcal{M}, \tau)$. Let $x = uh$ and $h = \int_0^\infty \lambda\, \mathrm{d}e(\lambda)$ be the polar and spectral decompositions. Set $e_n = e(n) - e(1/n)$, $n = 2, 3, \ldots$. We then have $\lim_{n\to\infty} \|x - xe_n\|_p = 0$ and $xe_n \in \mathfrak{m}_\tau$. Hence x belongs to E_p, that is $L^p(\mathcal{M}, \tau) \subset E_p$.

Fix an $f \in (E_q)^*$. We define f^* by

$$\langle x,\ f^* \rangle = \overline{\langle x^*,\ f \rangle}, \quad x \in \mathfrak{m}_\tau.$$

It follows that $f^* \in (E_q)^*$ and $\|f\| = \|f^*\|$. Now, if $e \in \mathrm{Proj}(\mathcal{M}) \cap \mathfrak{m}_\tau$, then the map: $x \in \mathcal{M} \mapsto \langle x,\ fe \rangle = \langle ex,\ f \rangle$ is σ-strongly* continuous, so that fe belongs to $L^1(\mathcal{M}, \tau) \subset \mathfrak{M}(\mathcal{M})$. Furthermore, the polar decomposition of fe and the spectral decomposition of $|fe|$, together with the Hölder type inequality yield the estimate:

$$\begin{aligned}
\|fe\|_p &= \sup\Big\{ \big|\tau(xfe)\big| : x \in \mathfrak{m}_\tau,\ \|x\|_q \leq 1 \Big\} \\
&= \sup\Big\{ \big|\langle x,\ fe \rangle\big| : x \in \mathfrak{m}_\tau,\ \|x\|_q \leq 1 \Big\} = \|fe\|_{(E_q)^*} < +\infty.
\end{aligned}$$

Thus fe belongs to $L^p(\mathcal{M}, \tau)$. Hence f gives rise to a densely defined closed operator $M(fe)$ for each $e \in \mathrm{Proj}(\mathcal{M}) \cap \mathfrak{m}_\tau$. Let $\mathfrak{D}_A = \bigcup\big\{\mathfrak{D}(M(fe)) \cap e\mathfrak{H} : e \in \mathrm{Proj}(\mathcal{M}) \cap \mathfrak{m}_\tau\big\}$. Since $f(e_1 \vee e_2)e_1 = fe_1$ and $f(e_1 \vee e_2)e_2 = fe_2$ for e_1 and e_2, there exists a linear operator A on $\mathfrak{D}$ such that $A\xi = M(fe)\xi$ if $\xi \in \mathfrak{D}\big(M(fe)\big) \cap e\mathfrak{H}$. It is clear that A commutes with every unitary in $\mathcal{M}'$, so that A is affiliated with $\mathcal{M}$. Based on f^*, we then define B by

$$B\eta = M(f^*g)\eta \quad \text{for} \quad \eta \in \mathfrak{D}_B = \bigcup\Big\{\mathfrak{D}\big(M(f^*g)\big) \cap g\mathfrak{H} : g \in \mathrm{Proj}(\mathcal{M}) \cap \mathfrak{m}_\tau\Big\}.$$

We observe here that $(fe)^* = ef^*$ since $L^p(\mathcal{M}, \tau) \subset E_p \subset (E_q)^*$. For each $\xi \in \mathfrak{D}\big(M(fe)\big) \cap e\mathfrak{H}$ and $\eta \in \mathfrak{D}\big(M(f^*g)\big) \cap g\mathfrak{H}$ with $e, g \in \mathrm{Proj}(\mathcal{M}) \cap \mathfrak{m}_\tau$, we have

$$\big(A\xi \,\big|\, \eta\big) = \big(fe\xi \,\big|\, g\eta\big) = \big(gfe\xi \,\big|\, \eta\big) = \big((f^*g)^*e\xi \,\big|\, \eta\big) = \big(e\xi \,\big|\, f^*g\eta\big) = \big(\xi \,\big|\, B\eta\big).$$

Thus, $A^* \supset B$ and $B^* \supset A$. In particular, A is preclosed. Identifying A^{**} with f, we view f as a densely defined closed operator affiliated with $\mathcal{M}$. We now consider the polar and spectral decompositions $f = u|f|$ and $|f| = \int_0^\infty \lambda \, \mathrm{d}e(\lambda)$. If $\tau\big(e(\lambda)^\perp\big) = +\infty$, $\lambda > 0$, then $\lambda e(\lambda)^\perp \leq |f|$ implies $\tau\big(|f|^p\big) = +\infty$. So we have $\sup\big\{\big|\tau(x|f|)\big| : x \in \mathfrak{m}_\tau,\ \|x\|_q \leq 1\big\} = +\infty$. But $|f| = u^*f \in (E_q)^*$, which is a contradiction. Thus, we conclude that $\tau\big(e(\lambda)^\perp\big) < +\infty$ for $\lambda > 0$. Therefore, f is measurable and

$$\|f\|_p = \tau\big(|f|^p\big)^{1/p} = \sup\Big\{\big|\tau(xf)\big| : x \in \mathfrak{m}_\tau,\ \|x\|_q \leq 1\Big\} = \|f\|_{(E_q)^*}.$$

Thus, f belongs to $L^p(\mathcal{M}, \tau)$. We now conclude that $(E_q)^* = L^p(\mathcal{M}, \tau)$; hence $L^p(\mathcal{M}, \tau) = \big(L^q(\mathcal{M}, \tau)\big)^*$. In particular, $L^p(\mathcal{M}, \tau)$ is a Banach space. Q.E.D.

In the case $p = 2$, $L^2(\mathcal{M}, \tau)$ is a Hilbert space which can canonically be identified with the Hilbert space $\mathfrak{H}_\tau$.

We will use the following result later:

Theorem 2.14. *Let $\{\mathcal{M}, \tau\}$ be as before, and let E_a be the characteristic function of $[a, +\infty[$. Then we have*

$$\int_0^\infty \big\|E_{\sqrt{a}}(h)\big\|_2^2 \, \mathrm{d}a = \|h\|_2^2, \quad h \in L^2(\mathcal{M}, \tau)_+; \tag{28}$$

$$\int_0^\infty \big\|E_{\sqrt{a}}(h) - E_{\sqrt{a}}(k)\big\|_2^2 \, \mathrm{d}a \leq \|h - k\|_2 \|h + k\|_2, \quad h, k \in L^2(\mathcal{M}, \tau). \tag{29}$$

PROOF: Let $X = (\mathbf{R}_+ \setminus \{0\}) \times (\mathbf{R}_+ \setminus \{0\})$. Let H and K be the projections of X to the first and second components respectively, i.e. $H(x, y) = x$ and $K(x, y) = y$. Let h and k be positive elements of $L^2(\mathcal{M}, \tau)$, and consider the von Neumann algebra $\mathcal{A}$ generated by h and JkJ, or more precisely by their spectral projections. Then $\mathcal{A}$ is an abelian von Neumann algebra. Let $\mathcal{K}$ be the algebra of all continuous functions on $]0, +\infty[$ with compact support, and consider the algebraic tensor product $\mathcal{K} \otimes \mathcal{K}$ identified with a subalgebra of $C_\infty(X)$. Let $\mathfrak{A}$ be the subset of $L^2(\mathcal{M}, \tau)$ consisting of all operators of the form:

$$\sum_{i=1}^{n} f_i(h) g_i(k) = \sum_{i=1}^{n} \big(f_i \otimes g_i\big)(h, k).$$

In $\mathfrak{A}$, we define a new product:

$$\left(\sum_{i=1}^{n} f_i(h) g_i(k)\right) \cdot \left(\sum_{j=1}^{m} f'_j(h) g'_j(k)\right) = \sum_{i,j} f_i(h) f'_j(h) g_i(k) g'_j(k),$$

and a new involution:

$$\left(\sum_{i=1}^{n} f_i(h) g_i(k)\right)^{\circ} = \sum_{i=1}^{n} \bar{f}_i(h) \bar{g}_i(k).$$

It then follows that $\mathfrak{A}$ is a commutative Hilbert algebra and its left von Neumann algebra $\mathcal{R}_\ell(\mathfrak{A})$ is isomorphic to $\mathcal{A}$. In fact, the operator corresponding to $\sum_{i=1}^{n} f_i(h) g_i(k)$ is exactly the restriction of $\sum_{i=1}^{n} f_i(h) J \bar{g}_i(k) J$ on the closure $\mathfrak{K}$ of $\mathfrak{A}$. Therefore the faithful semifinite normal trace on $\mathcal{A}$ corresponding to the Hilbert algebra $\mathfrak{A}$ gives rise to a measure μ on X such that

$$\left\| \sum_{i=1}^{n} f_i(h) g_i(k) \right\|_2^2 = \int_X \left| \sum_{i=1}^{n} f_i(x) g_i(y) \right|^2 \mathrm{d}\mu(x, y) .$$

Furthermore, the subspace $\mathfrak{K}$ is identified with $L^2(X, \mu)$ under the identification: $\sum_{i=1}^{n} f_i \otimes g_i \in L^2(X, \mu) \longleftrightarrow \sum_{i=1}^{n} f_i(h) g_i(k)$. Therefore we get for any Borel functions f and g:

$$\| f(h) - g(k) \|_2^2 = \int_X \big| f(x) - g(y) \big|^2 \mathrm{d}\mu(x, y) = \| f \circ H - g \circ K \|_{L^2(X,\mu)}^2.$$

Now, we compute as in the last part of the last section:

$$\|h\|_2^2 = \tau\left(\int_0^\infty E_a(h^2)\, \mathrm{d}a \right) = \int_0^\infty \tau\big(E_{\sqrt{a}}(h)\big)\, \mathrm{d}a = \int_0^\infty \| E_{\sqrt{a}}(h) \|_2^2\, \mathrm{d}a ;$$

$$\begin{aligned}
\int_0^\infty \left\| E_{\sqrt{a}}(h) - E_{\sqrt{a}}(k) \right\|_2^2 \, \mathrm{d}a &= \int_0^\infty \int_X \left| E_{\sqrt{a}}(x) - E_{\sqrt{a}}(y) \right|^2 \mathrm{d}\mu(x, y) \, \mathrm{d}a \\
&= \int_X \int_0^\infty \left| E_{\sqrt{a}}(x) - E_{\sqrt{a}}(y) \right| \mathrm{d}a \, \mathrm{d}\mu(x, y) \\
&= \int_X \left| x^2 - y^2 \right| \mathrm{d}\mu(x, y) \\
&\le \left(\int_X |x - y|^2 \, \mathrm{d}\mu(x, y) \right)^{\frac{1}{2}} \left(\int_X |x + y|^2 \, \mathrm{d}\mu(x, y) \right)^{\frac{1}{2}} \\
&= \|h - k\|_2 \, \|h + k\|_2. \qquad \text{Q.E.D.}
\end{aligned}$$

Exercise IX.2

Assume that $\{\mathcal{M}, \tau\}$ is a von Neumann algebra equipped with a faithful semi-finite normal trace τ.

1) Show that if $\mathcal{M}$ is a factor of type I, then the τ-measure convergence in $\mathcal{M}$ is precisely the norm convergence, so that $\mathfrak{M}(\mathcal{M}) = \mathcal{M}$.

2) (**a**) Show that the identity map of $\mathcal{M}$ is continuous as a map from $\mathcal{M}$ with the norm topology to $\mathcal{M}$ with the τ-topology, i.e. the norm topology is stronger than the τ-topology.

(**b**) Show that if $\mathcal{M}$ has no non-zero minimal projection and τ is finite, then there is no non-zero linear functional on $\mathfrak{M}(\mathcal{M})$ which is continuous relative to the τ-topology.

(**c**) Show that if $\mathcal{M}$ has no non-zero minimal projection and τ is infinite, then a necessary and sufficient condition for a linear functional φ on $\mathcal{M}$ to be τ-continuous is that $\varphi(\mathfrak{m}_\tau) = 0$.

3) Let $\mathcal{M} = \ell^\infty(\mathbf{N})$ be the von Neumann algebra of bounded sequences and τ be a faithful semi-finite normal trace. Show that τ is finite if and only if every sequence is τ-measurable.

4) Let $\mathcal{M} = L^\infty(X, \mu)$ with $\{X, \mu\}$ a σ-finite measure space and τ be the trace on $\mathcal{M}$ corresponding to the integration relative to the measure μ. Show that τ-measurable operators are precisely those functions f on X such that (i) f is μ-measurable as a function on X and (ii) $\mu\big(\big\{ x \in X : |f(x)| \ge n \big\}\big) \to 0$ as $n \to \infty$.

5) Let $\mathfrak{H}$ be the underlying Hilbert space of $\mathcal{M}$. A subspace $\mathfrak{D}$ of $\mathfrak{H}$, not necessarily closed, is said to be τ*-dense* if there exists an increasing sequence $\{p_n\}$ in $\mathrm{Proj}(\mathcal{M})$ such that (i) $\lim_{n\to\infty}\tau(p_n^{\perp}) = 0$ and (ii) $p_n\mathfrak{H} \subset \mathfrak{D}$.

(**a**) Show that a subspace $\mathfrak{D}$ of $\mathfrak{H}$ is τ-dense if and only if for any $\varepsilon > 0$ there exists $p \in \mathrm{Proj}(\mathcal{M})$ such that $\tau(p^{\perp}) < \varepsilon$ and $p\mathfrak{H} \subset \mathfrak{D}$.

(**b**) Show that if $\{\mathfrak{D}_n\}$ is a sequence of τ-dense subspaces then $\mathfrak{D} = \bigcap_{n=1}^{\infty}\mathfrak{D}_n$ is also τ-dense. (Hint: Use (a). For each $\varepsilon > 0$, let $p_n \in \mathrm{Proj}(\mathcal{M})$ be such that $\tau(p_n^{\perp}) < \varepsilon/2^n$ and $p_n\mathfrak{H} \subset \mathfrak{D}_n$. Set $p = \bigwedge_{n=1}^{\infty} p_n$ and observe $\tau(p^{\perp}) \le \sum_{n=1}^{\infty}\tau(p_n^{\perp}) < \varepsilon$.)

6) A sequence $\{A_n\}$ in $\mathfrak{M}(\mathcal{M})$ is said to converge τ*-nearly everywhere* if there exists a τ-dense subspace $\mathfrak{D} \subset \mathfrak{H}$ such that $\mathfrak{D} \subset \bigcap_{n=1}^{\infty}\mathfrak{D}(A_n)$ and $A_n\xi$ converges in norm for every $\xi \in \mathfrak{D}$. Define $A\xi = \lim_{n\to\infty} A_n\xi$ for those $\xi \in \bigcap_{n=1}^{\infty}\mathfrak{D}(A_n)$ such that $\lim_{n\to\infty} A_n\xi$ exists.

(**a**) Show that A is affiliated with $\mathcal{M}$.

(**b**) Show that if $p\mathfrak{H} \subset \mathfrak{D}$, $p \in \mathrm{Proj}(\mathcal{M})$, then $\|A_n p\| < +\infty$ and $\|Ap\| < +\infty$. (Hint: Use the closed graph theorem for the boundedness of $A_n p$ and the uniform boundedness theorem for $\{A_n p\}$).

(**c**) Prove that A is τ-measurable.

(**d**) Show that if $\tau(e) < +\infty$, $e \in \mathrm{Proj}(\mathcal{M})$, then for any $\varepsilon > 0$ and $\delta > 0$ there exists N and $p \in \mathrm{Proj}(\mathcal{M}_e)$ such that $\tau(e-p) < \delta$ and $\left\|(A_n - A)p\right\| < \varepsilon$, $n \ge N$. (Hint: $\tau(e - e\wedge q) = \tau(e\vee q - q) \le \tau(q^{\perp})$ for those q with $q\mathfrak{H} \subset \mathfrak{D}$. Set $p_1 = e\wedge q$ to get p_1 with $\tau(e - p_1) < \delta/2$. Observe that $A_n p_1$ converges strongly to Ap_1. Apply Lemma II.4.12 to $A_n p_1$.)

(**e**) Show that if $\tau(e) < +\infty$, $e \in \mathrm{Proj}(\mathcal{M})$, then for any $\varepsilon > 0$, there exists $p \in \mathrm{Proj}(\mathcal{M}_e)$ such that

$$\tau(e-p) < \varepsilon \qquad \text{and} \qquad \lim_{n\to\infty}\left\|(A_n - A)p\right\| = 0\,.$$

(**f**) Let $\mathcal{M} = L^{\infty}(X,\mu)$ with $\{X,\mu\}$ a σ-finite measure space and τ the integration relative to μ. Let $\{A_n\}$ be a sequence of measurable operators and $\{f_n\}$ be the corresponding sequence of functions on X. Show that if $\{f_n\}$ converges μ-almost everywhere, then $\{A_n\}$ converges τ-nearly everywhere; conversely if $\{A_n\}$ converges τ-nearly everywhere, then some subsequence $\left\{f_{n_k}\right\}$ of $\{f_n\}$ converges almost everywhere, and the limit function does not depend on the choice of a convergent subsequence.

(**g**) Show that if $\mathcal{M}$ is non-atomic, i.e. does not admit a non-zero minimal projection then the τ-measure convergence does not imply the τ-nearly everywhere convergence. (Hint: If $\mathcal{M}$ is non-atomic, then $L^{\infty}(0,1)$, the algebra of essentially bounded functions on $[0,1]$ relative to the Lebesgue measure, can be embedded as a von Neumann subalgebra of a reduced subalgebra $\mathcal{M}_e$ along with the trace on $L^{\infty}(0,1)$ given by the Lebesgue measure. Consider the sequence $\{I_n\}$ of intervals defined by

$$I_{j+k(k+1)/2} = \left[\frac{j-1}{k+1}, \frac{j}{k+1}\right], \quad 1 \le j \le n+1,$$

and set

$$f_n = |I_n|^{-2} \chi_{I_n} .$$

Then $\{ f_n \}$ converges to zero in measure, but not τ-nearly everywhere.)

(**h**) Show that if $\mathcal{M} = L^\infty(0, 1)$ and τ is the integration relative to the Lebesgue measure, then the τ-nearly everywhere convergence does not imply the almost everywhere convergence. (Hint: In the hint for (g), set $f_n = \chi_{I_n}$. Then $\{ f_n \}$ converges strongly to zero, so that it converges to zero τ-nearly everywhere. But it does not converge to zero at any point of [0, 1].)

(**i**) Show that the τ-nearly everywhere convergence of self-adjoint measurable operators implies the resolvent convergence. (cf. A.6.(iii).)

7) For a τ-measurable operator T, set

$$\lambda_t(T) = \tau\Big(E_{]t,+\infty[}\big(|T|\big)\Big), \quad t \geq 0 ,$$

where $E_{]t,+\infty[}\big(|T|\big)$ is the spectral projection of $|T|$ corresponding to the open half line $]t, +\infty[$;

$$\mu_t(T) = \inf\big\{ \|Te\| : e \in \operatorname{Proj}(\mathcal{M}),\ \tau(e^\perp) \leq t \big\}.$$

Following the steps described below, prove that

$$\mu_t(T) = \inf\big\{ s \geq 0 : \lambda_s(T) \leq t \big\}; \quad \lambda_{\mu_t(T)}(T) \leq t, \quad t > 0. \qquad [517]$$

(**a**) First, observe that $s \mapsto \lambda_s(T)$ is continuous from the right. With $a = \inf\big\{ s \geq 0 : \lambda_s(T) \leq t \big\}$ and $E = E_{[0,a]}(|T|)$, prove

$$\|TE\| \leq a \quad \text{and} \quad \tau(1 - E) \leq t.$$

(**b**) Given $\varepsilon > 0$, let $E \in \operatorname{Proj}(\mathcal{M})$ such that $\tau(1 - E) \leq t$ and

$$\|TE\| < \mu_t(T) + \varepsilon = \alpha .$$

Show that $E \wedge E_{]\alpha,\infty[}\big(|T|\big) = 0$. (Hint: If $\xi \in E\mathfrak{H} \cap E_{]\alpha,\infty[}(|T|)\mathfrak{H}$, $\|\xi\| = 1$, then $(T^*T\xi \mid \xi) \geq \alpha^2 > \|T\xi\|^2 = (T^*T\xi \mid \xi)$.)

(**c**) Show that $\tau(E_{]\alpha,\infty[}\big(|T|\big)) \leq t$. (Hint: $\lambda_\alpha(T) = \tau(E_{]\alpha,\infty[}(|T|)) \leq t$, so $a \leq \alpha$.)

(**d**) Show that if T is a positive τ-measurable operator, then

$$\tau(T) = \int_0^\infty \mu_t(T)\, \mathrm{d}t; \qquad \mu_t\big(f(T)\big) = f\big(\mu_t(T)\big)$$

for any continuous increasing function f on $[0, \infty[$ with $f(0) \geq 0$;

$$\mu_t(T) \leq \mu_t(S); \qquad \tau\big(f(T)\big) \leq \tau\big(f(S)\big) \quad \text{if } 0 \leq T \leq S \text{ in addition.}$$

8) Let $\{\mathcal{M}, \mathfrak{H}\}$ be a semi-finite von Neumann algebra equipped with a faithful semi-finite normal trace τ. Let $\mathcal{N}$ be the commutant $\pi_\tau(\mathcal{M})'$ of $\mathcal{M}$ represented on $L^2(\mathcal{M}, \tau)$ and let $\tau_{\mathcal{N}}$ be the trace on $\mathcal{N}$ opposite to τ.

(**a**) Show that there exists a unique faithful semi-finite normal trace $\tau_{\mathfrak{H}}$ on the commutant $\pi_{\mathfrak{H}}(\mathcal{M})'$ of $\mathcal{M}$ on $\mathfrak{H}$ such that

$$\tau_{\mathfrak{H}}(yy^*) = \tau_{\mathcal{N}}(y^*y)$$

for every $y \in \mathcal{L}\big(L^2(\mathcal{M}, \tau), \mathfrak{H}\big)$ such that $xy = y\pi_\tau(x),\ x \in \mathcal{M}$.

(**b**) Observe that for every $\xi \in \mathfrak{H}$ there exists a unique $h_\xi \in L^2(\mathcal{M}, \tau)_+$ such that $(x\xi \mid \xi) = \big(xh_\xi \mid h_\xi\big)_\tau,\ x \in \mathcal{M}$. Show that each $\xi \in \mathfrak{H}$ gives rise to an operator: $\eta_\tau(x) \mapsto x\xi,\ x \in \mathfrak{n}_\tau$, which is preclosed.

(**c**) Let $\mathfrak{K}$ be another Hilbert space on which $\mathcal{M}$ acts faithfully and normally. Normalize the faithful semi-finite normal trace $\tau_{\mathfrak{K}}$ on the commutant $\{\mathcal{M}'_{\mathfrak{K}}, \mathfrak{K}\}$ of $\pi_{\mathfrak{K}}(\mathcal{M})$ according to (a). Show that if $y \in \mathcal{L}(\mathfrak{H}, \mathfrak{K})$ intertwines the actions of $\mathcal{M}$, i.e. $y\pi_{\mathfrak{H}}(x) = \pi_{\mathfrak{K}}(x)y,\ x \in \mathcal{M}$, then $\tau_{\mathfrak{H}}(y^*y) = \tau_{\mathfrak{K}}(yy^*)$. Set $\mathcal{I}_{\mathcal{M}}(\mathfrak{H}, \mathfrak{K}) = \big\{ y \in \mathcal{L}(\mathfrak{H}, \mathfrak{K}) : y\pi_{\mathfrak{H}}(x) = \pi_{\mathfrak{K}}(x)y,\ x \in \mathcal{M} \big\}$.

(**d**) Define the measure topology on $\mathcal{I}_{\mathcal{M}}(\mathfrak{H}, \mathfrak{K})$ by the family of a neighborhood system $\big\{ N(\varepsilon, \delta) : \varepsilon > 0,\ \delta > 0 \big\}$ of 0 given by:

$$N(\varepsilon, \delta) = \big\{ y \in \mathcal{I}_{\mathcal{M}}(\mathfrak{H}, \mathfrak{K}) : \|yp\| < \varepsilon \text{ and } \tau_{\mathfrak{H}}(p^\perp) < \delta \text{ for some } p \in \mathrm{Proj}(\mathcal{M}'_{\mathfrak{H}}) \big\}.$$

Prove the statements on $\mathcal{I}_{\mathcal{M}}(\mathfrak{H}, \mathfrak{K})$ corresponding to Theorem 2.2 with respect to the measure topology defined above. But the measure topology on both $\mathfrak{H}$ and $\mathfrak{K}$ should be referred to $\big\{\mathcal{M}'_{\mathfrak{H}}, \tau_{\mathfrak{H}}\big\}$ and $\big\{\mathcal{M}'_{\mathfrak{K}}, \tau_{\mathfrak{K}}\big\}$ respectively instead of $\mathcal{M}$. (Hint: Consider the direct sum $\widetilde{\mathfrak{H}} = L^2(\mathcal{M}, \tau) \oplus \mathfrak{H} \oplus \mathfrak{K},\ \pi_{\widetilde{\mathfrak{H}}} = \pi_\tau \oplus \pi_{\mathfrak{H}} \oplus \pi_{\mathfrak{K}}$ and $\mathcal{M}'_{\widetilde{\mathfrak{H}}} = \pi_{\widetilde{\mathfrak{H}}}(\mathcal{M})'$ and identify $\mathcal{I}_{\mathcal{M}}(\mathfrak{H}, \mathfrak{K}) = p_{\mathfrak{K}}\mathcal{M}'_{\widetilde{\mathfrak{H}}}p_{\mathfrak{H}}$ where $p_{\mathfrak{H}}$ and $p_{\mathfrak{K}}$ are the projections from $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}$ and $\mathfrak{K}$ respectively.)

(**e**) Let $\mathfrak{M}_{\mathcal{M}}(\mathfrak{H}, \mathfrak{K})$ be the completion of $\mathcal{I}_{\mathcal{M}}(\mathfrak{H}, \mathfrak{K})$ relative to the measure topology. Prove the statements on $\mathfrak{M}_{\mathcal{M}}(\mathfrak{H}, \mathfrak{K})$ corresponding to Theorem 2.5.

(**f**) Prove that each $\xi \in \mathfrak{H}$ gives rise to an element $R(\xi) \in \mathfrak{M}_{\mathcal{M}}\big(L^2(\mathcal{M}, \tau), \mathfrak{H}\big)$ such that $R(\xi)\eta_\tau(x) = x\xi,\ x \in \mathfrak{n}_\tau$.

(**g**) Prove that $R(\xi)^*R(\xi) \in L^1(\mathcal{N}, \tau_{\mathcal{N}})$.

(**h**) Prove that every $R \in \mathfrak{M}_{\mathcal{M}}\big(L^2(\mathcal{M}, \tau), \mathfrak{H}\big)$ with $\tau_{\mathcal{N}}(R^*R) < \infty$, called *square integrable*, corresponds uniquely to a vector $\xi \in \mathfrak{H}$ such that $R = R(\xi)$. Let $L^2_{\mathcal{M}}(\mathfrak{H}) = \big\{ R \in \mathfrak{M}_{\mathcal{M}}\big(L^2(\mathcal{M}, \tau), \mathfrak{H}\big) : \tau_{\mathcal{N}}(R^*R)^{1/2} = \|R\|_2 < \infty \big\}$.

(**i**) Prove that $\xi \in \mathfrak{H} \mapsto R(\xi) \in L^2_{\mathcal{M}}(\mathfrak{H})$ is a surjective isometry of $\mathfrak{H}$ onto $L^2_{\mathcal{M}}(\mathfrak{H})$.

(**j**) Observe that on the Hilbert space $L^2_{\mathcal{M}}(\mathfrak{H}),\ \mathcal{M}'_{\mathfrak{H}}$ acts from the left and $\mathcal{N}$ acts from the right by operator multiplication.

(**k**) Observe that it is possible to do the same for $\mathcal{M}'_{\mathfrak{H}}$ so that $\mathcal{M}$ acts on $L^2_{\mathcal{M}'_{\mathfrak{H}}}(\mathcal{M}'_{\mathfrak{H}}, \tau_{\mathfrak{H}})$ from the left. This will be studied more in detail in the next section.

§ 3 Bimodules, Spatial Derivatives and Relative Tensor Products

We are going to consider a right action of a von Neumann algebra on a Hilbert space. To this end, we first set:

Definition 3.1.

(i) Given a von Neumann algebra $\mathcal{N}$, the *opposite* von Neumann algebra $\mathcal{N}^\circ$ means the von Neumann algebra obtained by reversing the product in $\mathcal{N}$, i.e. as a linear space equipped with the *-operation we take $\mathcal{N}$ to be $\mathcal{N}^\circ$, denote by x° the element in $\mathcal{N}^\circ$ corresponding to $x \in \mathcal{N}$ and then define the product in $\mathcal{N}^\circ$ by:

$$x^\circ y^\circ = (yx)^\circ, \quad x, y \in \mathcal{N}. \tag{1}$$

(ii) A right $\mathcal{N}$*-module* means a Hilbert space $\mathfrak{H}$ equipped with a normal anti-representation, $\pi'_{\mathfrak{H}}$, of $\mathcal{N}$ on $\mathfrak{H}$, equivalently a Hilbert space equipped with a normal representation of $\mathcal{N}^\circ$. To avoid uninteresting notational complexity, we consider only *faithful* right $\mathcal{N}$-modules $\mathfrak{H}$ in the sense that $\pi'_{\mathfrak{H}}(x) \neq 0$ for every non-zero $x \in \mathcal{N}$. We denote the right $\mathcal{N}$-module $\mathfrak{H}$ by $\mathfrak{H}_{\mathcal{N}}$ to emphasize that $\mathfrak{H}$ is being viewed as a right $\mathcal{N}$-module.

(iii) For a pair $\mathcal{M}$, $\mathcal{N}$ of von Neumann algebras, an $\mathcal{M}$-$\mathcal{N}$*-bimodule* means a Hilbert space $\mathfrak{H}$, denoted by ${}_{\mathcal{M}}\mathfrak{H}_{\mathcal{N}}$, equipped with a normal representation of π of $\mathcal{M}$ on $\mathfrak{H}$ and a normal anti-representation π' of $\mathcal{N}$ on $\mathfrak{H}$ such that $\pi(\mathcal{M})$ and $\pi'(\mathcal{N})$ commute. We write:

$$x\xi y = \pi(x)\pi'(y)\xi, \quad x \in \mathcal{M}, \quad y \in \mathcal{N}. \tag{2}$$

The commutativity of $\pi(\mathcal{M})$ and $\pi'(\mathcal{N})$ is equivalent to the associativity: $x(\xi y) = (x\xi)y$, $x \in \mathcal{M}$ and $y \in \mathcal{N}$. Once again, we are going to consider only faithful modules. In the case that $\pi(\mathcal{M})' = \pi'(\mathcal{N})$, the bimodule ${}_{\mathcal{M}}\mathfrak{H}_{\mathcal{N}}$ is said to be *full*.

Let us fix von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$. If $\mathfrak{H}$ is an $\mathcal{M}$-$\mathcal{N}$-bimodule, then its Banach space dual $\bar{\mathfrak{H}}$ is canonically an $\mathcal{N}$-$\mathcal{M}$-bimodule by the action:

$$x\bar{\xi}y = \overline{y^*\xi x^*}, \quad x \in \mathcal{M}, \quad y \in \mathcal{N}, \tag{3}$$

where $\bar{\xi}$ means the vector in $\mathfrak{H}$ corresponding to $\xi \in \mathfrak{H}$ by the pairing: $\langle \eta, \bar{\xi} \rangle = (\eta \mid \xi)$, $\eta \in \mathfrak{H}$ and $\bar{\xi} \in \bar{\mathfrak{H}}$. This $\mathcal{M}$-$\mathcal{N}$-module $\bar{\mathfrak{H}}$ will be called the *conjugate* bimodule or the bimodule *dual* to the original bimodule $\mathfrak{H}$.

Of special interest is a von Neumann algebra in a standard form. Let us fix a faithful semi-finite normal weight ψ on $\mathcal{N}$ and consider the standard form, which we will denote by $\left\{L^2(\mathcal{N}),\ L^2(\mathcal{N})_+,\ J\right\}$. The right action of $\mathcal{N}$ is given by:

$$\xi x = Jx^*J\xi, \quad x \in \mathcal{N}. \tag{4}$$

Thus we obtain an $\mathcal{N}$-$\mathcal{N}$-bimodule $L^2(\mathcal{N})$, which will be called the *standard bimodule*. Sometimes, we write ξ^* for $J\xi$, $\xi \in L^2(\mathcal{N})$. We state here the following easy but important proposition:

Proposition 3.2. *For a von Neumann algebra* $\mathcal{N}$, *the standard bimodule* $L^2(\mathcal{N})$ *is self-dual under the correspondence:* $\xi^* \longleftrightarrow \bar{\xi}$, $\xi \in L^2(\mathcal{N})$.

The proof is straightforward, so it will be left to the reader.

With ψ a faithful semi-finite normal weight on $\mathcal{N}$, the left action on $L^2(\mathcal{N})$ is nothing but the semi-cyclic representation π_ψ on $\mathfrak{H}_\psi$. The right action π'_ψ of $\mathcal{N}$ is then given by:

$$\pi'_\psi(x) = J\pi_\psi(x^*)J, \quad x \in \mathcal{N}. \tag{5}$$

It then follows from Lemma VIII.3.18 that the right action of $\mathcal{N}$ is given by the following:

$$\eta_\psi(x)b = \eta_\psi\big(x\sigma^\psi_{-\frac{i}{2}}(b)\big), \quad x \in \mathfrak{n}_\psi, \quad b \in \mathfrak{D}\big(\sigma^\psi_{-\frac{i}{2}}\big). \tag{6}$$

This twist on the right action suggests that we write $x\psi^{1/2}$ for $\eta_\psi(x)$, $x \in \mathfrak{n}_\psi$, viewing $\psi^{1/2}$ as an infinitely long vector "in" $L^2(\mathcal{N})$. Then the formula (6) is simply:

$$x\psi^{\frac{1}{2}}b = \big(x\psi^{\frac{1}{2}}b\psi^{-\frac{1}{2}}\big)\psi^{\frac{1}{2}} = \big(x\sigma^\psi_{-\frac{i}{2}}(b)\big)\psi^{\frac{1}{2}}, \quad x \in \mathfrak{n}_\psi, \quad b \in \mathfrak{D}\big(\sigma^\psi_{-\frac{i}{2}}\big). \tag{6'}$$

We then introduce a new notation:

$$\eta'_\psi(x) = J\eta_\psi(x^*), \quad x \in \mathfrak{n}^*_\psi, \tag{7}$$

which can be written as $\psi^{1/2}x$, $x \in \mathfrak{n}^*_\psi$. This new map $\eta'_\psi : x \in \mathfrak{n}^*_\psi \mapsto \eta'_\psi(x) \in L^2(\mathcal{N})$ allows us to write (5) in a simple form:

$$\pi'_\psi(b)\eta'_\psi(x) = \eta'_\psi(xb) = \eta'_\psi(x)b, \quad x \in \mathfrak{n}^*_\psi, \quad b \in \mathcal{N}. \tag{8}$$

We now consider a general right $\mathcal{N}$-module $\mathfrak{H}$. First, we set for a pair $\{\mathfrak{H}_1, \mathfrak{H}_2\}$ of right $\mathcal{N}$-modules:

$$\mathcal{L}\big(\mathfrak{H}_{1,\mathcal{N}}, \mathfrak{H}_{2,\mathcal{N}}\big) = \big\{ T \in \mathcal{L}(\mathfrak{H}_1, \mathfrak{H}_2) : T(\xi y) = (T\xi)y, \ y \in \mathcal{N} \big\} \tag{9}$$

and for $\mathcal{L}\big(\mathfrak{H}_\mathcal{N}, \mathfrak{H}_\mathcal{N}\big)$, we write simply $\mathcal{L}(\mathfrak{H}_\mathcal{N})$. With this notation, the right $\mathcal{N}$-module $\mathfrak{H}$ becomes canonically an $\mathcal{L}(\mathfrak{H}_\mathcal{N})$-$\mathcal{N}$-bimodule. Also, we note that $\mathcal{L}\big(L^2(\mathcal{N})_\mathcal{N}\big) = \mathcal{N}$, a fact that will be used heavily later. For the pair $\{\mathfrak{H}_1, \mathfrak{H}_2\}$, we consider the direct sum right $\mathcal{N}$-module: $\mathfrak{H}_\mathcal{N} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and let e_1 and e_2 be the projection of $\mathfrak{H}$ to $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively. Then we have $\mathcal{L}\big(\mathfrak{H}_{1,\mathcal{N}}, \mathfrak{H}_{2,\mathcal{N}}\big) = e_2\mathcal{L}(\mathfrak{H}_\mathcal{N})e_1$.

Now let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra. We want to study the relation between a semi-finite normal weight φ on $\mathcal{M}$ and a faithful semi-finite normal weight ψ' on $\mathcal{M}'$. Set $\mathcal{N} = (\mathcal{M}')^\circ$, which allows us to view $\mathfrak{H}$ as an $\mathcal{M}$-$\mathcal{N}$-bimodule. Let ψ be the weight on $\mathcal{N}$ defined by $\psi(y) = \psi'(y^\circ)$, $y \in \mathcal{N}$. We first relate the von Neumann algebra $\{\mathcal{M}, \mathfrak{H}\}$ to the one in a standard form. Let $\tilde{\mathfrak{H}} = L^2(\mathcal{N}) \oplus \mathfrak{H}$ as a right $\mathcal{N}$-module. Let $\mathcal{R} = \mathcal{L}(\tilde{\mathfrak{H}}_\mathcal{N})$. It is then straightforward to observe that

$\mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}},\, \mathfrak{H}_{\mathcal{N}}\big) = f\mathcal{R}e$, where e and f are the projections of $\tilde{\mathfrak{H}}$ onto $L^2(\mathcal{N})$ and $\mathfrak{H}$ respectively. The semi-finite normal weights ψ on $\mathcal{N}$ and φ on $\mathcal{M}$ give rise to a semi-finite normal weight ρ on $\mathcal{R}$ by: $\rho(x) = \psi(exe) + \varphi(fxf),\ x \in \mathcal{R}$. We set

$$\left.\begin{aligned} \mathfrak{n}_\psi(\mathfrak{H}) &= f\big(\mathcal{R} \cap \mathfrak{n}_\rho\big)e = \Big\{ x \in \mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}},\, \mathfrak{H}_{\mathcal{N}}\big) : \psi(x^*x) < \infty \Big\}; \\ \mathfrak{D}(\mathfrak{H}, \psi) &= \Big\{ \xi \in \mathfrak{H} : \|\xi x\| \le C_\xi \big\|\eta'_\psi(x)\big\|,\ x \in \mathfrak{n}^*_\psi \text{ for some } C_\xi \ge 0 \Big\}. \end{aligned}\right\} \tag{10}$$

Each $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$ gives rise to an operator, denoted by $L_\psi(\xi)$, which belongs to $\mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}},\, \mathfrak{H}_{\mathcal{N}}\big)$, defined by the equation:

$$L_\psi(\xi)\eta'_\psi(x) = \xi x, \quad x \in \mathfrak{n}^*_\psi, \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi). \tag{11}$$

Lemma 3.3.

(i) *We have*

$$\mathfrak{n}_\psi(\mathfrak{H}) = \mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}},\, \mathfrak{H}_{\mathcal{N}}\big)\mathfrak{n}_\psi \quad \textit{and} \quad \mathfrak{D}(\mathfrak{H}, \psi) = \mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}},\, \mathfrak{H}_{\mathcal{N}}\big)\mathfrak{B}_\psi,$$

where $\mathfrak{B}_\psi = \eta_\psi(\mathfrak{n}_\psi)$ *in* $L^2(\mathcal{N})$.

(ii) *The map* $\eta_\psi : (x, y) \in \mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}\big) \times \mathfrak{n}_\psi \mapsto x\eta_\psi(y) \in \mathfrak{D}(\mathfrak{H}, \psi)$ *gives rise to a map, denoted by* η_ψ *again, from* $\mathfrak{n}_\psi(\mathfrak{H})$ *onto* $\mathfrak{D}(\mathfrak{H}, \psi)$ *such that*

$$\left.\begin{aligned} \eta_\psi(ax) &= a\eta_\psi(x), \quad a \in \mathcal{M}, \quad x \in \mathfrak{n}_\psi(\mathfrak{H}); \\ \eta_\psi\big(x\sigma^\psi_{-\frac{i}{2}}(b)\big) &= \eta_\psi(x)b, \quad x \in \mathfrak{n}_\psi(\mathfrak{H}), \quad b \in \mathcal{D}\big(\sigma^\psi_{-\frac{i}{2}}\big). \end{aligned}\right\} \tag{12}$$

(iii) $\mathfrak{D}(\mathfrak{H}, \psi)$ *is dense in* $\mathfrak{H}$.

(iv) *The maps* $L_\psi : \xi \in \mathfrak{D}(\mathfrak{H}, \psi) \mapsto L_\psi(\xi) \in \mathfrak{n}_\psi(\mathfrak{H})$ *and* $\eta_\psi : x \in \mathfrak{n}_\psi(\mathfrak{H}) \mapsto \eta_\psi(x) \in \mathfrak{D}(\mathfrak{H}, \psi)$ *are the inverse of each other.*

(v)
$$L_\psi\big(\xi\sigma^\psi_{\frac{i}{2}}(b)\big) = L_\psi(\xi)b, \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi), \quad b \in \mathcal{D}\big(\sigma^\psi_{\frac{i}{2}}\big). \tag{12$'$}$$

(vi) *With the semi-finite normal weight* $\bar{\psi}$ *on* $\mathcal{R}$ *defined by*

$$\bar{\psi}(x) = \psi(exe) = \rho(exe), \quad x \in \mathcal{R},$$

we have $\mathfrak{n}_{\bar{\psi}} = \mathfrak{n}_\psi \oplus \mathfrak{n}_\psi(\mathfrak{H}) \oplus \mathcal{R}f$, $\mathcal{R}f \subset N_{\bar{\psi}}$ *where* $N_{\bar{\psi}}$ *means the left kernel* $\big\{ y \in \mathcal{N} : \bar{\psi}(y^*y) = 0 \big\}$ *of* $\bar{\psi}$ *and the action of* $\mathcal{R}$ *on* $\tilde{\mathfrak{H}}$ *is semi-cyclic relative to the semi-finite normal weight* $\bar{\psi}$ *under the identification:*

$$\eta_{\bar{\psi}}(x+y) \in \mathfrak{H}_{\bar{\psi}} \longleftrightarrow \eta_\psi(x) \oplus \eta_\psi(y) \in L^2(\mathcal{N}) \oplus \mathfrak{H}, \quad x \in \mathfrak{n}_\psi, \quad y \in \mathfrak{n}_\psi(\mathfrak{H}).$$

PROOF:

(i) If x is in $\mathfrak{n}_\psi(\mathfrak{H})$, then the absolute value $|x|$ belongs to $\mathfrak{n}_\psi$ by definition, so that the polar decomposition of x shows that x belongs to $\mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}},\, \mathfrak{H}\big)\mathfrak{n}_\psi$. Conversely, if $a \in \mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}},\, \mathfrak{H}_{\mathcal{N}}\big)$ and $x \in \mathfrak{n}_\psi$, then the inequality: $x^*a^*ax \le \|a\|^2x^*x$ implies that $ax \in \mathfrak{n}_\psi(\mathfrak{H})$. If $\xi \in \mathfrak{n}_\psi(\mathfrak{H})$, then $a = L_\psi(\xi)$ belongs to

$\mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}\big)$ and with the polar decomposition $a = uh$ we conclude first that h belongs to $\mathfrak{n}_\psi$ and also that $\xi = u\eta_\psi(h) \in \mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}\big)\mathfrak{B}_\psi$.

(ii) If $xy = 0$ with $x \in \mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}\big)$ and $y \in \mathfrak{n}_\psi$, then with $x = uh$ the polar decomposition we have $hy = 0$ and

$$x\eta_\psi(y) = uh\eta_\psi(y) = u\eta_\psi(hy) = 0.$$

This means that if $x_1y_1 = x_2y_2$ with $x_1, x_2 \in \mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}\big)$ and $y_1, y_2 \in \mathfrak{n}_\psi$, then we have $x_1\eta_\psi(y_1) = x_2\eta_\psi(y_2)$, so that the map η_ψ is well-defined. The rest follows easily.

(iii) From (i) it follows that

$$\big[\mathfrak{D}(\mathfrak{H}, \psi)\big] = \big[\mathfrak{D}(\mathfrak{H}, \psi)\mathfrak{B}_\psi\big] = \big[\mathfrak{D}(\mathfrak{H}, \psi)L^2(\mathcal{N})\big].$$

Let $\xi \in \mathfrak{H}$. Consider $\omega = \omega_\xi$ as a functional over $\mathcal{N}$. Let $\xi(\omega)$ be the representing vector in $L^2(\mathcal{N})_+$ of ω for the right action of $\mathcal{N}$ on $L^2(\mathcal{N})$. Then we have a partial isometry u in $\mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}\big)$ such that $u\xi(\omega) = \xi$. This means that $\mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}\big)L^2(\mathcal{N}) = \mathfrak{H}$; hence $\mathfrak{D}(\mathfrak{H}, \psi)$ is dense in $\mathfrak{H}$. The first formula of (9) follows from the construction of the map η_ψ.

(iv) Let $\xi = \eta_\psi(x)$ with $x \in \mathfrak{n}_\psi(\mathfrak{H})$ and $x = uh$ be the polar decomposition. Then $h \in \mathfrak{n}_\psi$ and $u \in \mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}\big)$ and $\xi = u\eta_\psi(h)$ by (ii). Now for each $y \in \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^* \cap \mathcal{D}\big(\sigma^\psi_{-\mathrm{i}/2}\big)$ such that $\sigma^\psi_{-\mathrm{i}/2}(y) \in \mathfrak{n}_\psi$, we have

$$\begin{aligned} L_\psi(\xi)J\eta_\psi(y^*) &= \xi y = u\eta_\psi(h)y = u\eta_\psi\big(h\sigma^\psi_{-\frac{\mathrm{i}}{2}}(y)\big) \qquad \text{by (5)} \\ &= uh\eta_\psi\big(\sigma^\psi_{-\frac{\mathrm{i}}{2}}(y)\big) = x\Delta_\psi^{\frac{1}{2}}\eta_\psi(y) = xJ\eta_\psi(y^*). \end{aligned}$$

Therefore, we have $x = L_\psi(\xi)$. Conversely, suppose $x = L_\psi(\xi)$ with $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$. With $x = uh$ polar decomposition, we have, for each $y \in \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^* \cap \mathcal{D}\big(\sigma^\psi_{-\mathrm{i}/2}\big)$ such that $\sigma^\psi_{-\mathrm{i}/2}(y) \in \mathfrak{n}_\psi$,

$$hJ\eta_\psi(y^*) = u^*xJ\eta_\psi(y^*) = u^*(\xi y) = (u^*\xi)y,$$

so that the vector $u^*\xi \in L^2(\mathcal{N})$ is left bounded relative to the left Hilbert algebra $\mathfrak{A}_\psi$ and $h = \pi_\ell(u^*\xi)$. This means that $h \in \mathfrak{n}_\psi$, and so $x \in \mathfrak{n}_\psi(\mathfrak{H})$. It is easy to see now that $\xi = \eta_\psi(x)$.

(v) This follows from (ii), (iv) and Lemma VIII.3.18.

(vi) By now, the assertion follows from a routine calculation of actions of $\mathcal{R}$ on $\tilde{\mathfrak{H}}$ and $\mathfrak{H}_{\bar{\psi}}$. Q.E.D.

We continue to study the action of $\mathcal{R}$ on $\tilde{\mathfrak{H}}$. The direct sum decomposition, $\tilde{\mathfrak{H}} = L^2(\mathcal{N}) \oplus \mathfrak{H}$, entails the matrix representation of $\mathcal{R}$:

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad \begin{aligned} &x_{11} \in \mathcal{N}, && x_{12} \in \mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}\big), \\ &x_{21} \in \mathcal{L}\big(\mathfrak{H}_{\mathcal{N}}, L^2(\mathcal{N})_{\mathcal{N}}\big), && x_{22} \in \mathcal{M} = \mathcal{L}(\mathfrak{H}_{\mathcal{N}}) \end{aligned}$$

for each $x \in \mathcal{R}$.

We have seen that $\mathfrak{a}_\psi = \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*$ or more precisely its image $\eta_\psi(\mathfrak{a}_\psi)$ form a left Hilbert algebra. Likewise, $\mathcal{A} = \mathcal{D}\big(\sigma^\psi_{-\mathrm{i}/2}\big) \cap \mathcal{D}\big(\sigma^\psi_{\mathrm{i}/2}\big) = \mathcal{D}\big(\sigma^\psi_{\mathrm{i}/2}\big) \cap \mathcal{D}\big(\sigma^\psi_{\mathrm{i}/2}\big)^*$ is a self-adjoint subalgebra of $\mathcal{N}$ which multiplies $\mathfrak{a}_\psi$ and $\mathfrak{n}_\psi^*$ from both sides. We then have the following tautological statement:

Lemma 3.4. *The anti-representation π'_ψ of $\mathcal{A}$ defined by:*

$$\pi'_\psi(b)\eta_\psi(x) = \eta_\psi\big(x\sigma^\psi_{-\frac{\mathrm{i}}{2}}(b)\big), \quad x \in \mathfrak{n}_\psi(\mathfrak{H}), \quad b \in \mathcal{A}, \tag{12''}$$

extends to the original right action of $\mathcal{N}$ on $\mathfrak{H}$.

PROOF: Our assertion follows directly from (6), (12) and (12′). We leave the detail to the reader. Q.E.D.

Observe that we have used only the semi-finite normal weight ψ on $\mathcal{N}$ and not at all the semi-finite normal weight φ on $\mathcal{M}$. The "balanced" weight $\rho = \psi \oplus \varphi$ on $\mathcal{R}$ then gives a semi-cyclic representation $\{\pi_\rho, \mathfrak{H}_\rho\}$ of $\mathcal{R}$. We want to identify the representation π_ρ in terms of $\mathfrak{H}$ and π_φ. First we consider the weights $\bar\psi$ and $\bar\varphi$ on $\mathcal{R}$ given by $\bar\psi(x) = \psi(exe)$ and $\bar\varphi(x) = \varphi(fxf)$, $x \in \mathcal{R}$. We have now:

$$\begin{aligned}\mathfrak{H}_\rho &= \begin{pmatrix} \eta_\rho\big(e\mathcal{R}e \cap \mathfrak{n}_\rho\big) + \eta_\rho\big(e\mathcal{R}f \cap \mathfrak{n}_\rho\big) \\ + \\ \eta_\rho\big(f\mathcal{R}e \cap \mathfrak{n}_\rho\big) + \eta_\rho\big(f\mathcal{R}f \cap \mathfrak{n}_\rho\big)\end{pmatrix} \\ &= \begin{pmatrix} L^2(\mathcal{N}) \oplus \big[\eta_\rho(e\mathcal{R}f \cap \mathfrak{n}_\rho)\big] \\ \oplus \\ \mathfrak{H} \quad \oplus \quad \mathfrak{H}_\varphi \end{pmatrix},\end{aligned} \tag{13}$$

where $[\,\cdot\,]$ stands for the closure in the Hilbert space as usual. We have seen already that $\eta_\rho\big(f\mathcal{R}e \cap \mathfrak{n}_\rho\big) = \mathfrak{D}(\mathfrak{H}, \psi)$ and $f\mathcal{R}e \cap \mathfrak{n}_\rho = \mathfrak{n}_\psi(\mathfrak{H})$. Also we knew that $f\mathcal{R}e = \mathcal{L}\big(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N}\big)$ and $e\mathcal{R}f = \mathcal{L}\big(\mathfrak{H}_\mathcal{N}, L^2(\mathcal{N})_\mathcal{N}\big)$. We now want to know about $\mathfrak{n}_\rho \cap e\mathcal{R}f$ and its image under the map η_ρ. As we did not assume the faithfulness for φ, we don't have symmetry between φ and ψ. At any rate, we have

$$\begin{aligned} e\mathcal{R}f \cap \mathfrak{n}_\rho &= \Big\{ y \in \mathcal{L}\big(\mathfrak{H}_\mathcal{N}, L^2(\mathcal{N})_\mathcal{N}\big) : \varphi(y^*y) < \infty \Big\} \\ &= \Big\{ x^* : x \in \mathcal{L}\big(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N}\big) \text{ with } \varphi(xx^*) < \infty \Big\}. \end{aligned} \tag{10'}$$

Following the decomposition (13), we naturally set:

$$\mathfrak{H}_{1,1} = L^2(\mathcal{N}), \qquad \mathfrak{H}_{1,2} = \big[\eta_\rho(e\mathcal{R}f \cap \mathfrak{n}_\rho)\big]$$
$$\mathfrak{H}_{2,1} = \mathfrak{H}, \qquad \mathfrak{H}_{2,2} = \mathfrak{H}_\varphi.$$

Lemma 3.5.

(i) *The restriction of π_ρ to the second column space of* (13), $\mathfrak{H}_{1,2} \oplus \mathfrak{H}_{2,2}$, *is semi-cyclic relative to the weight $\bar\varphi$.*

(ii) *The Hilbert space* $\mathfrak{H}_{1,2}$ *is isomorphic to* $\overline{s(\varphi)\mathfrak{H}_{2,1}}$ *dual to* $s(\varphi)\mathfrak{H}_{2,1}$ *as an* $\mathcal{N}$*-*$\mathcal{M}_{s(\varphi)}$*-bimodule under the natural map.*

PROOF: First consider the case that φ is faithful. Then with $\mathfrak{a}_\rho = \mathfrak{n}_\rho \cap \mathfrak{n}_\rho^*$ and $\mathfrak{A}_\rho = \eta_\rho(\mathfrak{a}_\rho)$, we have a left Hilbert algebra $\mathfrak{A}_\rho$. Furthermore, the $\mathcal{R}$-$\mathcal{R}$-bimodule $L^2(\mathcal{R})$ is naturally identified with $\mathfrak{H}_\rho$. Then the components of $\mathfrak{H}_\rho$ of (13) are identified with

$$\mathfrak{H}_{1,1} = eL^2(\mathcal{R})e, \qquad \mathfrak{H}_{1,2} = eL^2(\mathcal{R})f,$$

$$\mathfrak{H}_{2,1} = fL^2(\mathcal{R})e, \qquad \mathfrak{H}_{2,2} = fL^2(\mathcal{R})f.$$

The modular conjugation J implements the desired isomorphism between $\bar{\mathfrak{H}}_{2,1}$ and $\mathfrak{H}_{1,2}$. This proves the assertion (ii) for the faithful case. The assertion (i) follows from the symmetry between ψ on $\mathcal{N}$ and φ° on $\mathcal{M}^\circ$.

If φ is not faithful, then with $q = s(\varphi)$ we consider an auxiliary semi-finite normal weight φ_1 on $\mathcal{M}$ with $e_1 = s(\varphi_1) = 1_{\mathfrak{H}} - q \le e$ to have a faithful semi-finite normal weight $\rho_1 = \rho + \bar{\varphi}_1$ on $\mathcal{R}$ such that $s(\bar{\varphi}_1)$ belongs to $\mathcal{R}_{\rho_1}$ where $\bar{\varphi}_1$ is given by $\bar{\varphi}_1(x) = \varphi_1(exe)$, $x \in \mathcal{R}_+$. Observe that η_ρ and η_{ρ_1} agree on $e\mathcal{R}e$ and $f\mathcal{R}e$ and that we have $\eta_\rho(x) = \eta_{\rho_1}(xq)$, $x \in (e\mathcal{R}f + f\mathcal{R}f) \cap \mathfrak{n}_\rho$. Hence we get

$$\mathfrak{H}_{1,1} = eL^2(\mathcal{R})e, \qquad \mathfrak{H}_{2,1} = fL^2(\mathcal{R})e,$$

$$\mathfrak{H}_{1,2} = eL^2(\mathcal{R})q, \qquad \mathfrak{H}_{2,2} = fL^2(\mathcal{R})q.$$

Therefore, we have $Jq\mathfrak{H}_{2,1} = JqL^2(\mathcal{R})e = eL(\mathcal{R})q = \mathfrak{H}_{1,2}$. This completes the proof for (i). For (ii), we have $\mathfrak{H}_{1,2} \oplus \mathfrak{H}_{2,2} = L^2(\mathcal{R})q$ as an $\mathcal{N}$-$\mathcal{M}_q$-bimodule. Therefore, the representation $\{\pi_\rho, \mathfrak{H}_{1,2} \oplus \mathfrak{H}_{2,2}\}$ is nothing but the semi-cyclic representation $\{\pi_{\bar{\varphi}}, \mathfrak{H}_{\bar{\varphi}}, \eta_{\bar{\varphi}}\}$. Q.E.D.

If we look at the origin of the above proof, we recognize that the conjugation operator J comes from the conjugate linear operator

$$S_{\varphi,\psi}\colon \eta_\psi(x) \in \eta_\psi\big(\mathcal{L}\big(L^2(\mathcal{N})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}\big) \cap \mathfrak{n}_\rho \cap \mathfrak{n}_\rho^*\big) \mapsto \eta_\varphi(x^*) \in \mathfrak{H}_{1,2},$$

which is the restriction of the $S_{\bar{\varphi}_1,\bar{\psi}}$-map for ρ_1 to the smaller domain $qL^2(\mathcal{R})e$, i.e. $S_{\varphi,\psi}$ can be directly defined as the closure of the operator given by $S_{\varphi,\psi}\eta_\psi(x) = \eta_\varphi(x^*)$ for $x \in \mathfrak{n}_\psi(\mathfrak{H}) \cap \mathfrak{n}_\varphi(\mathfrak{H})^*$, where

$$\mathfrak{n}_\varphi(\mathfrak{H}) = \Big\{ x \in \mathcal{L}\big(\mathfrak{H}_{\mathcal{N}}, L^2(\mathcal{N})_{\mathcal{N}}\big) : \varphi(x^*x) < \infty \Big\}.$$

Thus we come to the following:

Definition 3.6. The absolute value $\Delta_{\varphi,\psi}$ of $S_{\varphi,\psi}$ is called *spatial derivative* of the semi-finite normal weight φ on $\mathcal{M}$ relative to the faithful semi-finite normal weight ψ' on the commutant $\mathcal{M}'$ and denoted by $\frac{d\varphi}{d\psi'}$ as it is determined by ψ' on $\mathcal{M}'$ and φ on $\mathcal{M}$.

Dualizing (10), we set:

$$\mathfrak{D}'(\mathfrak{H}, \varphi) = \left\{ \xi \in \mathfrak{H} : \begin{array}{l} \|x\xi\|^2 \le C_\xi \varphi(x^* x),\ x \in \mathcal{M}, \\ \text{for some constant } C_\xi \ge 0 \end{array} \right\}. \tag{10''}$$

To each $\xi \in \mathfrak{D}'(\mathfrak{H}, \varphi)$ there corresponds an operator $R_\varphi(\xi)$ defined by

$$R_\varphi(\xi)\eta_\varphi(x) = x\xi, \quad x \in \mathfrak{n}_\varphi,$$

which belongs to $\mathcal{L}\big({}_{\mathcal{M}}L^2(\mathcal{M}),\ {}_{\mathcal{M}}\mathfrak{H}\big)$. As φ is not assumed to be faithful, ψ and φ are not symmetric. In fact, we have the following:

Lemma 3.7. *The closure of $\mathfrak{D}'(\mathfrak{H}, \varphi)$ is the range of the projection $s(\varphi)$, i.e $\big[\mathfrak{D}'(\mathfrak{H}, \varphi)\big] = s(\varphi)\mathfrak{H}$.*

PROOF: If $\xi \in \mathfrak{D}'(\mathfrak{H}, \varphi)$, then we have

$$\big\|\big(1 - s(\varphi)\big)\xi\big\|^2 \le C_\xi \varphi\big((1 - s(\varphi)\big) = 0.$$

Hence $\mathfrak{D}'(\mathfrak{H}, \varphi) \subset s(\varphi)\mathfrak{H}$. Conversely, suppose $\xi \perp \mathfrak{D}'(\mathfrak{H}, \varphi)$. With $\Phi = \big\{\omega : \omega \le \varphi\big\}$, we know $\varphi = \sup \Phi$, and that

$$\eta \in \mathfrak{D}'(\mathfrak{H}, \varphi) \iff \omega_\eta \in \bigcup_{C>0} C\Phi.$$

Also every $\omega \in \Phi$ is a countable sum of ω_{η_n} with $\eta_n \in \mathfrak{D}'(\mathfrak{H}, \varphi)$, so that $s(\varphi) = \sup\big\{ s(\omega_\eta) : \eta \in \mathfrak{D}'(\mathfrak{H}, \varphi) \big\}$; thus we have $s(\varphi)\xi = 0$. Q.E.D.

We now state one of the main results of the section:

Theorem 3.8. *Let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra, φ a semi-finite normal weight on $\mathcal{M}$ and ψ' a faithful semi-finite normal weight on the commutant $\mathcal{M}'$. Then the spatial derivative $\mathrm{d}\varphi/\mathrm{d}\psi'$ has the following property:*

(i) *The support $s\big(\mathrm{d}\varphi/\mathrm{d}\psi'\big)$ of the spatial derivative $\mathrm{d}\varphi/\mathrm{d}\psi'$* [4] *is equal to $s(\varphi)$.*
(ii) *On the reduced von Neumann algebra $\big\{\mathcal{M}_{s(\varphi)}, s(\varphi)\mathfrak{H}\big\}$ and its commutant $\mathcal{M}'_{s(\varphi)}$, we have:*

$$\left.\begin{aligned} \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)^{\mathrm{i}t} x \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)^{-\mathrm{i}t} &= \sigma_t^\varphi(x), \quad x \in \mathcal{M}_{s(\varphi)}, \\ \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)^{\mathrm{i}t} y \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)^{-\mathrm{i}t} &= \sigma_{-t}^{\psi'}(y), \quad y \in \mathcal{M}'_{s(\varphi)}. \end{aligned}\right\} \tag{14}$$

(iii) *If φ_1 and φ_2 are faithful semi-finite normal weights on $\mathcal{M}$, then*

$$\left(\frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi'}\right)^{\mathrm{i}t} = \big(\mathrm{D}\varphi_2 : \mathrm{D}\varphi_1\big)_t \left(\frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi'}\right)^{\mathrm{i}t}. \tag{15}$$

[4] The support of a self-adjoint operator means the projection to the closure of the range.

(iv) *If* φ *is also faithful , then*

$$\frac{\mathrm{d}\psi'}{\mathrm{d}\varphi} = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)^{-1}. \tag{16}$$

(v) *With* $\mathcal{N} = (\mathcal{M}')^\circ$ *and* $\psi = (\psi')^\circ$*, the square root of the spatial derivative,* $\left(\mathrm{d}\varphi/\mathrm{d}\psi'\right)^{1/2}$*, is essentially self-adjoint on*

$$\mathfrak{D}_{\varphi,\psi}(\mathfrak{H}) = \left\{\xi \in \mathfrak{D}(\mathfrak{H}, \psi) : L_\psi(\xi) \in \mathfrak{n}_\varphi^*\right\}$$

and determined by:

$$\left(\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)^{\frac{1}{2}}\xi \,\middle|\, \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)^{\frac{1}{2}}\eta\right) = \varphi(L_\psi(\xi)L_\psi(\eta)^*), \quad \xi, \eta \in \mathfrak{D}_{\varphi,\psi}(\mathfrak{H}). \tag{17}$$

Therefore, the spatial derivative $\mathrm{d}\varphi/\mathrm{d}\psi'$ *of* φ *relative to* ψ' *is directly computable from* φ *and* ψ'.

PROOF: From the previous arguments to $\mathcal{M}$, $\mathcal{N}$, φ and ψ, we know that the spatial derivative $\mathrm{d}\varphi/\mathrm{d}\psi'$ is defined to be the relative modular operator $\Delta_{\varphi,\psi}$ on the subspace $s(\varphi)\mathfrak{H}$. Now we replace $\mathfrak{H}$ by $s(\varphi)\mathfrak{H}$ and assume that φ is faithful. Then the assertions (i), (ii), (iii) and (iv) are nothing but the statements about the relative modular operator which was already proven in the last chapter. Statement (v) also follows from the definition of the relative modular operator $\Delta_{\varphi,\psi}$ in the last chapter, i.e. $\Delta_{\psi,\varphi}^{1/2}$ is the absolute value of $S_{\varphi,\psi}$. Q.E.D.

From the statement (v) in the above theorem it follows that the spatial derivative $\mathrm{d}\varphi/\mathrm{d}\psi'$ is directly determined by φ and ψ' without making use of the auxiliary von Neumann algebra $\mathcal{R}$ in the preceding discussion.

Lemma 3.9. *The linear span* $\mathcal{J}_\psi$ *of* $L_\psi(\xi)L_\psi(\eta)^*$, $\xi, \eta \in \mathfrak{D}(\mathfrak{H}, \psi)$*, is a* σ*-weakly dense ideal of* $\mathcal{M}$ *and we have:*

$$\mathcal{J}_\psi^+ = \left\{\sum_{i=1}^n L_\psi(\xi_i)L_\psi(\xi_i)^* : \xi_i \in \mathfrak{D}(\mathfrak{H}, \psi),\ i = 1, \dots, n\right\}.$$

PROOF: It is easy to see that $L_\psi(a\xi) = aL_\psi(\xi)$ for any $a \in \mathcal{M}$ and $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$. Hence $\mathcal{J}_\psi$ is an ideal of $\mathcal{M}$. The rest follows from the polarization techniques which we have been using repeatedly. See for example, Chapter VII, §1. As $L_\psi\left(\mathfrak{D}(\mathfrak{H}, \psi)\right) = \mathfrak{n}_\psi(\mathfrak{H})$, for the σ-weak density of $\mathcal{J}_\psi$ it suffices to prove that $a\mathfrak{n}_\psi(\mathfrak{H}) = \{0\}$, $a \in \mathcal{M}$, implies that $a = 0$. Suppose $a\mathfrak{n}_\psi(\mathfrak{H}) = \{0\}$, $a \in \mathcal{M}$. This means that $aL_\psi(\xi) = 0$ for every $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$. Thus for every $x \in \mathfrak{n}_\psi^*$ we have

$$0 = aL_\psi(\xi)\eta'_\psi(x) = a(\xi x) = (a\xi)x.$$

As $\pi'_{\mathfrak{H}}(\mathfrak{n}_\psi^*)$ is σ-weakly dense in $\pi'_{\mathfrak{H}}(\mathcal{N})$, we have $a\xi = 0$. The density of $\mathfrak{D}(\mathfrak{H}, \psi)$ in $\mathfrak{H}$ implies $a = 0$. Q.E.D.

Proposition 3.10. *The spatial derivative* $(\mathrm{d}\varphi/\mathrm{d}\psi')$ *has the following properties further:*

(i) $$\varphi_1 \le \varphi_2 \iff \frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi'} \le \frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi'}.$$

(ii) *If* φ_1 *and* φ_2 *are both finite, then*
$$\frac{\mathrm{d}(\varphi_1+\varphi_2)}{\mathrm{d}\psi'} = \frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi'} + \frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi'}, \tag{18}$$
where the above sum is a form sum, see A.9.

(iii) *If* $a \in \mathcal{M}$ *is invertible, then*
$$\frac{\mathrm{d}(a\varphi a^*)}{\mathrm{d}\psi'} = a\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)a^*. \tag{19}$$

(iv) *The support of* $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}$ *is equal to the support* $s(\varphi)$ *of* φ, *where the support of* $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}$ *is the projection to the closure of the range of the operator* $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}$ *and denoted by* $s\big(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\big)$.

PROOF:

(i) Suppose $\varphi_1 \le \varphi_2$. Then we have
$$\left|\left(\frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi'}\right)^{\frac{1}{2}}\xi\right|^2 = \varphi_1\big(L_\psi(\xi)L_\psi(\xi)^*\big) \le \varphi_2\big(L_\psi(\xi)L_\psi(\xi)^*\big) = \left|\left(\frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi'}\right)^{\frac{1}{2}}\xi\right|^2$$
for each $\xi \in \mathfrak{D}(\mathfrak{H},\psi)$. Hence $\mathrm{d}\varphi_1/\mathrm{d}\psi \le \mathrm{d}\varphi_2/\mathrm{d}\psi$ follows.

Conversely, suppose $\mathrm{d}\varphi_1/\mathrm{d}\psi' \le \mathrm{d}\varphi_2/\mathrm{d}\psi'$. It then follows that $\varphi_1(a) \le \varphi_2(a)$ for every $a \in \mathcal{M}_+$ of the form $a = \sum_{i=1}^n L_\psi(\xi_i)L_\psi(\xi_i)^*$. Our assertion then follows from the last lemma.

(ii) Suppose that $\varphi_1, \varphi_2 \in \mathcal{M}_*^+$ and $\varphi = \varphi_1 + \varphi_2$. The boundedness of φ_1 and φ_2 imply that φ is bounded and the square roots of all the spatial derivatives $\frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi'}$, $\frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi'}$ and $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}$ are essentially self-adjoint on $\mathfrak{D}(\mathfrak{H},\psi)$. Let $H_1 = \mathrm{d}\varphi_1/\mathrm{d}\psi'$, $H_2 = \mathrm{d}\varphi_2/\mathrm{d}\psi'$ and $H = \mathrm{d}\varphi/\mathrm{d}\psi'$. Then we have $\|H^{1/2}\xi\|^2 = \|H_1^{1/2}\xi\|^2 + \|H_2^{1/2}\xi\|^2$, $\xi \in \mathfrak{D}(\mathfrak{H},\psi)$. Hence our assertion follows.

(iii) Let $H = \mathrm{d}\varphi/\mathrm{d}\psi'$. It follows that aHa^* is a positive self-adjoint operator with domain ${a^*}^{-1}\mathfrak{D}(H)$, and that for each $\xi \in \mathfrak{D}(\mathfrak{H},\psi)$
$$\begin{aligned}\|H^{\frac{1}{2}}a^*\xi\|^2 &= \varphi\big(L_\psi(a^*\xi)L_\psi(a^*\xi)^*\big) = \varphi\big(a^*L_\psi(\xi)L_\psi(\xi)^*a\big)\\ &= (a\varphi a^*)\big(L_\psi(\xi)L_\psi(\xi)^*\big).\end{aligned}$$ [5]

Since a is invertible, $\mathfrak{D}\big((aHa^*)^{1/2}\big) = \mathfrak{D}\big(H^{1/2}a^*\big)$ and the absolute value of $H^{1/2}a^*$ is precisely $(aHa^*)^{1/2}$. Hence we conclude (19).

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5 $\|H^{1/2}a^*\xi\|^2$ can be $+\infty$ if φ is not finite. In fact, $\|H^{1/2}a^*\xi\|^2 < +\infty$ if and only if $a^*\xi \in \mathfrak{D}\big(H^{1/2}\big)$.

(iv) Let p be the support of $H = \mathrm{d}\varphi/\mathrm{d}\psi'$ and $q = s(\varphi)$. Then p is characterized by the fact that $1 - p$ is the projection of $\mathfrak{H}$ onto the null space of H, i.e. onto the subspace: $\mathfrak{N} = \{\xi \in \mathfrak{H} : H\xi = 0\}$. Let $\mathfrak{A}_1$ be the maximal Tomita algebra associated with the left Hilbert algebra $\mathfrak{A}_{\rho_1} = \eta_{\rho_1}(\mathfrak{n}_{\rho_1} \cap \mathfrak{n}_{\rho_1}^*)$. As $f \in \mathcal{R}_{\rho_1}$, we have, with $f' = JfJ$, $f'\mathfrak{A}_1 = J\mathfrak{A}_1 \subset \mathfrak{A}_1$. If $\xi \in \mathfrak{N}$, then there exists a sequence $\{\xi_n\}$ in $\mathfrak{D}(\mathfrak{H}, \psi)$ such that $\xi = \lim \xi_n$ and $\lim H\xi_n = 0$ as $\mathfrak{D}(\mathfrak{H}, \psi)$ is a core for H. For each $\eta \in f'\mathfrak{A}_1$, we have $\pi_r(\eta)\xi = \lim_n \pi_r(\eta)\xi_n$ and

$$\begin{aligned}\|H^{\frac{1}{2}}\pi_r(\eta)\xi\|^2 &= \lim\|H^{\frac{1}{2}}\pi_r(\eta)\xi_n\|^2 = \lim\|H^{\frac{1}{2}}\pi_\ell(\xi_n)\eta\|^2\\ &= \lim\varphi\big(\pi_\ell(\xi_n\eta)\pi_\ell(\xi_n\eta)^*\big)\\ &\le \|\pi_\ell(\eta)\|^2\lim\varphi\big(\pi_\ell(\xi_n)\pi_\ell(\xi_n)^*\big) = 0,\end{aligned}$$

so that $\pi_r(\eta)\xi \in \mathfrak{N}$. Since $\{\pi_r(\eta)_e : \eta \in f'\mathfrak{A}_1\}$ is σ-weakly dense in $\mathcal{N}$, the projection p belongs to $\mathcal{M} = \mathcal{N}'$.

If $\xi \in (1 - q)\mathfrak{D}(\mathfrak{H}, \psi)$, then

$$\varphi\big(L_\psi(\xi)L_\psi(\xi)^*\big) = \varphi\big(qL_\psi(\xi)L_\psi(\xi)^*q\big) = \varphi\big(L_\psi(q\xi)L_\psi(q\xi)^*\big) = 0,$$

so $\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)^{1/2}\xi = 0$. If $\xi \in (1 - q)\mathfrak{H}$, then we choose a sequence $\xi_n \in \mathfrak{D}(\mathfrak{H}, \psi)$ with $\xi = \lim \xi_n$. It follows that $\xi = \lim(1 - q)\xi_n$ and $(\mathrm{d}\varphi/\mathrm{d}\psi')^{1/2}(1 - q)\xi_n = 0$, so that $\xi \in \mathfrak{D}((\mathrm{d}\varphi/\mathrm{d}\psi')^{1/2})$ and $(\mathrm{d}\varphi/\mathrm{d}\psi')^{1/2}\xi = 0$. Thus $1 - q \le 1 - p$, i.e. $p \le q$. On the other hand, φ is a faithful weight on $\mathcal{M}_q$. It is then easy to check that we can view the weight ψ as one on $\mathcal{N}_{s(\varphi)}$ without changing $\mathrm{d}\varphi/\mathrm{d}\psi'$ other than changing the space from $\mathfrak{H}$ to $s(\varphi)\mathfrak{H}$. By Theorem 3.8, $\mathrm{d}\varphi/\mathrm{d}\psi'$ is non-singular on $s(\varphi)\mathfrak{H}$, which means $p = s(\varphi)$. Q.E.D.

Theorem 3.11. *Let $\{\mathcal{M}, \mathfrak{H}\}$ and $\mathcal{N} = \mathcal{M}'$ be as before and fix a faithful weight ψ on $\mathcal{N}$. For a positive self-adjoint operator H on $\mathfrak{H}$, the following three conditions are equivalent:*

(i) *There exists a weight φ on $\mathcal{M}$ with $H = \mathrm{d}\varphi/\mathrm{d}\psi'$;*

(ii) *For every $y \in \mathcal{N}$, $H^{\mathrm{i}t}\sigma_t^\psi(y) = yH^{\mathrm{i}t}$, $t \in \mathbf{R}$, where $H^{\mathrm{i}t}$ is considered only on the closure of the range of H;*

(iii) *$\mathfrak{D}(\mathfrak{H}, \psi) \cap \mathfrak{D}(H^{1/2})$ is a core for $H^{1/2}$ and the scalar $\sum_{i=1}^n \|H^{1/2}\xi_i\|^2$ depends only on the operator:*

$$\sum_{i=1}^n L_\psi(\xi_i)L_\psi(\xi_i)^* \quad \textit{for } \xi_1, \ldots, \xi_n \in \mathfrak{D}(\mathfrak{H}, \psi) \cap \mathfrak{D}(H^{\frac{1}{2}}).$$

PROOF:

(i) $\iff$ (ii): Let p be the support of H. Each of conditions (i) and (ii) implies $p \in \mathcal{M}$. Hence we may and do assume the non-singularity of H. The implication (i) $\Longrightarrow$ (ii) follows from Theorem 3.8. Now assume (ii), and take any faithful weight φ_1 on $\mathcal{M}$. Put $H_1 = \mathrm{d}\varphi_1/\mathrm{d}\psi'$, and $u_t = H^{\mathrm{i}t}H_1^{-\mathrm{i}t}$, $t \in \mathbf{R}$. It then follows that u_t is a $\{\sigma_t^{\varphi_1}\}$-cocycle in $\mathcal{M}$. By Theorem VIII.3.7 there exists a faithful weight φ on $\mathcal{M}$ with $(\mathrm{D}\varphi : \mathrm{D}\varphi_1)_t = u_t,\ t \in \mathbf{R}$, and hence $H = \mathrm{d}\varphi/\mathrm{d}\psi'$.

(i) $\Longrightarrow$ (iii): By construction, we have

$$\sum_{i=1}^n \left\| H^{\frac{1}{2}}\xi_i \right\|^2 = \varphi\left(\sum_{i=1}^n L_\psi(\xi_i)L_\psi(\xi_i)^* \right), \quad \xi_1, \dots, \xi_n \in \mathfrak{D}(\mathfrak{H}, \psi),$$

so that the assertion follows.

(iii) $\Longrightarrow$ (i): We need only to construct a semi-finite normal weight φ on $\mathcal{M}$ such that

$$\left\| H^{\frac{1}{2}}\xi \right\|^2 = \varphi\big(L_\psi(\xi)L_\psi(\xi)^*\big), \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi) \cap \mathfrak{D}\big(H^{\frac{1}{2}}\big).$$

The weight with this property is automatically semi-finite because $\mathfrak{D}\big(H^{1/2}\big) \cap \mathfrak{D}(\mathfrak{H}, \psi)$ is dense in $\mathfrak{H}$ which means that $\big\{ L_\psi(\xi)L_\psi(\xi)^* : \xi \in \mathfrak{D}\big(H^{1/2}\big) \cap \mathfrak{D}(\mathfrak{H}, \psi) \big\}$ is non-degenerate. The rest of the proof then follows from the next result.

Q.E.D.

Lemma 3.12.

(i) *Let H be as in Theorem* 3.11.(iii). *Then there exists a weight*[6] φ_1 *on* $\mathcal{J}_\psi$ *such that*

$$\varphi_1\big(L_\psi(\xi)L_\psi(\xi)^*\big) = \left\| H^{\frac{1}{2}}\xi \right\|^2, \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi), \tag{20}$$

where $\left\| H^{1/2}\xi \right\|^2 = +\infty$ *if* $\xi \notin \mathfrak{D}\big(H^{1/2}\big)$. *The weight* φ_1 *has the property that for any net* $\{x_i\}$ *of* $\mathcal{M}$ *converging strongly to* 1

$$\liminf \varphi_1\big(x_i y x_i^*\big) \geq \varphi_1(y), \quad y \in \mathcal{J}_\psi^+. \tag{21}$$

(ii) *Any weight* φ_1 *on* $\mathcal{J}_\psi$ *with the above property extends to a normal weight* φ *on* $\mathcal{M}$.

PROOF:

(i) By assumption, φ_1 defined by (20) on $L_\psi(\xi)L_\psi(\xi)^*$ extends to a weight on $\mathcal{J}_\psi^+$ by Lemma 3.5.(ii), which is denoted again by φ_1. Suppose $y = \sum_{k=1}^n L_\psi(\xi_k)L_\psi(\xi_k)^* \in \mathcal{J}_\psi^+$. We then have

$$x_i y x_i^* = \sum_{k=1}^n L_\psi(x_i\xi_k)L_\psi(x_i\xi_k)^*$$

so that

$$\varphi(x_i y x_i^*) = \sum_{k=1}^n \left\| H^{\frac{1}{2}} x_i \xi_k \right\|^2.$$

Hence the inequality (21) follows from the lower semi-continuity of the positive quadratic form associated with H, see A.8.

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6 We do not consider the normality nor the semi-finiteness for φ_1 here.

(ii) Since $\mathcal{J}_\psi$ is a σ-weakly dense ideal of $\mathcal{M}$, every element of $\mathcal{M}_+$ is approximated by $\mathcal{J}_\psi^+$ from below. So we put

$$\varphi(x) = \sup\{\varphi_1(y) : y \in \mathcal{J}_\psi^+,\ y \leq x\}, \quad x \in \mathcal{M}_+.$$

It follows that φ agrees with φ_1 on $\mathcal{J}_\varphi^+$. Since $x_i \nearrow x$ and $y_i \nearrow y$ imply $(x_i + y_i) \nearrow (x + y)$, the additivity of φ follows from that of φ_1. We only have to check the normality of φ. Suppose that $x_i \nearrow x$ in $\mathcal{M}_+$. Then $x_i^{1/2} = a_i x^{1/2}$ for a unique $a_i \in \mathcal{M}$ with $s_r(a_i) \leq s(x)$. Put $b_i = a_i + (1 - s(x))$. Then $x_i = b_i x b_i^*$ and $\{b_i\}$ converges strongly to 1. For any $y \in \mathcal{J}_\psi^+$ with $y \leq x$ we must show that $\sup \varphi(x_i) \geq \varphi_1(y)$. But we have, by (21),

$$\varphi_1(y) \leq \liminf \varphi_1(b_i y b_i^*) \leq \liminf \varphi(b_i x b_i^*),$$

and $b_i x b_i^* = x_i$. Q.E.D.

Corollary 3.13. *Let $\mathcal{M}$, $\mathcal{N}$ and $\mathfrak{H}$ be as before, and ψ' a fixed faithful weight on $\mathcal{M}'$. If $\{\varphi_n\}$ is an increasing sequence of faithful weights on $\mathcal{M}$ and if $\varphi = \sup \varphi_n$ is semi-finite, then $\{\mathrm{d}\varphi_n / \mathrm{d}\psi'\}$ is increasing and converges to $\mathrm{d}\varphi / \mathrm{d}\psi'$ in the strong resolvent sense, and therefore $\{\sigma_t^{\varphi_n}\}$ converges to $\{\sigma_t^{\varphi}\}$ in $\mathrm{Aut}(\mathcal{M})$ uniformly on any finite interval of t.*

PROOF: Let $H_n = \mathrm{d}\varphi_n / \mathrm{d}\psi'$. By Proposition 3.10, $\{H_n\}$ is increasing and bounded by $H = \mathrm{d}\varphi / \mathrm{d}\psi'$ from above. Hence $\{H_n\}$ converges to a positive self-adjoint operator K in the strong resolvent sense by A.11. Since $H_n^{\mathrm{i}t} y H_n^{-\mathrm{i}t} = \sigma_{-t}^{\psi}(y)$ for every $y \in \mathcal{N}$, $K^{\mathrm{i}t} y K^{\mathrm{i}t} = \sigma_{-t}^{\psi}(y)$, $y \in \mathcal{N}$. By Theorem 3.11, there exists uniquely a weight ρ on $\mathcal{M}$ with $K = \mathrm{d}\rho / \mathrm{d}\psi'$. The inequalities, $H_n \leq K \leq H$, show that $\varphi_n \leq \rho \leq \varphi$. Hence $\rho = \varphi$ and $K = H$. The rest follows from the general fact about monotone convergence, see A.11. Q.E.D.

We are now going to study a very special case that $\mathcal{M}$ and $\mathcal{N}$ are both finite.

Theorem 3.14. *Suppose that a finite von Neumann algebra $\{\mathcal{M}, \mathfrak{H}\}$ has finite commutant $\mathcal{M}' = \mathcal{N}^\circ$. If τ and τ' are both faithful normal tracial states on $\mathcal{M}$ and on $\mathcal{M}'$ respectively which agree on the center $\mathcal{Z} = \mathcal{M} \cap \mathcal{M}'$, then the spatial derivative $\frac{\mathrm{d}\tau}{\mathrm{d}\tau'}$ is equal to the coupling operator $c_{\mathcal{M}}$ in the sense of §3 of Chapter V.*

PROOF: To avoid possible confusions, we use the notations φ for τ and ψ' for τ' and set $\psi = \psi'^\circ$ on $\mathcal{N}$. First we note that as the modular automorphism groups of φ and ψ' are both trivial, the spatial derivative $\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)^{\mathrm{i}t}$ must belong to the center $\mathcal{Z}$ of $\mathcal{M}$. Hence $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}$ is affiliated with $\mathcal{Z}$.

As in the previous arguments, we consider the von Neumann algebra $\mathcal{R} = \mathcal{L}((L^2(\mathcal{N}) \oplus \mathfrak{H})_{\mathcal{N}})$. The projections e and f of $\tilde{\mathfrak{H}} = L^2(\mathcal{N}) \oplus \mathfrak{H}$ onto $L^2(\mathcal{N})$ and $\mathfrak{H}$ respectively are both finite in $\mathcal{R}$. As $1 = e + f$, we conclude by Theorem V.1.37 that $\mathcal{R}$ is finite and σ-finite. Let $\tilde{\rho}$ be a normalized faithful normal trace on $\mathcal{R}$. Then there exist non-singular positive self-adjoint operators a and b affiliated with $\mathcal{Z}$ such that $\bar{\varphi} = \tilde{\rho}(ae\,\cdot\,)$ and $\bar{\psi} = \tilde{\rho}(bf\,\cdot\,)$, where $\bar{\varphi}$ and $\bar{\psi}$ mean the ones we have defined above. Recall also the decomposition (13) of $\mathfrak{H}_\rho$ with $\rho = \bar{\varphi} + \bar{\psi}$.

Choose $\xi \in \mathfrak{H}$ and consider its polar decomposition: $\xi = u|\xi|$ in $L^2(\mathcal{R})$. As $\xi \in \mathfrak{H} = \mathfrak{H}_{2,1}$, its absolute value $|\xi|$ belongs to $L^2(\mathcal{N})_+$ and $|\xi^*| \in L^2(\mathcal{M})_+$. The initial projection $p = u^*u$ of u is equal to $\pi_\psi\big(s(\omega_\xi|_{\mathcal{M}'})^\circ\big)$ and the final projection $q = uu^*$ of u is equal to $\pi_\varphi\big(s(\omega_\xi|_{\mathcal{M}})\big)$. Hence, with $T_{\mathcal{R}}$ the center valued trace of $\mathcal{R}$, we have $T_{\mathcal{R}}(p) = T_{\mathcal{R}}(q)$, so that $\varphi\big(s(\omega_\xi|_{\mathcal{M}})a^{-1}x\big) = \psi\big(s(\omega_\xi|_{\mathcal{M}'})^\circ b^{-1}x\big),\ x \in \mathcal{Z}$. Thus we have

$$T_{\mathcal{M}}\big(s(\omega_\xi|_{\mathcal{M}})a^{-1}\big) = T_{(\mathcal{M}')^\circ}\big(s(\omega_\xi|_{\mathcal{M}'})^\circ b^{-1}\big) = T_{\mathcal{M}'}\big(s(\omega_\xi|_{\mathcal{M}'})b^{-1}\big).$$

Therefore, we conclude that $c_{\mathcal{M}} = ab^{-1}$.

Suppose now that $\xi, \eta \in \mathfrak{D}_{\psi,\varphi}(\mathfrak{H})$. Let $\{z_n\}$ be an increasing sequence of projections in $\mathcal{Z}$ converging σ-strongly to $1_{\mathcal{R}}$ such that $az_n \in \mathcal{Z}$, $bz_n \in \mathcal{Z}$ and $\big(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\big)z_n \in \mathcal{Z}$ for each $n \in \mathbf{N}$. Then we have

$$\begin{aligned}
\overline{\rho}\left(b\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)z_n L_\psi(\eta)^* L_\psi(\xi)\right) &= \overline{\psi}\left(\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)z_n L_\psi(\eta)^* L_\psi(\xi)\right)\\
&= \left(\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)\eta_{\overline{\psi}}\big(L_\psi(\xi)\big)\;\middle|\;\eta_{\overline{\psi}}\big(L_\psi(\xi)\big)\right)\\
&= \left(\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)z_n\xi\;\middle|\;\eta\right) = \varphi\big(z_n L_\psi(\xi)L_\psi(\eta)^*\big)\\
&= \overline{\rho}\big(az_n L_\psi(\xi)L_\psi(\eta)^*\big).
\end{aligned}$$

As $\big(\mathfrak{n}_\rho \cap \mathfrak{n}_\rho^* \cap f\mathcal{R}e\big)^*\big(\mathfrak{n}_\rho \cap \mathfrak{n}_\rho^* \cap f\mathcal{R}e\big)$ is σ-weakly dense in $\mathcal{R}_e$ and also the same is true for $\big(\mathfrak{n}_\rho \cap \mathfrak{n}_\rho^* \cap f\mathcal{R}e\big)\big(\mathfrak{n}_\rho \cap \mathfrak{n}_\rho^* \cap f\mathcal{R}e\big)^*$ in $\mathcal{R}_f$, the above calculation shows that $b\big(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\big)z_n = az_n$. Hence we have $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'} = ab^{-1} = c_{\mathcal{M}}$. Q.E.D.

We now move on to relative tensor products of a right module and a left module over the same von Neumann algebra $\mathcal{N}$. Unlike the ordinary tensor product case, the construction of the relative tensor product depends on the choice of a faithful semi-finite normal weight on $\mathcal{N}$ as mentioned above. Furthermore, the tensor product of an arbitrary pair of vectors of $\mathfrak{H}$ and $\mathfrak{K}$ cannot be defined. It is restricted to certain pairs of vectors of $\mathfrak{H}$ and $\mathfrak{K}$ depending on the faithful semi-finite normal weight. It behaves like the product of closed unbounded operators. Let us prepare some notations first.

As in the case of right modules, for two left $\mathcal{N}$-modules ${}_{\mathcal{N}}\mathfrak{K}_1$ and ${}_{\mathcal{N}}\mathfrak{K}_2$ we consider $\mathcal{L}({}_{\mathcal{N}}\mathfrak{K}_1, {}_{\mathcal{N}}\mathfrak{K}_2) = \big\{T \in \mathcal{L}(\mathfrak{K}_1, \mathfrak{K}_2) : Ta\eta = aT\eta,\ \eta \in \mathfrak{K}_1,\ a \in \mathcal{N}\big\}$. For $\mathcal{L}({}_{\mathcal{N}}\mathfrak{K}, {}_{\mathcal{N}}\mathfrak{K})$ we write simply $\mathcal{L}({}_{\mathcal{N}}\mathfrak{K})$. Throughout the rest of this section, we denote by $\mathfrak{H}$ a right $\mathcal{N}$-module and by $\mathfrak{K}$ a left $\mathcal{N}$-module. Observe that a right $\mathcal{N}$-module $\mathfrak{H}$ means canonically an $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})$-$\mathcal{N}$-bimodule and a left $\mathcal{N}$-module $\mathfrak{K}$ means canonically an $\mathcal{N}$-$\mathcal{L}({}_{\mathcal{N}}\mathfrak{K})^{\circ}$-bimodule. We are now going to construct the relative tensor product $\mathfrak{H} \otimes_{\psi} \mathfrak{K}$ of a right $\mathcal{N}$-module $\mathfrak{H}$ and a left $\mathcal{N}$-module $\mathfrak{K}$ which will depend on the choice of a faithful semi-finite normal weight ψ on $\mathcal{N}$. To put things in perspective, let us consider a standard von Neumann algebra $\{\mathcal{R}, L^2(\mathcal{R})\}$ and a pair of projections $e,\ f \in \mathcal{R}$ such that $\{\mathcal{N}, \mathfrak{K}\} \cong \{\mathcal{R}_{eJfJ},\ eL^2(\mathcal{R})f\}$ and $\mathcal{N} \cong \mathcal{R}_e$. Here we are considering the case that the both $\mathfrak{H}$ and $\mathfrak{K}$ are faithful modules, so that the central support $c(e)$ and $c(f)$ are both the identity. Thus after identifying $\mathcal{N}$ with $\mathcal{R}_e$ as an abstract von Neumann algebra, the left $\mathcal{N}$-module $\mathfrak{K}$ is identified with $eL^2(\mathcal{R})f$. This identification identifies also $\mathcal{L}({}_{\mathcal{N}}\mathfrak{K})$ with $\{\mathcal{R}'_{eJfJ},\ eL^2(\mathcal{R})f\}$. Also the right $\mathcal{N}$-module $\mathfrak{H}$ is identified with $gL^2(\mathcal{R})e$ for another projection $g \in \mathcal{R}$ which yields the identification of $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})$ with $\{\mathcal{R}_g,\ gL^2(\mathcal{R})e\}$. The relative tensor product of $\mathfrak{H}$ and $\mathfrak{K}$ means roughly the Hilbert space $g[L^2(\mathcal{R})eL^2(\mathcal{R})]f = gL^2(\mathcal{R})f$ with the $\mathcal{R}_g$-$\mathcal{R}_f$-bimodule structure. Here we observe that we cannot multiply an arbitrary pair $\xi \in gL^2(\mathcal{R})e$ and $\eta \in gL^2(\mathcal{R})f$ to have a vector "$\xi\eta$" $\in gL(\mathcal{R})f$. To associate the third vector "$\xi \otimes_{\mathcal{N}} \eta$" in $gL^2(\mathcal{R})f$, one needs to convert ξ into an operator $\pi_\ell(\xi)$, which belongs to $\mathcal{N}$, after fixing a faithful semi-finite normal weight ψ on $\mathcal{N}$, and then set $\xi \otimes_{\mathcal{N}} \eta$ to be $\pi_\ell(\xi)\eta$. In doing this, we see that only certain vectors ξ can be converted to $\pi_\ell(\xi)$, i.e. ξ must be left bounded relative to a certain left Hilbert algebra. However, in this way we will have too many choices. First, the projections $e,\ f, g$ are determined only up to equivalence. The modules $\mathfrak{H}$ and $\mathfrak{K}$ alone can not determine them. So we have to take an approach which relates the construction more directly to the modules $\mathfrak{H}$ and $\mathfrak{K}$.

We fix a von Neumann algebra $\mathcal{N}$, a right $\mathcal{N}$-module $\mathfrak{H}$ and a left $\mathcal{N}$-module $\mathfrak{K}$. We also need a faithful semi-finite normal weight ψ on $\mathcal{N}$ to be fixed. We know from Lemma 3.3 that the right module $\mathfrak{H}$ can be recovered from $\mathfrak{D}(\mathfrak{H}, \psi)$ and that the left module $\mathfrak{K}$ is also recoverable from $\mathfrak{D}'(\mathfrak{K}, \psi)$ (observe that ψ and ψ° are symmetric as they are both faithful, unlike the previous case of φ). We collect here a few facts about $\mathfrak{D}(\mathfrak{H}, \psi)$ (resp. $\mathfrak{D}'(\mathfrak{H}, \psi)$) and L_ψ (resp. R_ψ) which have been implicit in the previous arguments:

$$\left.\begin{aligned}
(\xi_1 \mid \xi_2) &= \psi\big(L_\psi(\xi_2)^* L_\psi(\xi_1)\big), && \xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi)\\
\big(\text{resp. } (\eta_1 \mid \eta_2) &= \psi\big(JR_\psi(\eta_1)^* R_\psi(\eta_2)J\big), && \eta_1, \eta_2 \in \mathfrak{D}'(\mathfrak{K}, \psi)\big);\\
\eta_\psi\big(L_\psi(\xi_2)^* L_\psi(\xi_1)\big) &= L_\psi(\xi_2)^*\xi_1, && \xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi)\\
\big(\text{resp. } \eta'_\psi\big(JR_\psi(\eta_1)^* R_\psi(\eta_2)J\big) &= R_\psi(\eta_2)^*\eta_1, && \eta_1, \eta_2 \in \mathfrak{D}'(\mathfrak{K}, \psi)\big).
\end{aligned}\right\} \quad (22)$$

It is easy to see that if $\mathfrak{H} = L^2(\mathcal{N}, \psi)$ with the right action of $\mathcal{N}$ (resp. $\mathfrak{K} = L^2(\mathcal{N}, \psi)$ with the left action of $\mathcal{N}$) as a right (resp. left) module, then

$$\begin{aligned}
&\mathfrak{D}(\mathfrak{H}, \psi) = \eta_\psi(\mathfrak{n}_\psi) = \mathfrak{B}_\psi, \quad && L_\psi(\xi) = \pi_\ell(\xi), \quad \xi \in \mathfrak{B}_\psi\\
\big(\text{resp.} \quad &\mathfrak{D}'(\mathfrak{K}, \psi) = \mathfrak{B}'_\psi, && R_\psi(\eta) = \pi_r(\eta), \quad \eta \in \mathfrak{B}'_\psi\big),
\end{aligned}$$

where $\mathfrak{B}_\psi$ (resp. $\mathfrak{B}'_\psi$) means the algebra of all left (resp. right) bounded vectors in $L^2(\mathcal{N}, \psi)$.

Proposition 3.15.

(i) *The sesquilinear form, say* B*, on the algebraic tensor product* $\mathfrak{D}(\mathfrak{H}, \psi) \otimes_{\mathrm{alg}} \mathfrak{K}$ *determined by:*

$$B\big(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\big) = \Big(\pi_{\mathfrak{K}}\big(L_\psi(\xi_2)^* L_\psi(\xi_1)\big)\eta_1 \;\Big|\; \eta_2\Big) \in \mathbf{C} \tag{23}$$

is positive, so that it defines an inner product on $\mathfrak{D}(\mathfrak{H}, \psi) \otimes_{\mathrm{alg}} \mathfrak{K}$*, which is in many cases degenerated.*

(ii) *If* $\xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi)$ *and* $\eta_1, \eta_2 \in \mathfrak{D}'(\mathfrak{K}, \psi)$*, then*

$$\Big(\pi_{\mathfrak{K}}\big(L_\psi(\xi_2)^* L_\psi(\xi_1)\big)\eta_1 \;\Big|\; \eta_2\Big) = \Big(\pi'_{\mathfrak{H}}\big(J R_\psi(\eta_1)^* R_\psi(\eta_2) J\big)\xi_1 \;\Big|\; \xi_2\Big). \tag{24}$$

(i$'$) *Dual to* (i)*, the sesquilinear form* B' *on* $\mathfrak{H} \otimes_{\mathrm{alg}} \mathfrak{D}'(\mathfrak{K}, \psi)$ *determined by:*

$$B'\big(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\big) = \Big(\pi_{\mathfrak{K}}\big(J R_\psi(\eta_1)^* R_\psi(\eta_2) J\big)\xi_1 \;\Big|\; \xi_2\Big) \in \mathbf{C} \tag{23$'$}$$

is positive and agrees with B *on* $\mathfrak{D}(H, \psi) \otimes_{\mathrm{alg}} \mathfrak{D}'(\mathfrak{K}, \psi)$.

PROOF:

(i) Suppose that $\xi_1, \dots, \xi_n \in \mathfrak{D}(\mathfrak{H}, \psi)$. Let $a_{k,j} = L_\psi(\xi_k)^* L_\psi(\xi_j)$, $k, j = 1, \dots, n$, and $a = [a_{k,j}]$ be an $n \times n$ matrix over $\mathcal{N}$. If $x_1, \dots, x_n \in \mathcal{A} = \mathcal{D}(\sigma^\psi_{\mathrm{i}/2}) \cap \mathcal{D}(\sigma^\psi_{-\mathrm{i}/2})$, then we have, by (12$'$),

$$\begin{aligned}\sum_{k,j=1}^n x_j^* L_\psi(\xi_j)^* L_\psi(\xi_k) x_k &= \sum_{k,j=1}^n L_\psi\big(\xi_j \sigma^\psi_{\frac{\mathrm{i}}{2}}(x_j)\big)^* L_\psi\big(\xi_k \sigma^\psi_{\frac{\mathrm{i}}{2}}(x_k)\big) \\ &= \left(\sum_{k=1}^n L_\psi\big(\xi_k \sigma^\psi_{\frac{\mathrm{i}}{2}}(x_k)\big)\right)^* \left(\sum_{k=1}^n L_\psi\big(\xi_k \sigma^\psi_{\frac{\mathrm{i}}{2}}(x_k)\big)\right) \geq 0.\end{aligned}$$

As $\mathcal{A}$ is σ-weakly dense in $\mathcal{N}$, the matrix a is positive in $M_n(\mathcal{N})$, so that there exists $b = [b_{k,j}] \in M_n(\mathcal{N})$ such that $a = b^* b$, i.e.,

$$a_{k,j} = \sum_{\ell=1}^n b^*_{\ell,k} b_{\ell,j}, \quad k, j = 1, \dots, n.$$

We then have, for $\eta_1, \dots, \eta_n \in \mathfrak{K}$,

$$B\left(\sum_{k=1}^n \xi_k \otimes \eta_k, \sum_{k=1}^n \xi_k \otimes \eta_k\right) = \sum_{k,j=1}^n \big(a_{k,j}\eta_j \mid \eta_k\big) = \sum_{k=1}^n \left\| \sum_{j=1} b_{k,j}\eta_j \right\|^2 \geq 0.$$

Hence the sesquilinear form B is positive.

(ii) Suppose that ξ's and η's are as in the Proposition. We then have, as $L_\psi(\xi_2)^* L_\psi(\xi_1) \in \mathfrak{m}_\psi \subset \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*$ and also $J R_\psi(\eta_2)^* R_\psi(\eta_1) J \in \mathfrak{m}_\psi$,

$$\begin{aligned}
&\Big(\pi_{\mathfrak{K}}\big(L_\psi(\xi_2)^* L_\psi(\xi_1)\big)\eta_1 \;\Big|\; \eta_2\Big) = \Big(R_\psi(\eta_1)\eta_\psi\big(L_\psi(\xi_2)^* L_\psi(\xi_1)\big) \;\Big|\; \eta_2\Big)\\
&\quad = \big(L_\psi(\xi_2)^*\xi_1 \mid R_\psi(\eta_1)^*\eta_2\big) = \Big(L_\psi(\xi_2)^*\xi_1 \;\Big|\; \eta'_\psi\big(JR_\psi(\eta_2)^* R_\psi(\eta_1)J\big)\Big)\\
&\quad = \Big(\xi_1 \;\Big|\; L_\psi(\xi_2)\eta'_\psi\big(JR_\psi(\eta_2)^* R_\psi(\eta_1)J\big)\Big)\\
&\quad = \Big(\xi_1 \;\Big|\; \pi'_{\mathfrak{H}}\big(JR_\psi(\eta_2)^* R_\psi(\eta_1)J\big)\xi_2\Big) = \Big(\pi'_{\mathfrak{H}}\big(JR_\psi(\eta_1)^* R_\psi(\eta_2)J\big)\xi_1 \;\Big|\; \xi_2\Big).
\end{aligned}$$

(i′) The positivity follows from (i) by symmetry. The second assertion follows from (ii). Q.E.D.

Definition 3.16. Let $\mathfrak{N}$ be the subspace of $\mathfrak{D}(\mathfrak{H}, \psi) \otimes_{\mathrm{alg}} \mathfrak{K}$ consisting of those vectors ζ with $B(\zeta, \zeta) = 0$. The Hilbert space obtained as the completion of the quotient space $\mathfrak{D}(\mathfrak{H}, \psi) \otimes_{\mathrm{alg}} \mathfrak{K}/\mathfrak{N}$ relative to the inner product induced by the positive sequilinear form B will be called the *relative tensor product* of the right $\mathcal{N}$-module $\mathfrak{H}$ and the left $\mathcal{N}$-module $\mathfrak{K}$ with respect to the faithful semi-finite normal weight ψ and will be written as $\mathfrak{H} \otimes_\psi \mathfrak{K}$ and the image of $\xi \otimes \eta$ as $\xi \otimes_\psi \eta$ for $\xi \in \mathfrak{D}(\mathfrak{H}, \psi),\ \eta \in \mathfrak{K}$. By Proposition 3.15, the relative tensor product $\mathfrak{H} \otimes_\psi \mathfrak{K}$ can also be obtained as the completion of the quotient space of the algebraic tensor product $\mathfrak{H} \otimes_{\mathrm{alg}} \mathfrak{D}'(\mathfrak{K}, \psi)$ by the subspace $\mathfrak{N}'$ consisting of null vectors with respect to the positive sequilinear form B'. In this way, we can consider the tensor product $\xi \otimes_\psi \eta$ for a pair $\xi \in \mathfrak{H},\ \eta \in \mathfrak{D}'(\mathfrak{K}, \psi)$.

Theorem 3.17. *Let $\mathcal{N}$ be a von Neumann algebra equipped with a faithful semi-finite normal weight ψ, $\mathfrak{H}$ a right $\mathcal{N}$-module and $\mathfrak{K}$ a left $\mathcal{N}$-module. Set $\mathcal{P} = \mathcal{L}(\mathfrak{H}_{\mathcal{N}})$ and $\mathcal{Q} = \mathcal{L}({}_{\mathcal{N}}\mathfrak{K})$. Consider the direct sum: $\tilde{\mathfrak{H}} = L^2(\mathcal{N}, \psi) \oplus \mathfrak{H} \oplus \bar{\mathfrak{K}}$ as a right $\mathcal{N}$-module and also $\mathcal{R} = \mathcal{L}(\tilde{\mathfrak{H}}_{\mathcal{N}})$ together with the "balanced" faithful semi-finite normal weight $\rho = \psi \oplus \varphi \oplus \nu$ where φ is a faithful semi-finite normal weight on $\mathcal{P}$ and ν is a faithful semi-finite normal weight on $\mathcal{Q}$. Let e, f and g be respectively the projections of $\tilde{\mathfrak{H}}$ onto $L^2(\mathcal{N}, \psi)$, $\mathfrak{H}$ and $\bar{\mathfrak{K}}$, which belong to $\mathcal{R}$. Represent the standard Hilbert space $\mathfrak{H}_\rho$ as the space of 3×3 matrices:*

$$\mathfrak{H}_\rho = \begin{pmatrix} L^2(\mathcal{N}, \psi) & \bar{\mathfrak{H}} & \mathfrak{K} \\ \mathfrak{H} & L^2(\mathcal{P}, \varphi) & \mathfrak{H}_{2,3} \\ \bar{\mathfrak{K}} & \mathfrak{H}_{3,2} & L^2(\mathcal{Q}, \nu) \end{pmatrix}. \tag{13′}$$

Then there exists a natural isomorphism between $\mathfrak{H} \otimes_\psi \mathfrak{K}$ and $\mathfrak{H}_{2,3}$.

PROOF: Let $\mathfrak{A}(= \mathfrak{A}_\rho)$ be the left Hilbert algebra associated with ρ, $\mathfrak{B}\ (= \mathfrak{B}_\rho)$ the algebra of left bounded elements in $L^2(\mathcal{R}, \rho)$ and $\mathfrak{n}_\rho = \big\{x \in \mathcal{R} : \varphi(x^* x) < \infty\big\}$. As e and f are both in $\mathcal{R}_\rho$, $\mathfrak{A}$ and $\mathfrak{B}$ are both decomposed into the matrix direct sum relative to (13′):

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{A}_{11} & \mathfrak{A}_{12} & \mathfrak{A}_{13} \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} & \mathfrak{A}_{23} \\ \mathfrak{A}_{31} & \mathfrak{A}_{32} & \mathfrak{A}_{33} \end{pmatrix}; \qquad \mathfrak{B} = \begin{pmatrix} \mathfrak{B}_{11} & \mathfrak{B}_{12} & \mathfrak{B}_{13} \\ \mathfrak{B}_{21} & \mathfrak{B}_{22} & \mathfrak{B}_{23} \\ \mathfrak{B}_{31} & \mathfrak{B}_{32} & \mathfrak{B}_{33} \end{pmatrix}. \tag{13″}$$

It follows from Lemma 3.3 that $\mathfrak{B}_{21} = \mathfrak{D}(\mathfrak{H}, \psi)$ and $\mathfrak{B}_{31} = \mathfrak{D}(\bar{\mathfrak{K}}, \psi)$. Also we have $L_\psi(\xi) = \pi_\ell(\xi)|_{\mathfrak{H}_{11}}$, $\xi \in \mathfrak{D}(\mathfrak{H}, \psi) = \mathfrak{B}_{21}$ and $L_\psi(\bar{\eta}) = \pi_\ell(\bar{\eta})|_{\mathfrak{H}_{12}}$, $\bar{\eta} \in \mathfrak{D}(\bar{\mathfrak{K}}, \psi) = \mathfrak{B}_{31}$, where π_ℓ means the left multiplication representation of $\mathfrak{B}$ on $\mathfrak{H}_\rho$. At this point, one should note that the right Hilbert algebra $\mathfrak{A}'$ and the algebra $\mathfrak{B}'$ of right bounded vectors admit also the similar matrix decompositions and that $\mathfrak{B}'_{21} = \mathfrak{D}'(\bar{\mathfrak{H}}, \psi)$, $\mathfrak{B}'_{31} = \mathfrak{D}'(\mathfrak{K}, \psi)$ and $R_\psi(\eta) = \pi_r(\eta)|_{\mathfrak{H}_{11}}$, $\eta \in \mathfrak{D}'(\mathfrak{K}, \psi)$.

We claim that $\xi \otimes_\psi \eta$, with $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$ and $\eta \in \mathfrak{K}$, is identified with $\pi_\ell(\xi)\eta \in \mathfrak{H}_{23}$. Let U_0 be the map from $\mathfrak{D}(\mathfrak{H}, \psi) \otimes_{\mathrm{alg}} \mathfrak{K}$ into $\mathfrak{H}_\rho$ determined by $U_0(\xi \otimes \eta) = \pi_\ell(\xi)\eta$ for $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$, $\eta \in \mathfrak{K}$. As $\xi \in \mathfrak{B}_{21}$ and $\eta \in \mathfrak{K} = \mathfrak{H}_{13}$, $\pi_\ell(\xi)\eta$ belongs to $\mathfrak{H}_{23}$. Now we have, for $\xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi)$ and $\eta_1, \eta_2 \in \mathfrak{K}$,

$$\begin{aligned}\big(U_0(\xi_1 \otimes \eta_1) \,\big|\, U_0(\xi_2 \otimes \eta_2)\big) &= \big(\pi_\ell(\xi_1)\eta_1 \,\big|\, \pi_\ell(\xi_2)\eta_2\big) = \big(\pi_\ell(\xi_2)^*\pi_\ell(\xi_1)\eta_1 \,\big|\, \eta_2\big)\\ &= \Big(\pi_{\mathfrak{K}}\big(L_\psi(\xi_2)^* L_\psi(\xi_1)\big)\eta_1 \,\Big|\, \eta_2\Big) = \big(\xi_1 \otimes_\psi \eta_1 \,\big|\, \xi_2 \otimes_\psi \eta_2\big).\end{aligned}$$

Therefore, the map U_0 gives rise to an isometry U of $\mathfrak{H} \otimes_\psi \mathfrak{K}$ into $\mathfrak{H}_{23}$. Let $\mathfrak{M} = U(\mathfrak{H} \otimes_\psi \mathfrak{K}) = \big[\pi_\ell(\mathfrak{B}_{21})\mathfrak{K}\big]$. First, we observe $\mathfrak{H}_{23} = eL^2(\mathcal{R}, \psi)g$, $\mathcal{P} = \mathcal{R}_e$ and $\mathcal{Q} = \mathcal{R}_g$. Hence $\pi_{\mathfrak{H}_{23}}(\mathcal{P})' = \pi'_{\mathfrak{H}_{23}}(\mathcal{Q})$. We know that $\mathfrak{M}$ is invariant under the right action of $\mathcal{Q}$. Hence the projection p of $\mathfrak{H}_{23}$ onto $\mathfrak{M}$ belongs to $\pi_{\mathfrak{H}_{23}}(\mathcal{P})$, i.e. p can be identified with the left multiplication by a projection in $\mathcal{P}$, which will be denoted by p again. This means that $\mathfrak{M} = p\mathfrak{H}_{23}$ with $p \in (\mathrm{Proj}\,\mathcal{P})$. But as $\pi_\ell(a\xi) = a\pi_\ell(\xi)$ for $a \in \mathcal{P}$ and $\xi \in \mathfrak{D}(\mathfrak{H}, \psi) = \mathfrak{B}_{21}$, $\mathfrak{B}_{23}$ is invariant under the left multiplication by $\mathcal{P}$, which in turn implies the invariance of $\mathfrak{M}$ under the left multiplication by $\mathcal{P}$. Hence, the projection p belongs to the center $\mathcal{C}_{\mathcal{P}}$ of $\mathcal{P}$, which is of the form: $\mathcal{C}_{\mathcal{P}} = (\mathcal{C}_{\mathcal{R}})_f$. Thus p can be viewed as a projection in $\mathcal{C}_{\mathcal{R}}$. Now we have $(1-p)\mathfrak{M} = \{0\}$, so that $0 = (1-p)\pi_\ell(\xi)\eta = \pi_\ell\big((1-p)\xi\big)\eta$ for every $\xi \in \mathfrak{B}_{21}$ and $\eta \in \mathfrak{K}$. Thus, $\pi_{\mathfrak{K}}\big(\pi_\ell((1-p)\xi)^*\pi_\ell((1-p)\xi)\big) = 0$. As $\mathfrak{K}$ is a faithful left $\mathcal{N}$-module, we have $\pi_\ell\big((1-p)\xi\big) = 0$, $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$, which means that $1 - p = 0$. Therefore, we have $\mathfrak{M} = \mathfrak{H}_{23}$.

Thus under the isometry U, $\mathfrak{H} \otimes_\psi \mathfrak{K}$ is identified with $\mathfrak{H}_{23}$. Q.E.D.

Corollary 3.18.

(i) *If $\mathfrak{H}$ and $\mathfrak{K}$ are respectively a right $\mathcal{N}$-module and a left $\mathcal{N}$-module for a von Neumann algebra $\mathcal{N}$ equipped with a faithful semi-finite normal weight ψ, then the relative tensor product $\mathfrak{H} \otimes_\psi \mathfrak{K}$ is naturally an $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})$-$\mathcal{L}({}_{\mathcal{N}}\mathfrak{K})^\circ$-bimodule which is determined by:*

$$\begin{aligned}&a(\xi \otimes_\psi \eta)b = (a\xi) \otimes_\psi (\eta b),\\ &\quad a \in \mathcal{L}(\mathfrak{H}_{\mathcal{N}}), \quad b \in \mathcal{L}({}_{\mathcal{N}}\mathfrak{K})^\circ, \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi), \quad \eta \in \mathfrak{K}.\end{aligned} \tag{25}$$

(ii) *In terms of operators acting from the left as usual, if $x \in \mathcal{L}(\mathfrak{H}_{\mathcal{N}})$ and $y \in \mathcal{L}({}_{\mathcal{N}}\mathfrak{K})$, then there exists a unique operator $x \otimes_\psi y \in \mathcal{L}(\mathfrak{H} \otimes_\psi \mathfrak{K})$ such that:*

$$(x\xi) \otimes_\psi (y\eta) = (x \otimes_\psi y)(\xi \otimes_\psi \eta), \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi), \quad \eta \in \mathfrak{K}. \tag{26}$$

The map: $(x, y) \in \mathcal{L}(\mathfrak{H}_{\mathcal{N}}) \times \mathcal{L}({}_{\mathcal{N}}\mathfrak{K}) \mapsto x \otimes_\psi y \in \mathcal{L}(\mathfrak{H} \otimes_\psi \mathfrak{K})$ extends canonically to an injective *-homomorphism from the algebraic tensor product, $\mathcal{L}(\mathfrak{H}_{\mathcal{N}}) \otimes_{\mathrm{alg}} \mathcal{L}({}_{\mathcal{N}}\mathfrak{K})$, into $\mathcal{L}(\mathfrak{H} \otimes_\psi \mathfrak{K})$.

(iii) *Although* $\mathcal{N}$ *does not act on the relative tensor product* $\mathfrak{H} \otimes_\psi \mathfrak{K}$*, we have:*

$$(\xi b) \otimes_\psi \eta = \xi \otimes_\psi \left(\sigma^\psi_{\frac{i}{2}}(b)\eta\right), \quad b \in \mathcal{D}\left(\sigma^\psi_{\frac{i}{2}}\right), \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi), \quad \eta \in \mathfrak{K}. \tag{27}$$

We leave the proof to the reader as an exercise.

Summarizing the above arguments, we restate the matrix decomposition of $\mathfrak{H}_\rho$ in the following form:

$$L^2(\mathcal{R}, \rho) = \begin{pmatrix} L^2(\mathcal{N}, \psi) & \bar{\mathfrak{H}} & \mathfrak{K} \\ \mathfrak{H} & L^2(\mathcal{P}, \varphi) & \mathfrak{H} \otimes_\psi \mathfrak{K} \\ \bar{\mathfrak{K}} & \bar{\mathfrak{K}} \otimes_\psi \bar{\mathfrak{H}} & L^2(\mathcal{Q}, \nu) \end{pmatrix}. \tag{28}$$

Proposition 3.19.

(i) *Viewing* $L^2(\mathcal{N}, \psi)$ *as a right* $\mathcal{N}$*-module, the map*

$$V^\psi_{\mathfrak{K}}: \ \eta_\psi(y) \otimes_\psi \eta \in L^2(\mathcal{N}, \psi) \otimes_\psi \mathfrak{K} \mapsto y\eta \in \mathfrak{K}, \quad y \in \mathfrak{n}_\psi, \quad \eta \in \mathfrak{K},$$

gives rise to an isomorphism of $L^2(\mathcal{N}, \psi) \otimes_\psi \mathfrak{K}$ *onto* $\mathfrak{K}$ *as* $\mathcal{N}$-$\mathcal{L}({}_\mathcal{N}\mathfrak{K})^\circ$*-bimodules.*

(ii) *If we look at* $L^2(\mathcal{N}, \psi)$ *as a left* $\mathcal{N}$*-module, then the map* $U^\psi_{\mathfrak{H}}$ *:*

$$\xi \otimes_\psi \eta'_\psi(y) \in \mathfrak{H} \otimes_\psi L^2(\mathcal{N}, \psi) \mapsto \xi y \in \mathfrak{H}, \quad \xi \in \mathfrak{H}, \quad y \in \mathfrak{n}^*_\psi$$

extends to an isomorphism of $\mathfrak{H} \otimes_\psi L^2(\mathcal{N}, \psi)$ *onto* $\mathfrak{H}$ *as* $\mathcal{L}(\mathfrak{H}_\mathcal{N})$-$\mathcal{N}$*-bimodules.*

The proof is now routine, so we leave it to the reader.

Also routine arguments show the following identities with canonical identifications:

$$\left.\begin{aligned} \left(\mathfrak{H}_1 \oplus \mathfrak{H}_2\right) \otimes_\psi \mathfrak{K} &= \left(\mathfrak{H}_1 \otimes_\psi \mathfrak{K}\right) \oplus \left(\mathfrak{H}_2 \otimes_\psi \mathfrak{K}\right); \\ \mathfrak{H} \otimes_\psi \left(\mathfrak{K}_1 \oplus \mathfrak{K}_2\right) &= \left(\mathfrak{H} \otimes_\psi \mathfrak{K}_1\right) \oplus \left(\mathfrak{H} \otimes_\psi \mathfrak{K}_2\right), \end{aligned}\right\} \tag{29}$$

where $\mathfrak{H}$, $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are right $\mathcal{N}$-modules whilst $\mathfrak{K}$, $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are left $\mathcal{N}$-modules.

Theorem 3.20. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be two von Neumann algebras equipped with faithful semi-finite normal weights* φ *and* ψ *respectively. If* $\mathfrak{H}$*,* $\mathfrak{K}$ *and* $\mathfrak{M}$ *are respectively a right* $\mathcal{M}$*-module, an* $\mathcal{M}$-$\mathcal{N}$*-bimodule and a left* $\mathcal{N}$*-module. Then under a natural identification we have*

$$\left(\mathfrak{H} \otimes_\varphi \mathfrak{K}\right) \otimes_\psi \mathfrak{M} = \mathfrak{H} \otimes_\varphi \left(\mathfrak{K} \otimes_\psi \mathfrak{M}\right) \tag{30}$$

as $\mathcal{L}(\mathfrak{H}_\mathcal{M})$-$\mathcal{L}({}_\mathcal{N}\mathfrak{M})^\circ$*-bimodules.*

PROOF: For each $\xi \in \mathfrak{D}(\mathfrak{H}, \varphi)$, $\eta \in \mathfrak{K}$ and $\zeta \in \mathfrak{D}'(\mathfrak{M}, \psi)$, set

$$U\big((\xi \otimes_\varphi \eta) \otimes_\psi \zeta\big) = \xi \otimes_\varphi (\eta \otimes_\psi \zeta).$$

Let ξ's, η's and ζ's be as above. We want to show that:

$$\Big(U\big((\xi_1 \otimes_\varphi \eta_1) \otimes_\psi \zeta_1\big) \Big| U\big((\xi_2 \otimes_\varphi \eta_2) \otimes_\psi \zeta_2\big)\Big) = \Big(\big(\xi_1 \otimes_\varphi \eta_1\big) \otimes_\psi \zeta_1 \Big| \big(\xi_2 \otimes_\varphi \eta_2\big) \otimes_\psi \zeta_2\Big),$$

as this shows that U is well-defined and a unitary. Let us compute:

$$\begin{aligned}
&\Big(U\big((\xi_1 \otimes_\varphi \eta_1) \otimes_\psi \zeta_1\big) \,\Big|\, U\big((\xi_2 \otimes_\varphi \eta_2) \otimes_\psi \zeta_2\big)\Big)\\
&= \big(\xi_1 \otimes_\varphi (\eta_1 \otimes_\psi \zeta_1) \,\big|\, \xi_2 \otimes_\varphi (\eta_2 \otimes_\psi \zeta_2)\big)\\
&= \Big(\pi_{\mathfrak{K} \otimes_\psi \mathfrak{M}}\big(L_\varphi(\xi_2)^* L_\varphi(\xi_1)\big)(\eta_1 \otimes_\psi \zeta_1) \,\Big|\, \eta_2 \otimes_\psi \zeta_2\Big)\\
&= \Big(\Big(\pi_{\mathfrak{K}}\big(L_\varphi(\xi_2)^* L_\varphi(\xi_1)\big)\eta_1\Big) \otimes_\psi \zeta_1 \,\Big|\, \eta_2 \otimes_\psi \zeta_2\Big)\\
&= \Big(\pi'_{\mathfrak{K}}\big(J R_\psi(\zeta_1)^* R_\psi(\zeta_2) J\big)\pi_{\mathfrak{K}}\big(L_\varphi(\xi_2)^* L_\varphi(\xi_1)\big)\eta_1 \,\Big|\, \eta_2\Big) \qquad \text{by (24)}\\
&= \Big(\pi_{\mathfrak{K}}\big(L_\varphi(\xi_2)^* L_\varphi(\xi_1)\big)\pi'_{\mathfrak{K}}\big(J R_\psi(\zeta_1)^* R_\psi(\zeta_2) J\big)\eta_1 \,\Big|\, \eta_2\Big)\\
&= \Big(\xi_1 \otimes_\varphi \Big(\pi'_{\mathfrak{K}}\big(J R_\psi(\zeta_1)^* R_\psi(\zeta_2) J\big)\eta_1 \,\Big|\, \xi_2 \otimes_\varphi \eta_2\Big)\\
&= \Big(\pi'_{\mathfrak{H} \otimes_\varphi \mathfrak{K}}\big(J R_\psi(\zeta_1)^* R_\psi(\zeta_2) J\big)\big(\xi_1 \otimes_\varphi \eta_1\big) \,\Big|\, \xi_2 \otimes_\varphi \eta_2\Big)\\
&= \Big(\big(\xi_1 \otimes_\varphi \eta_1\big) \otimes_\psi \zeta_1 \,\Big|\, \big(\xi_2 \otimes_\varphi \eta_2\big) \otimes_\psi \zeta_2\Big).
\end{aligned}$$

Now we have, for each $a \in \mathcal{L}(\mathfrak{H}_{\mathcal{M}})$ and $b \in \mathcal{L}({}_{\mathcal{N}}\mathfrak{M})^\circ$,

$$\begin{aligned}
U\Big(a\big((\xi \otimes_\varphi \eta) \otimes_\psi \zeta\big)b\Big) &= U\Big(\big((a\xi) \otimes_\varphi \eta\big) \otimes_\psi (\zeta b)\Big) = (a\xi) \otimes_\varphi \big(\eta \otimes_\psi (\zeta b)\big)\\
&= a\big(\xi \otimes_\varphi (\eta \otimes_\psi \zeta)\big)b = a\Big(U\big((\xi \otimes_\varphi \eta) \otimes_\psi \zeta\big)\Big)b.
\end{aligned}$$

Therefore, U is indeed an isomorphism of $(\mathfrak{H} \otimes_\varphi \mathfrak{K}) \otimes_\psi \mathfrak{M}$ onto $\mathfrak{H} \otimes_\varphi (\mathfrak{K} \otimes_\psi \mathfrak{M})$ as $\mathcal{L}(\mathfrak{H}_{\mathcal{M}})$-$\mathcal{L}({}_{\mathcal{N}}\mathfrak{M})^\circ$-bimodules. Q.E.D.

We now want to study what happens on relative tensor products when we change the reference faithful semi-finite normal weight ψ on a von Neumann algebra $\mathcal{N}$. To this end, we first fix a couple of notations. Let $\mathfrak{W}(\mathcal{M})$ denote the set of all semi-finite normal weights on a von Neumann algebra $\mathcal{M}$, and $\mathfrak{W}_0(\mathcal{M})$ the set of all faithful semi-finite normal weights on $\mathcal{M}$. With a fixed von Neumann algebra $\mathcal{N}$, we continue to study the relative tensor product of a right $\mathcal{N}$-module $\mathfrak{H}$ and a left $\mathcal{N}$-module $\mathfrak{K}$ relative to a faithful semi-finite normal weight ψ.

Theorem 3.21. *Let $\mathcal{N}$ be a von Neumann algebra, $\mathfrak{H}$ and $\mathfrak{K}$ be respectively right and left $\mathcal{N}$-modules. To each pair $(\psi_1, \psi_2) \in \mathfrak{W}_0(\mathcal{N}) \times \mathfrak{W}(\mathcal{N})$, there corresponds uniquely an $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})$-$\mathcal{L}({}_{\mathcal{N}}\mathfrak{K})^\circ$-bimodule isomorphism $U^{\psi_2,\psi_1}_{\mathfrak{H},\mathfrak{K}}$ of $\mathfrak{H} \otimes_{\psi_1} \mathfrak{K}$ onto $\mathfrak{H} \otimes_{\psi_2} \mathfrak{K}$ which makes the following diagram commutative for every pair $(a_i, b_i) \in \mathcal{L}\big(\mathfrak{H}_{\mathcal{N}}, L^2(\mathcal{N}, \psi_i)_{\mathcal{N}}\big) \times \mathcal{L}\big({}_{\mathcal{N}}L^2(\mathcal{N}, \psi_i), {}_{\mathcal{N}}\mathfrak{K}\big)$, $i = 1, 2$, such that $a_2 = U_{\psi_2,\psi_1} a_1$, $b_2 = b_1 U_{\psi_2,\psi_1}$ with U_{ψ_2,ψ_1} the canonical unitary implementing the equivalence of the standard forms $\big\{\mathcal{N}, L^2(\mathcal{N}, \psi_1), \mathfrak{P}_{\psi_1}, J_{\psi_1}\big\}$ and $\big\{\mathcal{N}, L^2(\mathcal{N}, \psi_2), \mathfrak{P}_{\psi_2}, J_{\psi_2}\big\}$:*

$$\left.\begin{array}{ccc}
\mathfrak{H} \otimes_{\psi_1} \mathfrak{K} & \xrightarrow{U^{\psi_2,\psi_1}_{\mathfrak{H},\mathfrak{K}}} & \mathfrak{H} \otimes_{\psi_2} \mathfrak{K} \\
{\scriptstyle (a_1 \otimes_{\psi_1} b_1^*)}\big\downarrow & & {\scriptstyle (a_2 \otimes_{\psi_2} b_2^*)}\big\downarrow \\
L^2(\mathcal{N}, \psi_1) \otimes_{\psi_1} L^2(\mathcal{N}, \psi_1) & & L^2(\mathcal{N}, \psi_2) \otimes_{\psi_2} L^2(\mathcal{N}, \psi_2) \\
{\scriptstyle U^{\psi_1}_{L^2(\mathcal{N},\psi_1)}}\big\downarrow & & {\scriptstyle U^{\psi_2}_{L^2(\mathcal{N},\psi_2)}}\big\downarrow \\
L^2(\mathcal{N}, \psi_1) & \xrightarrow{U_{\psi_2,\psi_1}} & L^2(\mathcal{N}, \psi_2)
\end{array}\right\} \tag{31}$$

The correspondence: $(\psi_1, \psi_2) \in \mathfrak{W}_0(\mathcal{N}) \times \mathfrak{W}_0(\mathcal{N}) \mapsto U^{\psi_2,\psi_1}_{\mathfrak{H},\mathfrak{K}}$ satisfies the chain rule:

$$U^{\psi_3,\psi_2}_{\mathfrak{H},\mathfrak{K}} U^{\psi_2,\psi_1}_{\mathfrak{H},\mathfrak{K}} = U^{\psi_3,\psi_1}_{\mathfrak{H},\mathfrak{K}}, \quad \psi_1, \psi_2, \psi_3 \in \mathfrak{W}_0(\mathcal{N}). \tag{32}$$

PROOF: Existence: We use the notations of Theorem 3.17. Choose faithful semi-finite normal weights $\varphi \in \mathfrak{W}_0(\mathcal{P})$ and $\nu \in \mathfrak{W}(\mathcal{Q})$ for $i = 1, 2$ and set $\rho_i = \psi_i \oplus \varphi \oplus \nu$, $i = 1, 2$ on $\mathcal{R}$, where one should observe that the construction of $\mathcal{R}$ does not depend on the choice of the faithful semi-finite normal weights ψ's. We then have the canonical isometry U_{ρ_2,ρ_1} from $L^2(\mathcal{R}, \rho_1)$ onto $L^2(\mathcal{R}, \rho_2)$ which implements an $\mathcal{R}$-$\mathcal{R}$-bimodule isomorphism. As the projections e, f and g commute with the faithful semi-finite normal weights ρ_1 and ρ_2, it is easy to check that the unitary U_{ρ_2,ρ_1} preserve the matrix decompositions (13$'$) of $L^2(\mathcal{R}, \rho_i)$, $i = 1, 2$. With J the conjugation operator: $\bar\eta \in \bar{\mathfrak{K}} \mapsto \eta \in \mathfrak{K}$, set $b_i^\circ = J b_i^* J \in \mathcal{L}\big(\bar{\mathfrak{K}}_{\mathcal{N}}, L^2(\mathcal{N}, \psi_i)\big)$, $i = 1, 2$. Then we have $A_i = \begin{pmatrix} 0 & a_i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B_i = \begin{pmatrix} 0 & 0 & b_i^\circ \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{R}$, $i = 1, 2$. We then see that the restriction of the operator $\pi_{\rho_i}(A_i)\pi'_{\rho_i}(B_i)^*$ to the (2,3)-component of $L^2(\mathcal{R}, \rho_i)$ is equal to $U^{\psi_i}_{L^2(\mathcal{N},\psi_i)}\big(a \otimes_{\psi_i} b_i^*\big)$, with π'_{ρ_i} the semi-cyclic anti-representation of $\mathcal{R}$ defined by $\pi'_{\rho_i}(x) = J\pi_{\rho_i}(x)^* J$, $x \in \mathcal{R}$. As U_{ρ_2,ρ_1} is an $\mathcal{R}$-$\mathcal{R}$-bimodule isomorphism of $L^2(\mathcal{R}, \rho_1)$ onto $L^2(\mathcal{R}, \rho_2)$ and carries the matrix decomposition (13$'$) of $L^2(\mathcal{R}, \rho_1)$ onto that of $L^2(\mathcal{R}, \rho_2)$, its restrictions to the (1,1)-component and (2,3)-component of (13$'$) gives U_{ψ_2,ψ_1} and the one for $U^{\psi_2,\psi_1}_{\mathfrak{H},\mathfrak{K}}$ which satisfy (31).

Uniqueness: Let $\mathfrak{A}_i\ (= \mathfrak{A}_{\psi_i})$, $i = 1, 2$, be the left Hilbert algebra associated wtih $\{\mathcal{N}, \psi_i\}$ and $\mathfrak{A}_i^0$ be the associated Tomita algebra. Set $\mathfrak{a}_i = \mathfrak{n}_{\psi_i} \cap \mathfrak{n}_{\psi_i}^* = \pi_\ell(\mathfrak{A}_i)$

and $\mathfrak{a}_i^0 = \pi_\ell(\mathfrak{A}_i^0)$, $i = 1, 2$. For each $\xi \in \mathfrak{D}(\mathfrak{H}, \psi_1)$, $\eta \in \mathfrak{D}'(\mathfrak{K}, \psi_1)$ and each $y_1, y_2 \in \mathfrak{a}_i^0$, we have with $a_1 = L_{\psi_1}(\xi)^*$ and $b_1 = R_{\psi_1}(\eta)$ in (31):

$$\begin{aligned}
U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}\big(\xi y_1 \otimes_{\psi_1} y_2\eta\big) &= U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}\big(L_{\psi_1}(\xi)\eta'_{\psi_1}(y_1) \otimes_{\psi_1} R_{\psi_1}(\eta)\eta_{\psi_1}(y_2)\big)\\
&= U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}\big(L_{\psi_1}(\xi) \otimes_{\psi_1} R_{\psi_1}(\eta)\big)\big(\eta'_{\psi_1}(y_1) \otimes_{\psi_1} \eta_{\psi_1}(y_2)\big)\\
&= U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}\big(a_1^* \otimes_{\psi_1} b_1\big)\big(U_{L^2(\mathcal{N},\psi_1)}^{\psi_1}\big)^*\big(\sigma_{-\frac{\mathrm{i}}{2}}^{\psi_1}(y_1)\eta_{\psi_1}(y_2)\big)\\
&= \big(a_2^* \otimes_{\psi_2} b_2\big)\big(U_{L^2(\mathcal{N},\psi_2)}^{\psi_2}\big)^* U_{\psi_2,\psi_1}\big(\sigma_{-\frac{\mathrm{i}}{2}}^{\psi_1}(y_1)\eta_{\psi_1}(y_2)\big).
\end{aligned}$$

This means that $U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}$ is uniquely determined on the vectors of the form: $\xi y_1 \otimes_{\psi_1} y_2\eta$ with ξ, η, y_1 and y_2 as above, which are dense in $\mathfrak{H} \otimes_{\psi_1} \mathfrak{K}$. Hence, $U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}$ is uniquely determined by the commutative diagram of (31).

The chain rule (32) follows from the uniqueness of $U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}$. Q.E.D.

Remark 3.22. The bimodule isomorphism $U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}$ does not send $\xi \otimes_{\psi_1} \eta$ into $\xi \otimes_{\psi_2} \eta$ for $\xi \in \mathfrak{H}$ and $\eta \in \mathfrak{K}$.

At this point, one might puzzle if one can construct the relative tensor products $\mathfrak{H} \otimes_{\mathcal{N}} \mathfrak{K}$ directly from the right $\mathcal{N}$-module $\mathfrak{H}$ and the left $\mathcal{N}$-module $\mathfrak{K}$. It is in fact possible to do so if one gives up the tensor product $\xi \otimes_{\mathcal{N}} \eta$ of vectors. We will discuss a weight free construction of the relative tensor product $\mathfrak{H} \otimes_{\mathcal{N}} \mathfrak{K}$ in the exercise.

Before closing the section, we discuss an example. Let $\mathcal{A}$ be an abelian separable von Neumann algebra and let it act on a separable Hilbert space $\mathfrak{H}$. As $\mathcal{A}$ is abelian, $\mathfrak{H}$ can be viewed as two sided module over $\mathcal{A}$. Let $\mathfrak{K}$ be another separable Hilbert space on which $\mathcal{A}$ acts. Now we view $\mathfrak{H}$ as a right $\mathcal{A}$-module and $\mathfrak{K}$ as a left $\mathcal{A}$-module. Let ψ be a faithful semi-finite normal weight on $\mathcal{A}$ and consider $\mathfrak{H} \otimes_\psi \mathfrak{K}$. We want to identify $\mathfrak{H} \otimes_\psi \mathfrak{K}$ more explicitly. First, consider the direct disintegration of $\mathfrak{H}$ and $\mathfrak{K}$ relative to $\mathcal{A}$:

$$\mathfrak{H} = \int_\Gamma^\oplus \mathfrak{H}(\gamma)\, \mathrm{d}\mu(\gamma); \qquad \mathfrak{K} = \int_\Gamma^\oplus \mathfrak{K}(\gamma)\, \mathrm{d}\mu(\gamma),$$

where $\mathcal{A} = L^\infty(\Gamma, \mu)$ is the measure theoretic spectral representation of $\mathcal{A}$ and μ is the measure corresponding to the weight ψ.

Proposition 3.23. *Under the above situation:*

$$\mathfrak{H} \otimes_\psi \mathfrak{K} = \int_\Gamma^\oplus \mathfrak{H}(\gamma) \otimes \mathfrak{K}(\gamma)\, \mathrm{d}\mu(\gamma)$$

and

$$\xi \otimes_\psi \eta = \int_\Gamma^\oplus \xi(\gamma) \otimes \eta(\gamma)\, \mathrm{d}\mu(\gamma)$$

if

$$\xi = \int_{\Gamma}^{\oplus} \xi(\gamma)\, \mathrm{d}\mu(\gamma) \in \mathfrak{H}; \qquad \eta = \int_{\Gamma}^{\oplus} \eta(\gamma)\, \mathrm{d}\mu(\gamma) \in \mathfrak{K}$$

are such that $\int_{\Gamma} \left\|\xi(\gamma)\right\|^2 \left\|\eta(\gamma)\right\|^2 \mathrm{d}\mu(\gamma) < \infty$.

We leave the proof to the reader as it is an easy good exercise. Here the reader should note why the relative tensor product of an arbitrary pair of vectors does not make sense and also why one need to fix a faithful semi-finite normal weight on $\mathcal{A}$ before considering the relative tensor product of vectors.

Exercise IX.3

Throughout this exercise, let $\mathcal{M}$ be a fixed von Neumann algebra.

1) In the positive cone $\mathcal{M}_*^+$, denote each element of $\mathcal{M}_*^+$ by $\varphi^{1/2}$, $\varphi \in \mathcal{M}_*^+$. Define for each pair $\varphi, \psi \in \mathcal{M}_*^+$:

$$\varphi^{\frac{1}{2}} + \psi^{\frac{1}{2}} = \chi,$$

where $\chi \in \mathcal{M}_*^+$ is given by

$$\chi(x) = (\varphi + \psi)(a^* x a), \quad x \in \mathcal{M},$$

$$a = \left(\mathrm{D}\varphi : \mathrm{D}(\varphi + \psi)\right)_{-\frac{\mathrm{i}}{2}} + \left(\mathrm{D}\psi : \mathrm{D}(\varphi + \psi)\right)_{-\frac{\mathrm{i}}{2}};$$

$$\lambda \cdot \varphi^{\frac{1}{2}} = \left(\lambda^2 \varphi\right)^{\frac{1}{2}}, \quad \lambda \geq 0;$$

$$\left(\varphi^{\frac{1}{2}} \mid \psi^{\frac{1}{2}}\right) = (\varphi + \psi)(a^* a).$$

(a) Let $\{\mathcal{M}, \mathfrak{H}, J, \mathcal{P}\}$ be a standard form and, for each $\varphi \in \mathcal{M}_*^+$, $\xi(\varphi)$ be the representing vector of φ in $\mathcal{P}$. By making use of the map: $\varphi \in \mathcal{M}_*^+ \to \xi(\varphi) \in \mathcal{P}$, show that

$$\varphi^{\frac{1}{2}} + \psi^{\frac{1}{2}} = \psi^{\frac{1}{2}} + \varphi^{\frac{1}{2}};$$

$$\varphi^{\frac{1}{2}} + 0^{\frac{1}{2}} = \varphi^{\frac{1}{2}};$$

$$\left(\varphi^{\frac{1}{2}} + \psi^{\frac{1}{2}}\right) + \chi^{\frac{1}{2}} = \varphi^{\frac{1}{2}} + \left(\psi^{\frac{1}{2}} + \chi^{\frac{1}{2}}\right);$$

$$\varphi^{\frac{1}{2}} + \chi^{\frac{1}{2}} = \psi^{\frac{1}{2}} + \chi^{\frac{1}{2}} \iff \varphi = \psi;$$

$$0 \cdot \varphi^{\frac{1}{2}} = 0^{\frac{1}{2}};$$

$$(\lambda\mu) \cdot \varphi^{\frac{1}{2}} = \lambda \cdot \left(\mu \cdot \varphi^{\frac{1}{2}}\right);$$

$$(\lambda + \mu) \cdot \varphi^{\frac{1}{2}} = \lambda \cdot \varphi^{\frac{1}{2}} + \mu \cdot \varphi^{\frac{1}{2}};$$

$$\lambda \cdot \left(\varphi^{\frac{1}{2}} + \psi^{\frac{1}{2}}\right) = \lambda \cdot \varphi^{\frac{1}{2}} + \lambda \cdot \varphi^{\frac{1}{2}}.$$

(**b**) Show that the relation: $\left(\varphi_1^{1/2}, \psi_1^{1/2}\right) \sim \left(\varphi_2^{1/2}, \psi_2^{1/2}\right)$ in $\mathcal{M}_*^+ \times \mathcal{M}_*^+$ defined by $\varphi_1^{1/2} \boldsymbol{+} \psi_2^{1/2} = \varphi_2^{1/2} \boldsymbol{+} \psi_1^{1/2}$ is an equivalence relation. Denote the equivalence class of $\left(\varphi^{1/2}, \psi^{1/2}\right)$ by $\varphi^{1/2} \boldsymbol{-} \psi^{1/2}$.

(**c**) Show that the quotient space $\mathcal{M}_*^+ \times \mathcal{M}_*^+ \big/$"$\sim$" is a real Hilbert space under the inner product:

$$\left(\varphi_1^{\frac{1}{2}} \boldsymbol{-} \psi_1^{\frac{1}{2}} \mid \varphi_2^{\frac{1}{2}} \boldsymbol{-} \psi_2^{\frac{1}{2}}\right) = \left(\varphi_1^{\frac{1}{2}} \mid \varphi_2^{\frac{1}{2}}\right) - 2\left(\varphi_1^{\frac{1}{2}} \mid \psi_2^{\frac{1}{2}}\right) + \left(\psi_1^{\frac{1}{2}} \mid \psi_2^{\frac{1}{2}}\right),$$

and the multiplication by scalars:

$$(\lambda - \mu) \cdot \left(\varphi^{\frac{1}{2}} \boldsymbol{-} \psi^{\frac{1}{2}}\right) = \lambda \cdot \varphi^{\frac{1}{2}} \boldsymbol{+} \mu \cdot \psi^{\frac{1}{2}} \boldsymbol{-} \lambda \cdot \psi^{\frac{1}{2}} \boldsymbol{-} \mu \cdot \varphi^{\frac{1}{2}}$$

for $\lambda, \mu \geq 0$. Denote this real Hilbert space by $L^2_{\mathbf{R}}(\mathcal{M})$, and the complexified Hilbert space: $L^2_{\mathbf{R}}(\mathcal{M}) \otimes_{\mathbf{R}} \mathbf{C}$ simply by $L^2(\mathcal{M})$.

(**d**) Observe that the canonical Hilbert space $L^2(\mathcal{M})$ constructed above is indeed canonical to the von Neumann algebra $\mathcal{M}$ which depends solely on $\mathcal{M}$ alone but nothing else.

(**e**) Let $L^2(\mathcal{M})_+$ denote the convex cone $\left\{\varphi^{1/2} : \varphi \in \mathcal{M}_*^+\right\}$ in $L^2(\mathcal{M})$, and define:

$$J_{\mathcal{M}}(\xi + \mathrm{i}\eta) = \xi - \mathrm{i}\eta, \quad \xi, \eta \in L^2_{\mathbf{R}}(\mathcal{M}).$$

2) Considering a standard form $\left\{\mathcal{M}, \mathfrak{H}, J, \mathcal{P}\right\}$, proceed the following.

(**a**) For a fixed $\varphi \in \mathcal{M}_*^+$ and $x, y \in \mathcal{M}$, show that the function:

$$f^{\varphi}_{x,y}(t) = \varphi\left(\sigma^{\varphi}_{-t}(x)\, x^*\, y\, x\, \sigma^{\varphi}_t(x)\right), \quad t \in \mathbf{R},$$

can be extended to a function $f^{\varphi}_{x,y} \in \mathcal{A}\left(\mathbf{D}_{1/2}\right)$, where $\mathcal{D}_{1/2}$ is the horizontal strip bounded by $\mathbf{R}$ and $\mathbf{R} - \frac{\mathrm{i}}{2}$ and $\mathcal{A}\left(\mathbf{D}_{1/2}\right)$ is the space of functions homomorphic on $\mathbf{D}_{1/2}$, bounded and continuous on the closure $\overline{\mathbf{D}_{1/2}}$.

(**b**) Show that the map: $y \in \mathcal{M} \to f^{\varphi}_{x,y}\left(-\frac{\mathrm{i}}{2}\right) = \left(\alpha(x)\varphi\right)(y)$ is a positive linear functional of y, so that $\alpha(x)\varphi \in \mathcal{M}_*^+$. (Hint: $\xi(\alpha(x)\varphi) = x\xi(\varphi)x^*$.)

(**c**) Show that the map $\rho(x)\colon \varphi^{1/2} \in L^2(\mathcal{M})_+ \to \left(\alpha(x)\varphi\right)^{1/2} \in L^2(\mathcal{M})_+$ can be uniquely extended to a bounded linear operator, denoted again by $\rho(x)$, on $L^2(\mathcal{M})$.

(**d**) Show that $\rho(x_1x_2) = \rho(x_1)\rho(x_2)$ for $x_1, x_2 \in \mathcal{M}$ and $\rho(1) = 1$.

(**e**) Show that for each $x \in \mathcal{M}$, $\left\{\rho(\exp tx)\right\}_{t\in\mathbf{R}}$ is a norm continuous one parameter group in $\mathcal{L}\left(L^2(\mathcal{M})\right)$. (Hint: $\rho(\exp tx)$ corresponds to $\exp(tx)J\exp(tx)J$.)

(**f**) Set $\delta(x) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho(\exp tx)\right)\Big|_{t=0}$, $x \in \mathcal{M}$. Show that δ is a real linear Lie algebra homomorphism of $\mathcal{M}$ into $\mathcal{L}(L^2)\big)$. (Hint: Recall Exercise IX.1.)

(**g**) Set

$$\pi(x) = \delta(x) - \mathrm{i}\delta(\mathrm{i}x), \quad x \in \mathcal{M};$$
$$\pi'(x) = \delta(x^*) + \mathrm{i}\delta(\mathrm{i}x^*).$$

Prove that π (resp. π') is a faithful normal representation (resp. anti-representation) of $\mathcal{M}$ on $L^2(\mathcal{M})$ such that $\{\pi(\mathcal{M}),\ L^2(\mathcal{M}),\ J_{\mathcal{M}},\ L^2(\mathcal{M})_+\}$ is a standard from of $\mathcal{M}$ and

$$J_{\mathcal{M}}\pi(x)^* J_{\mathcal{M}} = \pi'(x), \quad x \in \mathcal{M}.$$

We call $\{\pi(\mathcal{M}),\ L^2(\mathcal{M}),\ J_{\mathcal{M}},\ L^2(\mathcal{M})_+\}$ *the* standard form of $\mathcal{M}$.

3) For a semi-finite normal weight φ on $\mathcal{M}$, the map $\eta_\varphi : x \in \mathfrak{n}_\varphi \to \eta_\varphi(x) \in \mathfrak{H}_\varphi$ can be interpreted as $x\varphi^{1/2}$, viewing $\varphi^{1/2}$ as an infinitely long vector of $L^2(\mathcal{M})$ unless φ is finite, i.e.

$$\left\|\varphi^{\frac{1}{2}}\right\|_{L^2(\mathcal{M})} = \varphi(1)^{\frac{1}{2}} = +\infty.$$

Observe that this new interpretation of $\eta_\varphi(x)$ as $x\varphi^{1/2}$ is consistent with the previously established concepts and notations.

4†) It is also possible to construct two sided Banach $\mathcal{M}$-modules $L^p(\mathcal{M}),\ p \geq 1$, from $\left\{\varphi : \varphi \in \mathcal{M}_*^+\right\}$, [596].

5) Let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra and set $\mathcal{N} = (\mathcal{M}')^\circ$. View $\mathfrak{H}$ as an $\mathcal{M}$-$\mathcal{N}$-bimodule. Let ψ' be a faithful semi-finite normal weight on $\mathcal{M}'$ and $\psi = (\psi')^\circ$ on $\mathcal{N}$. With φ a faithful semi-finite normal weight on $\mathcal{M}$, let $\tilde{\mathfrak{H}}$, $\mathcal{R}$, ρ and others as in the earlier part of the section. Prove that if $\xi \in \mathfrak{D}(\mathfrak{H}, \psi) = \mathfrak{H} \cap \mathfrak{B}$, then $\pi_\ell(\xi) = f\pi_\ell(\xi)e$ and $L_\psi(\xi) = \pi_\ell(\xi)\big|_{L^2(\mathcal{N},\psi)}$.

6) In the previous problem, consider the relative tensor product: $\mathfrak{H} \otimes_\psi \bar{\mathfrak{H}}$. Let J be the conjugate linear involution of $\mathfrak{H} \otimes_\psi \bar{\mathfrak{H}}$ defined by $J(\xi \otimes_\psi \bar{\eta}) = (\eta \otimes_\psi \bar{\xi})$, $\xi, \eta \in \mathfrak{D}(\mathfrak{H}, \psi)$ and $(\mathfrak{H} \otimes_\psi \bar{\mathfrak{H}})_+$ be the closed convex cone in $\mathfrak{H} \otimes_\psi \bar{\mathfrak{H}}$ generated by $\left\{\xi \otimes_\psi \bar{\xi} : \xi \in \mathfrak{D}(\mathfrak{H}, \psi)\right\}$. Prove that $\left\{\mathfrak{H} \otimes_\psi \bar{\mathfrak{H}},\ (\mathfrak{H} \otimes_\psi \bar{\mathfrak{H}})_+,\ J\right\}$ is isormorphic to the standard form $\left\{L^2(\mathcal{M}),\ L^2(\mathcal{M})_+,\ J\right\}$ of $\mathcal{M}$ as an $\mathcal{M}$-$\mathcal{M}$-bimodule.

7) Keep the set-up of the last two problems. Let $L^2(\mathcal{N})$ be the standard form of $\mathcal{N}$. Let $\tilde{\mathfrak{H}} = L^2(\mathcal{N}) \oplus \mathfrak{H}$ and $\mathcal{R} = \mathcal{L}(\tilde{\mathfrak{H}}_{\mathcal{N}})$. Notice that in doing this we do not pick up a semi-finite normal weight ψ on $\mathcal{N}$ unlike the last problem or in the main part of the section. Following the steps described below, show that $\tilde{\mathfrak{H}}$ can be canonically embedded into the standard form $L^2(\mathcal{R})$ such a way that $J\tilde{\mathfrak{H}} = L^2(\mathcal{R}) \oplus \bar{\mathfrak{H}}$: $J(\xi \oplus \eta) = \xi^* \oplus \bar{\eta},\ \xi \in L^2(\mathcal{N}),\ \eta \in \mathfrak{H}$, where ξ^* means the modular conjugation in $L^2(\mathcal{N})$ and $\bar{\eta}$ means the element in $\bar{\mathfrak{H}}$ corresponding to $\eta \in \mathfrak{H}$ under the canonical conjugation.

(**a**) To each $\xi \in \tilde{\mathfrak{H}}$, there corresponds a unique element $|\xi| \in L^2(\mathcal{N})_+$ and a partial isometry $u \in \mathcal{R}$ such that $(\xi x \mid \xi) = \big(|\xi|x \bigm| |\xi|\big),\ x \in \mathcal{N},\ \xi = u|\xi|$ and u^*u is the projection from $\tilde{\mathfrak{H}}$ onto $\big[|\xi|\mathcal{N}\big]$, the smallest closed linear subspace containing $|\xi|\mathcal{N}$.

(**b**) As $\mathcal{N} = \mathcal{R}_e$, where e is the projection of $\tilde{\mathfrak{H}}$ onto $L^2(\mathcal{N})$ as before, we have $L^2(\mathcal{N}) = eL^2(\mathcal{R})e$.

(c) Set $U\xi = u|\xi|,\ \xi \in \tilde{\mathfrak{H}}$ and show that U is the required isometry embedding $\tilde{\mathfrak{H}}$ into $L^2(\mathcal{R})$ and $U\mathfrak{H} = fL^2(\mathcal{R})e$.

8) Let $\mathcal{N}$ be a von Neumann algebra, $\mathfrak{H}$ a right $\mathcal{N}$-module and $\mathfrak{K}$ a left $\mathcal{N}$-module. Let $\tilde{\mathfrak{H}} = L^2(\mathcal{N}) \oplus \mathfrak{H} \oplus \bar{\mathfrak{K}}$ and $\mathcal{R} = \mathcal{L}(\tilde{\mathfrak{H}}_{\mathcal{N}})$. By (7), we view $L^2(\mathcal{N})$, $\mathfrak{H}$ and $\bar{\mathfrak{K}}$ as closed subspaces of $L^2(\mathcal{R})$ and let $e,\ f$ and g be respectively the projection from $L^2(\mathcal{R})$ onto these subspaces. With this identification, we have $\mathfrak{H} = fL^2(\mathcal{R})e$ and $\bar{\mathfrak{K}} = gL^2(\mathcal{R})e$, which means that $\mathfrak{K} = eL^2(\mathcal{R})g$. Define the relative tensor product of $\mathfrak{H}$ and $\mathfrak{K}$ with respect to $\mathcal{N}$ in the following way:

$$\mathfrak{H} \otimes_{\mathcal{N}} \mathfrak{K} = fL^2(\mathcal{R})g,$$

which is an $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})$-$\mathcal{L}({}_{\mathcal{N}}\mathfrak{K})^\circ$-bimodule in the natural fashion. In this way, we eliminated the dependence of the relative tensor product on faithful semi-finite normal weights on $\mathcal{N}$. Prove that there exists a natural bimodule isomorphism of $\mathfrak{H} \otimes_{\psi} \mathfrak{K}$ onto $\mathfrak{H} \otimes_{\mathcal{N}} \mathfrak{K}$.

9) Assume that $\mathcal{N}$ is equipped with faithful semi-finite normal trace $\tau_{\mathcal{N}}$. Let $\mathfrak{H}$ be a right $\mathcal{N}$-module and $\mathcal{M} = \mathcal{L}(\mathfrak{H}_{\mathcal{N}})$. Consider $\tilde{\mathfrak{H}}$ and $\mathcal{R}$ as before. Normalize the trace $\tau_{\mathcal{M}}$ on $\mathcal{M}$ so that

$$\tau_{\mathcal{M}}(xx^*) = \tau'_{\mathcal{N}}(x^*x), \quad x \in \mathcal{L}\big(L^2(\mathcal{N}, \tau_{\mathcal{N}})_{\mathcal{N}}, \mathfrak{H}_{\mathcal{N}}\big), \qquad \tau_{\mathcal{R}} = \tau_{\mathcal{N}} \oplus \tau_{\mathcal{M}},$$

and

$$\tau_{\mathcal{R}}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \tau_{\mathcal{N}}(x_{11}) + \tau_{\mathcal{M}}(x_{22}).$$

Identify $\mathfrak{H}$ with the (2, 1)-component of $L^2(\mathcal{R}, \tau_{\mathcal{R}})$ and $L^2(\mathcal{M}, \tau_{\mathcal{M}})$ with the (2, 2)-component of $L^2(\mathcal{R}, \tau_{\mathcal{R}})$ so that $\mathfrak{H}$ can be viewed as the L^2-space of measurable operators from $L^2(\mathcal{N}, \tau_{\mathcal{N}})$ into $L^2(\mathcal{M}, \tau_{\mathcal{M}})$ relative to the trace $\tau_{\mathcal{R}}$.

10) Consider a standard von Neumann algebra $\{\mathcal{N},\ L^2(\mathcal{N}),\ J,\ L^2(\mathcal{N})_+\}$ and the standard bimodule ${}_{\mathcal{N}}L^2(\mathcal{N})_{\mathcal{N}}$. For each $\alpha \in \mathrm{Aut}(\mathcal{N})$, let $\mathfrak{H}(\alpha)$ be the $\mathcal{N}$-$\mathcal{N}$ full bimodule obtained by:

$$x\xi y = xJ\alpha(y^*)J\xi, \quad x, y \in \mathcal{N}, \quad \xi \in L^2(\mathcal{N}).$$

Prove that if $\alpha, \beta \in \mathrm{Aut}(\mathcal{N})$, then $\mathfrak{H}(\alpha) \otimes_{\mathcal{N}} \mathfrak{H}(\beta) = \mathfrak{H}(\alpha\beta)$.

11) Show that if $\mathfrak{H}$ is a full $\mathcal{N}$-bimodule, then there exists $\alpha \in \mathrm{Aut}(\mathcal{N})$ such that $\mathfrak{H} = \mathfrak{H}(\alpha)$.

§ 4 Conditional Expectations and Operator Valued Weights

In probability theory, conditional expectations play a fundamental role. We discuss a non-commutative analogue of conditional expectations. In fact, it will play a key role in the structure theory of factors.

Definition 4.1. Let φ be a faithful (semi-finite normal) weight on a von Neumann algebra $\mathcal{M}$ and $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ such that the restriction $\varphi|_{\mathcal{N}}$ of φ to $\mathcal{N}$ is semi-finite. A linear map $\mathcal{E}$ of $\mathcal{M}$ onto $\mathcal{N}$ is called the *conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ with respect to φ* if the following conditions are satisfied:

$$\|\mathcal{E}(x)\| \leq \|x\|, \quad x \in \mathcal{M}; \tag{1}$$

$$\mathcal{E}(x) = x, \qquad x \in \mathcal{N}; \tag{2}$$

$$\varphi = \varphi \circ \mathcal{E}. \tag{3}$$

By Theorem III.3.4, a conditional expectation $\mathcal{E}$ of $\mathcal{M}$ onto $\mathcal{N}$ enjoys the following properties:

$$\left.\begin{aligned} \mathcal{E}(x^*x) &\geq 0, && x \in \mathcal{M}; \\ \mathcal{E}(axb) &= a\mathcal{E}(x)b, && a, b \in \mathcal{N}, \quad x \in \mathcal{M}; \\ \mathcal{E}(x)^*\mathcal{E}(x) &\leq \mathcal{E}(x^*x), && x \in \mathcal{M}. \end{aligned}\right\} \tag{4}$$

In fact, a conditional expectation is completely positive.

Theorem 4.2. *Let $\mathcal{M}$, $\mathcal{N}$ and φ be as in Definition* 4.1. *The existence of a conditional expectation $\mathcal{E}$ of $\mathcal{M}$ onto $\mathcal{N}$ with respect to φ is equivalent to the global invariance, $\sigma_t^{\varphi}(\mathcal{N}) = \mathcal{N}$, $t \in \mathbf{R}$, of $\mathcal{N}$ under the modular automorphism group. If this is the case, then the conditional expectation $\mathcal{E}$ is normal and uniquely determined by φ.*

PROOF: Existence $\Longrightarrow$ Invariance: Assume that $\mathcal{E}$ exists and $\varphi = \varphi \circ \mathcal{E}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the full left Hilbert algebras corresponding to $\{\mathcal{M}, \varphi\}$, and $\left\{\mathcal{N}, \varphi|_{\mathcal{N}}\right\}$. It follows that $\mathfrak{B}$ is a self-adjoint subalgebra of $\mathfrak{A}$. Let $\mathfrak{H}$ and $\mathfrak{K}$ be the completions of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, and E be the projection of $\mathfrak{H}$ onto $\mathfrak{K}$. For each $x \in \mathfrak{n}_{\varphi}$ and $y \in \mathcal{N} \cap \mathfrak{n}_{\varphi}$, we have

$$\begin{aligned} \big(\eta_{\varphi}(x) \mid \eta_{\varphi}(y)\big) &= \varphi(y^*x) = \varphi\big(\mathcal{E}(y^*x)\big) = \varphi\big(y^*\mathcal{E}(x)\big) \\ &= \Big(\eta_{\varphi}\big(\mathcal{E}(x)\big) \Bigm| \eta_{\varphi}(y)\Big) \qquad \text{by (4)}, \end{aligned}$$

so that

$$E\eta_{\varphi}(x) = \eta_{\varphi}\big(\mathcal{E}(x)\big), \quad x \in \mathfrak{n}_{\varphi}.$$

Since $\mathcal{E}$ preserves the *-operation, we have

$$E\xi^{\sharp} = (E\xi)^{\sharp}, \quad \xi \in \mathfrak{A}.$$

Employing the notations $\mathfrak{D}^{\sharp}(\mathfrak{A})$, $\mathfrak{D}^{\sharp}(\mathfrak{B}), \ldots$ to indicate the natural objects corresponding to $\mathfrak{A}$ and $\mathfrak{B}$, we have

$$ES\xi = SE\xi, \quad \xi \in \mathfrak{D}^{\sharp}(\mathfrak{A}).$$

Namely, E leaves $\mathfrak{D}^\sharp(\mathfrak{A})$ invariant globally and $ES \subset SE$. Hence we get $(1-2E)S \subset S(1-2E)$. Applying $(1-2E)$ from the right, we get $(1-2E)S(1-2E) \subset S$, so that

$$S = (1-2E)^2 S(1-2E)^2 \subset (1-E)S(1-E) \subset S,$$

hence we conclude

$$S = (1-2E)S(1-2E), \tag{5}$$

consequently

$$F = (1-2E)F(1-2E), \qquad \Delta = (1-2E)\Delta(1-2E). \tag{6}$$

Hence Δ and E commute, so Δ^{it} leaves $\mathfrak{K}$ invariant. Since $E\eta_\varphi(\mathfrak{n}_\varphi) = \eta_\varphi\big(\mathfrak{n}_\varphi \cap \mathcal{N}\big)$ by (3), $E\mathfrak{A} = \mathfrak{B}$ and

$$\Delta^{it}\mathfrak{B} = \Delta^{it}E\mathfrak{A} = E\Delta^{it}\mathfrak{A} = E\mathfrak{A} = \mathfrak{B}.$$

Hence Δ^{it} leaves $\mathfrak{B}$ invariant. Hence with the equation:

$$\eta_\varphi\big(\sigma_t^\varphi(x)\big) = \Delta^{it}\eta_\varphi(x), \quad x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^* \cap \mathcal{N},$$

we conclude that $\{\sigma_t^\varphi\}$ leaves $\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^* \cap \mathcal{N}$ invariant. Thus the semi-finiteness of φ on $\mathcal{N}$ yields the invariance of $\mathcal{N}$ under $\{\sigma_t^\varphi\}$ by the σ-weak density of $\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^* \cap \mathcal{N}$ in $\mathcal{N}$.

Invariance $\Longrightarrow$ Existence: Suppose $\sigma_t^\varphi(\mathcal{N}) = \mathcal{N}$, $t \in \mathbf{R}$. We use the notations: $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{H}$, $\mathfrak{K}$, $\Delta_{\mathfrak{A}}$, $\Delta_{\mathfrak{B}}$ and so on as before. For each $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$, we have $\Delta_{\mathfrak{A}}^{it}\eta_\varphi(x) = \eta_\varphi\big(\sigma_t^\varphi(x)\big)$, so that $\Delta_{\mathfrak{A}}^{it}\mathfrak{B} = \mathfrak{B}$. Hence $\Delta_{\mathfrak{A}}^{it}\mathfrak{K} = \mathfrak{K}$, so the projection E of $\mathfrak{H}$ onto $\mathfrak{K}$ and $\Delta_{\mathfrak{A}}^{it}$ commute. Since the modular automorphism group of $\varphi|_{\mathcal{N}}$ is the restriction of $\{\sigma_t^\varphi\}$ to $\mathcal{N}$, $\Delta_{\mathfrak{B}}^{it}$ is nothing but the restriction of $\Delta_{\mathfrak{A}}^{it}$ to $\mathfrak{K}$. Hence

$$\mathfrak{D}^\sharp(\mathfrak{B}) = \mathfrak{D}\big(\Delta_{\mathfrak{B}}^{\frac{1}{2}}\big) = \mathfrak{D}\big(\Delta_{\mathfrak{A}}^{\frac{1}{2}}\big) \cap \mathfrak{K} = \mathfrak{D}^\sharp(\mathfrak{A}) \cap \mathfrak{K}, \tag{7}$$

which means that $\Delta_{\mathfrak{B}}^{1/2}$ is precisely the restriction of $\Delta_{\mathfrak{A}}^{1/2}$ to $\mathfrak{K}$ and $S_{\mathfrak{B}} = S_{\mathfrak{A}}\big|_{\mathfrak{K}}$. Therefore, we get $J_{\mathfrak{B}} = J_{\mathfrak{A}}$. Thus, we denote them simply by S and J. Furthermore, we have

$$\mathfrak{D}^\flat(\mathfrak{B}) = \mathfrak{D}^\flat(\mathfrak{A}) \cap \mathfrak{K} = \mathfrak{D}\big(\Delta_{\mathfrak{A}}^{-\frac{1}{2}}\big) \cap \mathfrak{K}. \tag{8}$$

Therefore, we can safely omit the subscripts $\mathfrak{A}$ and $\mathfrak{B}$ from Δ, $\mathfrak{K}$ and others as well.

We now prove

$$\mathfrak{B} = \mathfrak{A} \cap \mathfrak{K}, \qquad \mathfrak{B}' = \mathfrak{A}' \cap \mathfrak{K}. \tag{9}$$

Clearly $\mathfrak{B} \subset \mathfrak{A} \cap \mathfrak{K}$. Suppose $\xi \in \mathfrak{A}' \cap \mathfrak{K}$. There exists a constant $\gamma > 0$ such that $\big\|\pi_\ell^{\mathfrak{A}}(\eta)\xi\big\| \le \gamma\|\eta\|$ for every $\eta \in \mathfrak{A}$. In particular, $\big\|\pi_\ell^{\mathfrak{B}}(\eta)\xi\big\| \le \gamma\|\eta\|$ for every $\eta \in \mathfrak{B}$. Hence ξ is right bounded with respect to $\mathfrak{B}$. Since $\xi \in \mathfrak{D}^\flat(\mathfrak{A}) \cap \mathfrak{K} = \mathfrak{D}^\flat(\mathfrak{B})$, we have $\xi \in \mathfrak{B}'$. Thus $\mathfrak{A}' \cap \mathfrak{K} \subset \mathfrak{B}'$. Applying J, we get

$$\mathfrak{A} \cap \mathfrak{K} = J(\mathfrak{A}' \cap \mathfrak{K}) \subset J\mathfrak{B}' = \mathfrak{B}.$$

Thus $\mathfrak{A} \cap \mathfrak{K} = \mathfrak{B}$. Applying J once again, we conclude $\mathfrak{B}' = \mathfrak{A}' \cap \mathfrak{K}$.

Now, we consider the associated Tomita algebras $\mathfrak{A}_0$ and $\mathfrak{B}_0$ respectively. By definition, we have

$$\mathfrak{A}_0 = \left\{ \xi \in \bigcap_{n \in \mathbf{Z}} \mathfrak{D}(\Delta^n) : \Delta^n \xi \in \mathfrak{A} \right\}.$$

Hence $\mathfrak{B}_0 = \mathfrak{A}_0 \cap \mathfrak{K}$ follows from the previous discussion. We now prove

$$\left.\begin{aligned} &\mathfrak{B} = E\mathfrak{A}, \qquad \mathfrak{B}' = E\mathfrak{A}', \qquad \mathfrak{B}_0 = E\mathfrak{A}_0; \\ &E(\xi\eta) = \xi E\eta, \qquad \xi \in \mathfrak{B}, \quad \eta \in \mathfrak{A}; \\ &E(\xi\eta) = (E\xi)\eta, \quad \xi \in \mathfrak{A}, \quad \eta \in \mathfrak{B}'. \end{aligned}\right\} \tag{10}$$

First, we note that $\mathfrak{K}$ is invariant under $\pi_\ell^{\mathfrak{A}}(\mathfrak{B})$ and $\pi_\ell^{\mathfrak{A}}(\mathfrak{B})$ and $\pi_r^{\mathfrak{A}}(\mathfrak{B}')$ by (9). Hence E and $\pi_\ell^{\mathfrak{A}}(\mathfrak{B})$ (resp. $\pi_r^{\mathfrak{A}}(\mathfrak{B}')$) commute, which means precisely the last two identities in (10). If $\xi \in \mathfrak{A}$ and $\eta \in \mathfrak{B}'$, then

$$\left\|\pi_r^{\mathfrak{B}}(\eta)E\xi\right\| = \left\|E\pi_r^{\mathfrak{A}}(\eta)\xi\right\| = \left\|E\pi_\ell^{\mathfrak{A}}(\xi)\eta\right\| \le \left\|\pi_\ell^{\mathfrak{A}}(\xi)\right\|\|\eta\|,$$

so that $E\xi$ is left bounded with respect to $\mathfrak{B}$. Since $\mathfrak{B} = E\mathfrak{B} \subset E\mathfrak{A}$, we conclude $E\mathfrak{A} = \mathfrak{B}$. Applying J, we get $E\mathfrak{A}' = \mathfrak{B}'$. Now, the last claim $E\mathfrak{A}_0 = \mathfrak{B}_0$ in (10) follows from the construction of Tomita algebras and the above established facts. Thus (10) now follows.

The second formula in (10) means

$$\pi_\ell^{\mathfrak{B}}(\xi)E = E\pi_\ell^{\mathfrak{A}}(\xi)E, \quad \xi \in \mathfrak{B}. \tag{11}$$

Hence we have $E\mathcal{R}_\ell(\mathfrak{A})E\big|_{\mathfrak{K}} = \mathcal{R}_\ell(\mathfrak{B})$. With $\pi = \pi_\varphi\big|_{\mathcal{N}}$, we set

$$\mathcal{E}(x) = \pi^{-1}\big(E\pi_\varphi(x)E\big), \quad x \in \mathcal{M}. \tag{12}$$

It then follows that $\mathcal{E}$ is a normal projection of norm one from $\mathcal{M}$ onto $\mathcal{N}$.

If $x \in \mathcal{M}$ and $y, z \in \mathfrak{n}_\varphi \cap \mathcal{N}$, then

$$\begin{aligned} \varphi(z^*xy) &= \big(\pi_\varphi(x)\eta_\varphi(y) \mid \eta_\varphi(z)\big) = \big(E\pi_\varphi(x)E\eta_\varphi(y) \mid \eta_\varphi(z)\big) \\ &= \Big(\pi_\varphi\big(\mathcal{E}(x)\big)\eta_\varphi(y) \Bigm| \eta_\varphi(z)\Big) = \varphi\big(z^*\mathcal{E}(x)y\big), \end{aligned}$$

so we get

$$\varphi(z^*xy) = \varphi\big(z^*\mathcal{E}(x)y\big), \quad x \in \mathcal{M}, \quad y, z \in \mathfrak{n}_\varphi \cap \mathcal{N}.$$

We now prove

$$\varphi(x) = \sup\Big\{ \big(\pi_\varphi(x)\eta \mid \eta\big) : \eta \in \mathfrak{B}', \ \left\|\pi_r^{\mathfrak{B}}(\eta)\right\| < 1 \Big\}, \quad x \in \mathfrak{m}_\varphi^+. \tag{13}$$

Since $\mathfrak{B}' = \mathfrak{A}' \cap \mathfrak{K}$ and $\left\|\pi_r^{\mathfrak{B}}(\eta)\right\| = \left\|\pi_r^{\mathfrak{A}}(\eta)\right\|$ for $\eta \in \mathfrak{B}'$, it follows from the proof of Theorem VII.1.17 that the left hand side of (13) majorizes the right hand side of (13). If $\xi \in \mathfrak{m}_\varphi^+$, then the non-degeneracy of $\pi_r^{\mathfrak{A}}(\mathfrak{B}')$ yields

$$\begin{aligned}\varphi(x) = \left\|\eta_\varphi(x^{\frac{1}{2}})\right\|^2 &= \sup\left\{ \left\|\pi_r^{\mathfrak{A}}(\eta)\eta_\varphi(x^{\frac{1}{2}})\right\|^2 : \eta \in \mathfrak{B}',\ \left\|\pi_r^{\mathfrak{A}}(\eta)\right\| < 1 \right\}\\ &= \sup\left\{ \left\|\pi_\varphi(x^{\frac{1}{2}})\eta\right\|^2 : \eta \in \mathfrak{B}',\ \left\|\pi_r^{\mathfrak{B}}(\eta)\right\| < 1 \right\}\\ &= \sup\left\{ \left(\pi_\varphi(x)\eta \mid \eta\right) : \eta \in \mathfrak{B}',\ \left\|\pi_r^{\mathfrak{B}}(\eta)\right\| < 1 \right\}.\end{aligned}$$

Hence (13) follows.

Now, if $x \in \mathfrak{m}_\varphi^+$, then we have

$$\begin{aligned}\varphi(x) &= \sup\left\{ \left(\pi_\varphi(x)\eta \mid \eta\right) : \eta \in \mathfrak{B}',\ \left\|\pi_r^{\mathfrak{B}}(\eta)\right\| < 1 \right\}\\ &= \sup\left\{ \left(E\pi_\varphi(x)E\eta \mid \eta\right) : \eta \in \mathfrak{B}',\ \left\|\pi_r^{\mathfrak{B}}(\eta)\right\| < 1 \right\}\\ &= \sup\left\{ \left(\pi_\varphi\big(\mathcal{E}(x)\big)\eta \,\Big|\, \eta\right) : \eta \in \mathfrak{B}',\ \left\|\pi_r^{\mathfrak{B}}(\eta)\right\| < 1 \right\} = \varphi\circ\mathcal{E}(x).\end{aligned}$$

If $\mathcal{E}(x) = 0$ for an $x \in \mathcal{M}_+$, then $E\pi_\varphi(x)E = 0$. Since $\pi_\varphi(x) \ge 0$, $\pi_\varphi(x)\mathfrak{K} = \{0\}$. Namely, $xy = 0$ for every $y \in \mathfrak{n}_\varphi \cap \mathcal{N}$. But $\mathfrak{n}_\varphi \cap \mathcal{N}$ is σ-weakly dense in $\mathcal{N}$, so that 1 is in the σ-weak closure of $\mathfrak{n}_\varphi \cap \mathcal{N}$; thus $x = 0$. Therefore, $\mathcal{E}$ is faithful.

Finally, we complete the proof of the theorem by showing $\varphi = \varphi\circ\mathcal{E}$. To this end, set $\psi = \varphi\circ\mathcal{E}$. The commutativity of Δ and $\mathcal{E}$ implies the invariance of ψ under $\left\{\sigma_t^\varphi\right\}$. Hence Corollary VIII.3.6 implies the existence of a non-singular positive self-adjoint operator h affiliated with $\mathcal{M}_\varphi$ such that $\psi = \varphi_h$. As seen above, $\varphi(x) = \psi(x)$ for every $x \in \mathfrak{m}_\varphi$, so that for each $x \in \mathfrak{n}_\varphi$ we have, with $h_\varepsilon = h\,(1+\varepsilon h)^{-1}$,

$$\lim_{\varepsilon\to 0}\big(\pi_\varphi(h_\varepsilon)\eta_\varphi(x) \mid \eta_\varphi(x)\big) = \varphi_h(x^*x) = \psi(x^*x) = \varphi(x^*x) = \left\|\eta_\varphi(x)\right\|^2.$$

Hence $\eta_\varphi(\mathfrak{n}_\varphi) \subset \mathfrak{D}\big(\pi_\varphi(h)^{1/2}\big)$ and $\left\|\pi_\varphi(h)^{1/2}\eta_\varphi(x)\right\|^2 = \left\|\eta_\varphi(x)\right\|^2,\ x \in \mathfrak{n}_\varphi$. Hence $h = 1$, so $\varphi = \psi$. Q.E.D.

Proposition 4.3. *Let $\mathcal{M}$ be a von Neumann algebra, and $\mathcal{N}$ be a von Neumann subalgebra such that $\mathcal{N}' \cap \mathcal{M} = \mathcal{C}_\mathcal{N} \subset \mathcal{N}$. If $\mathcal{E}$ is a normal projection of norm one from $\mathcal{M}$ onto $\mathcal{N}$, then $\mathcal{E}$ is necessarily faithful and unique. Furthermore, if ψ is a faithful semi-finite normal weight on $\mathcal{N}$, then $\mathcal{E}$ is the conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ relative to $\varphi = \psi\circ\mathcal{E}$.*

PROOF: Let ψ be a faithful semi-finite normal weight on $\mathcal{N}$ and put $\varphi = \psi\circ\mathcal{E}$. Then φ is semi-finite and normal. Let N_φ be the left kernel of φ, i.e. $N_\varphi = \left\{ x \in \mathcal{M} : \varphi(x^*x) = 0 \right\}$. Then with $e = s(\varphi)$, the support of φ, we have $N_\varphi = \mathcal{M}(1-e)$. As ψ is faithful, $x \in \mathcal{M}$ belongs to N_φ if and only if $\mathcal{E}(x^*x) = 0$. If $b \in \mathcal{N}$, then for any $x \in N_\varphi$ we have $\mathcal{E}\big((xb)^*(xb)\big) = b^*\mathcal{E}(x^*x)b = 0$, so that N_φ is invariant under the multiplication of $\mathcal{N}$ from the right. This means that e commutes with $\mathcal{N}$, so that by the assumption on the relative commutant of $\mathcal{N}$, e belongs to the center $\mathcal{C}_\mathcal{N}$ of $\mathcal{N}$. But ψ and φ agree on $\mathcal{N}$, so that $e = 1$. Thus φ is faithful.

Suppose $\mathcal{E}_1$ is another normal projection of norm one from $\mathcal{M}$ onto $\mathcal{N}$. Set $\varphi_1 = \psi \circ \mathcal{E}_1$. As $\sigma_t^{\varphi}\big|_{\mathcal{N}} = \sigma_t^{\psi} = \sigma_t^{\varphi_1}\big|_{\mathcal{N}}$, we have $\left(\mathrm{D}\varphi_1 : \mathrm{D}\varphi\right)_t \in \mathcal{N}' \cap \mathcal{M} = \mathcal{C}_{\mathcal{N}}$. As σ_t^{φ} acts trivially on $\mathcal{C}_{\mathcal{N}}$, being the same as σ_t^{ψ} on $\mathcal{C}_{\mathcal{N}}$, $\left\{(\mathrm{D}\varphi_1 : \mathrm{D}\varphi)_t\right\}$ is a one parameter unitary group in $\mathcal{C}_{\mathcal{N}}$, so that there exists a self-adjoint non-singular positive operator h affiliated with $\mathcal{C}_{\mathcal{N}}$ such that $\left(\mathrm{D}\varphi_1 : \mathrm{D}\varphi\right)_t = h^{\mathrm{i}t}$, which means that $\varphi_1 = \varphi(h\,\cdot\,)$. But φ_1 and φ are precisely ψ on $\mathcal{N}$, so that $h = 1$. Hence $\varphi_1 = \varphi$. Thus we have $\psi \circ \mathcal{E}_1 = \psi \circ \mathcal{E}$. Theorem 4.2 implies that $\mathcal{E}_1 = \mathcal{E}$. Q.E.D.

In this second volume, we have learned that the study of unbounded operators and unbounded functionals – weights – provides powerful tools as well as good insight of the subject. In this section, we just established a criteria for the existence of conditional expectations. In this new domain, it is also true that the extension of our study to "unbounded conditional expectations" will give us useful tools.

To study weights, we considered the extended positive real numbers $\mathbf{R}_+ \cup \{\infty\}$. To study "unbounded conditional expectations", we need to consider the "extended positive part" $\widehat{\mathcal{N}}_+$ of the von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$. We begin by the following:

Definition 4.4. For a von Neumann algebra $\mathcal{M}$, the *extended positive cone* $\widehat{\mathcal{M}}_+$ of $\mathcal{M}$ is the set of maps $m\colon \mathcal{M}_*^+ \mapsto [0, \infty]$ with the following properties:

(i) $m(\lambda\varphi) = \lambda m(\varphi), \quad \varphi \in \mathcal{M}_*^+, \quad \lambda \geq 0,$
(ii) $m(\varphi + \psi) = m(\varphi) + m(\psi), \quad \varphi, \psi \in \mathcal{M}_*^+,$
(iii) m is lower semi-continuous.

Clearly, the positive part $\mathcal{M}_+$ of $\mathcal{M}$ is a subset of $\widehat{\mathcal{M}}_+$. It is easy to see that $\widehat{\mathcal{M}}_+$ is closed under addition, multiplication by non-negative scalars and increasing limits.

Example 4.5. Let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra and A a positive self-adjoint operator on $\mathfrak{H}$ affiliated with $\mathcal{M}$. Suppose that

$$A = \int_0^{\infty} \lambda \, \mathrm{d}e(\lambda) \tag{14}$$

is the spectral decomposition of A. For each $\varphi \in \mathcal{M}_*^+$, put

$$m_A(\varphi) = \int_0^{\infty} \lambda \, \mathrm{d}\varphi\big(e(\lambda)\big)\,. \tag{15}$$

Then m_A satisfies the conditions (i), (ii) and (iii) of Definition 4.4. The last condition, the lower semi-continuity, follows from

$$m_A(\varphi) = \sup_n \varphi(A_n) \quad \text{with} \quad A_n = \int_0^{n} \lambda \, \mathrm{d}e(\lambda) \in \mathcal{M}_+.$$

It now follows that

$$m_A(\omega_\xi) = \int_0^\infty \lambda \, \mathrm{d}\big(e(\lambda)\xi \mid \xi\big) = \begin{cases} \|A^{\frac{1}{2}}\xi\|^2, & \xi \in \mathfrak{D}\big(A^{\frac{1}{2}}\big), \\ +\infty, & \xi \notin \mathfrak{D}\big(A^{\frac{1}{2}}\big). \end{cases} \tag{16}$$

Hence if B is another positive self-adjoint operator on $\mathfrak{H}$ affiliated with $\mathcal{M}$, then the equality $m_A = m_B$ means precisely $A = B$. Hence the map: $A \mapsto m_A \in \widehat{\mathcal{M}}_+$ is injective. Thus, the set of positive self-adjoint operators affiliated with $\mathcal{M}$ can be identified with a subset of the extended positive cone $\widehat{\mathcal{M}}_+$.

Definition 4.6. For $m, n \in \widehat{\mathcal{M}}_+$, $\lambda \geq 0$ and $a \in \mathcal{M}$, we define the following operations:

$$\left.\begin{aligned} (\lambda m)(\varphi) &= \lambda m(\varphi), && \varphi \in \mathcal{M}_*^+, \\ (m+n)(\varphi) &= m(\varphi) + n(\varphi), && \varphi \in \mathcal{M}_*^+, \\ (a^* m a)(\varphi) &= m(a\varphi a^*), && \varphi \in \mathcal{M}_*^+. \end{aligned}\right\} \tag{17}$$

We also note here that $\sup m_i$ of an increasing net in $\widehat{\mathcal{M}}_+$ can be naturally defined.

Lemma 4.7. *Let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra. To each $m \in \widehat{\mathcal{M}}_+$, there corresponds uniquely a pair $\{A, \mathfrak{K}\}$ of a closed subspace $\mathfrak{K}$ of $\mathfrak{H}$ and a positive self-adjoint operator on $\mathfrak{K}$ such that:*

(i) *$\mathfrak{K}$ is affiliated with $\mathcal{M}$, in the sense that the projection to $\mathfrak{K}$ belongs to $\mathcal{M}$, and A is affiliated with $\mathcal{M}$;*

(ii)

$$m(\omega_\xi) = \begin{cases} \big\|A^{\frac{1}{2}}\xi\big\|^2, & \xi \in \mathfrak{D}\big(A^{\frac{1}{2}}\big), \\ +\infty, & \textit{otherwise}, \end{cases} \tag{18}$$

where ω_ξ means, of course, the functional $x \in \mathcal{M} \mapsto (x\xi \mid \xi)$.

PROOF: Consider the extended positive real valued function $q : \xi \in \mathfrak{H} \mapsto m(\omega_\xi)$. Then it follows easily that q is a lower semi continuous quadratic form invariant under the unitary group $\mathcal{U}(\mathcal{M}')$ of $\mathcal{M}'$, cf. A.7–A.8. Let $\mathfrak{K}$ be the closure of $\mathfrak{D}(q)$. Then $\mathfrak{K}$ is affiliated with $\mathcal{M}$ by the invariance of q under $\mathcal{U}(\mathcal{M}')$. Then by A.8, there exists a densely defined self-adjoint positive operator A on $\mathfrak{K}$ such that $\mathfrak{D}\big(A^{1/2}\big) = \mathfrak{D}(q)$ and $\big\|A^{1/2}\xi\big\|^2 = q(\xi)$, $\xi \in \mathfrak{K}$. Once again, the invariance of q implies that A is affiliated with $\mathcal{M}$. Q.E.D.

We say that an element $m \in \widehat{\mathcal{M}}_+$ is *semi-finite* if $\big\{\varphi \in \mathcal{M}_*^+ : m(\varphi) < +\infty\big\}$ is dense in $\mathcal{M}_*^+$; *faithful* if $m(\varphi) > 0$ for every non-zero $\varphi \in \mathcal{M}_*^+$.

Theorem 4.8. *Let $\mathcal{M}$ be a von Neumann algebra. Each $m \in \widehat{\mathcal{M}}_+$ has a unique spectral decomposition of the form:*

$$m(\varphi) = \int_0^\infty \lambda \, \mathrm{d}\varphi\big(e(\lambda)\big) + \infty\, \varphi(p), \quad \varphi \in \mathcal{M}_*^+, \tag{19}$$

where $\{ e(\lambda) : \lambda \in \mathbf{R}_+ \}$ *is an increasing family of projections in* $\mathcal{M}$ *which is* σ*-strongly continuous from the right, and* $p = 1 - \lim_{\lambda\to\infty} e(\lambda)$. *Furthermore,* $e(0) = 0$ *if and only if* m *is faithful, and* $p = 0$ *if and only if* m *is semi-finite.*

PROOF: Representing $\mathcal{M}$ on a Hilbert space $\mathfrak{H}$, we apply Lemma 4.7 to m, so that we obtain the pair $\{A, \mathfrak{K}\}$ of the lemma. Let $A = \int_0^{+\infty} \lambda \, \mathrm{d}e(\lambda)$ be the spectral decomposition of A and let p be the projection of $\mathfrak{H}$ onto $\mathfrak{K}^\perp$. Since $e(\lambda) \in \mathcal{M}_\mathfrak{K}$, $e(\lambda)$ may be regarded as a projection of $\mathcal{M}$. Now we have

$$A^{\frac12} = \int_0^\infty \sqrt{\lambda}\, \mathrm{d}e(\lambda),$$

$$\mathfrak{D}\big(A^{\frac12}\big) = \left\{ \xi \in \mathfrak{K} : \int_0^\infty \lambda \, \mathrm{d}\big(e(\lambda)\xi \mid \xi\big) < \infty \right\},$$

$$\big\| A^{\frac12}\xi \big\|^2 = \int_0^\infty \lambda \, \mathrm{d}\big(e(\lambda)\xi \mid \xi\big), \quad \xi \in \mathfrak{D}\big(A^{\frac12}\big).$$

Hence, we get

$$m(\omega_\xi) = \int_0^\infty \lambda \, \mathrm{d}\big(e(\lambda)\xi \mid \xi\big), \quad \xi \in \mathfrak{K},$$

and if $p\xi \neq 0$, then $m(\omega_\xi) = +\infty$, so that we obtain the formula:

$$m(\omega_\xi) = \int_0^\infty \lambda \, \mathrm{d}\big(e(\lambda)\xi \mid \xi\big) + \infty \big(p\xi \mid \xi\big), \quad \xi \in \mathfrak{H}.$$

If we choose the semi-cyclic representation induced by a faithful semi-finite normal weight for $\{\mathcal{M}, \mathfrak{H}\}$, then every $\varphi \in \mathcal{M}_*^+$ is a vectorial functional by Theorem VIII.3.2. Thus the formula (19) follows.

The uniqueness of $\{e(\lambda)\}$ follows from the uniqueness of $\{A, \mathfrak{K}\}$ and the uniqueness of the spectral decomposition of A.

Now, we have the implications:

$$e(0) = 0 \iff A \text{ is nonsingular} \iff m(\omega_\xi) > 0, \quad \xi \neq 0.$$

If $p = 0$, then $\mathfrak{K} = \mathfrak{H}$; so $\mathfrak{D}\big(A^{1/2}\big)$ is dense in $\mathfrak{H}$; hence $\{ \varphi : m(\varphi) < +\infty \}$ is dense in $\mathcal{M}_*^+$, which means the semi-finiteness of m. If $p \neq 0$, then $m(\varphi) = \infty$ for every non-zero $\varphi \in p\mathcal{M}_*^+ p$. But we have

$$\{ \psi \in \mathcal{M}_*^+ : m(\psi) < +\infty \} \subset (1-p)\mathcal{M}_*^+(1-p),$$

so that m is not semi-finite. Q.E.D.

To simplify the notation, we write

$$m = h + \infty p, \qquad h = \int_0^\infty \lambda \, \mathrm{d}e(\lambda), \tag{19$'$}$$

when m has the form of (19). We keep the convention: $0 \cdot (+\infty) = 0$. Although we consider h as an operator affiliated with $\mathcal{M}$, we use the following abuse notation:

$$\mathfrak{D}\big(h^{\frac{1}{2}}\big) = \big\{ \xi \in \mathfrak{H} : m(\omega_\xi) < +\infty \big\},$$

as long as the circumstance allows us to do this.

Corollary 4.9. *Any normal weight φ on $\mathcal{M}$ has a unique extension, denoted by φ again, to $\widehat{\mathcal{M}}_+$ such that*

$$\begin{aligned} \varphi(\lambda m) &= \lambda \varphi(m), && \lambda \geq 0, \quad m \in \widehat{\mathcal{M}}_+; \\ \varphi(m+n) &= \varphi(m) + \varphi(n), && m, n \in \widehat{\mathcal{M}}_+; \\ \varphi\big(\sup m_i\big) &= \sup \varphi(m_i) \end{aligned}$$

for any increasing net $\{m_i\}$ in $\widehat{\mathcal{M}}_+$ in the sense that $\big\{m_i(\omega)\big\}$ is increasing for every $\omega \in \mathcal{M}_*^+$.

We leave the proof to the reader as an exercise.

We observed that the extended positive cone $\widehat{\mathcal{M}}_+$ of $\mathcal{M}$ is nothing but the set of pairs $(A, \mathfrak{K})$ of closed subspaces $\mathfrak{K}$ and positive self-adjoint operators A, both of which are affiliated with $\mathcal{M}$. One of the main advantages of $\widehat{\mathcal{M}}_+$ over the collection of pairs $\{A, \mathfrak{K}\}$ is that one can freely add elements in $\widehat{\mathcal{M}}_+$ without worrying about domains. Of course, the sum of two semi-finite elements of $\widehat{\mathcal{M}}_+$ need not be semi-finite. In the theory of unbounded operators, there are several ways to define the "sum" of two positive self-adjoint operators A and B. When we define $A \dotplus B$ as the operator corresponding to $m_A + m_B$, the operator $A \dotplus B$ is called the *form sum* of A and B, which is not, in general, the closure of the algebraic sum $A + B$. It is completely possible that $A \dotplus B$ has no definition domain other than just $\{0\}$.

Now, for each $m = h + \infty p$, we put

$$m_0 = (1+h)^{-1}(1-p), \tag{20}$$

$$m_\varepsilon = h\,(1+\varepsilon h)^{-1}(1-p) + \frac{1}{\varepsilon} p, \quad \varepsilon > 0. \tag{21}$$

We notice that both m_0 and m_ε are bounded.

Lemma 4.10.

(i) *For each $m, n \in \widehat{\mathcal{M}}_+$, we have the following equivalence:*

$$m \leq n \iff m_0 \geq n_0 \iff m_\varepsilon \leq n_\varepsilon, \quad \varepsilon > 0. \tag{22}$$

(ii) *Let* $\{m_i\} = \{h_i + \infty p_i\}$ *be an increasing net in* $\widehat{\mathcal{M}}_+$ *and* $m = h + \infty p$. *Then we have*

$$m_i \nearrow m \iff (m_i)_0 \searrow m_0 \iff (m_i)_\varepsilon \nearrow m_\varepsilon, \quad \varepsilon > 0. \tag{23}$$

PROOF:

(i) Suppose $m \le n$. Let $m = h + \infty p$ and $n = k + \infty q$. By assumption, we have $p \le q$. Let $\mathfrak{K} = (1-p)\mathfrak{H}$ and $\mathfrak{M} = (1-q)\mathfrak{H}$, where $\mathfrak{H}$ means of course the underlying Hilbert space of $\mathcal{M}$. For any $\xi \in \mathfrak{M}$, set $\eta = (\varepsilon + h)^{-1}\xi \in \mathfrak{D}(h)$ and $\zeta = (\varepsilon + k)^{-1}\xi \in \mathfrak{D}(k)$. We then have

$$\begin{aligned}
\big((\varepsilon+k)^{-1}\xi \mid \xi\big)^2 &= \big(\xi \mid (\varepsilon+k)^{-1}\xi\big)^2 = \big((\varepsilon+h)\eta \mid \zeta\big)^2 \\
&= \big((\varepsilon+h)^{\frac12}\eta \mid (\varepsilon+h)^{\frac12}\zeta)^2 \\
&\qquad \Big(\zeta \in \mathfrak{D}(k) \subset \mathfrak{D}\big(k^{\frac12}\big) \subset \mathfrak{D}\big(h^{\frac12}\big) = \mathfrak{D}\big((\varepsilon+h)^{\frac12}\big)\Big) \\
&\le \big\|(\varepsilon+h)^{\frac12}\eta\big\|^2 \big\|(\varepsilon+h)^{\frac12}\zeta\big\|^2 \\
&= \big((\varepsilon+h)\eta \mid \eta\big)\big((\varepsilon+h)\zeta \mid \zeta\big) \\
&\le \big((\varepsilon+h)\eta \mid \eta\big)\big((\varepsilon+k)\zeta \mid \zeta\big) \\
&= \big(\xi \mid (\varepsilon+h)^{-1}\xi\big)\big(\xi \mid (\varepsilon+k)^{-1}\xi\big),
\end{aligned}$$

so that

$$\big((\varepsilon+k)^{-1}\xi \mid \xi\big) \le \big((\varepsilon+h)^{-1}\xi \mid \xi\big), \quad \xi \in \mathfrak{M}.$$

Hence we get $n_0(\omega_\xi) \le m_0(\omega_\xi)$ for any $\xi \in \mathfrak{M}$ by setting $\varepsilon = 1$. If $\xi \in \mathfrak{M}^\perp$, then $n_0(\omega_\xi) = 0$. Thus we conclude $n_0 \le m_0$.

Suppose $n_0 \le m_0$. It follows that $p \le q$. Let $\mathfrak{K}$ and $\mathfrak{M}$ be as before. The assumption means that

$$\big((1+h)^{-1}\xi \mid \xi\big) \ge \big((1+k)^{-1}\xi \mid \xi\big), \quad \xi \in \mathfrak{H}.$$

Setting $(1+k)^{-1}\xi = 0$ for $\xi \in \mathfrak{M}^\perp \cap \mathfrak{K}$, we view $(1+k)^{-1}$ as an operator on $\mathfrak{K}$ and have $(1+h)^{-1} \ge (1+k)^{-1}$. Then we have $(1+h)^{-1/2} \ge (1+k)^{-1/2}$ (see Proposition I.6.3), and

$$\mathfrak{D}\big(h^{\frac12}\big) = (1+h)^{-\frac12}\mathfrak{K} \supset (1+k)^{\frac12}\mathfrak{K} = \mathfrak{D}\big(k^{\frac12}\big).$$

The argument in the first paragraph shows

$$\begin{aligned}
(1+h)\big(1+\varepsilon(1+h)^{-1}\big)^{-1} &= \big(\varepsilon + (1+h)^{-1}\big)^{-1} \\
&\le \big(\varepsilon + (1+k)^{-1}\big)^{-1} = (1+k)\big(1+\varepsilon(1+k)^{-1}\big).
\end{aligned}$$

If $\xi \in \mathfrak{D}\big(k^{1/2}\big)$, we have

$$\begin{aligned}\big\|(1+k)^{\frac{1}{2}}\xi\big\|^2 &= \lim_{\varepsilon\to 0}\Big\|(1+k)^{\frac{1}{2}}\big(1+\varepsilon(1+k)^{-1}\big)^{-\frac{1}{2}}\xi\Big\|^2\\ &\geq \lim_{\varepsilon\to 0}\Big\|(1+h)^{\frac{1}{2}}\big(1+\varepsilon(1+h)^{-1}\big)^{-\frac{1}{2}}\xi\Big\|^2 = \big\|(1+h)^{\frac{1}{2}}\xi\big\|^2,\end{aligned}$$

so that we conclude $1+n \geq 1+m$, equivalently $n \geq m$.

Now we have seen the equivalence: $m \leq n \iff m_0 \geq n_0$. For a fixed $\varepsilon > 0$, we have then

$$\begin{aligned} m \leq n &\iff \varepsilon m \leq \varepsilon n \iff (\varepsilon m)_0 \geq (\varepsilon n)_0\\ &\iff 1-(\varepsilon m)_0 \leq 1-(\varepsilon n)_0\\ &\iff m_\varepsilon = \frac{1}{\varepsilon}\big(1-(\varepsilon m)_0\big) \leq \frac{1}{\varepsilon}\big(1-(\varepsilon n)_0\big) = n_\varepsilon.\end{aligned}$$

(ii) By (i), the net $\{(m_i)_0\}$ is decreasing. If $\ell = \inf(m_i)_0$, then there exists $n \in \widehat{\mathcal{M}}_+$ such that $n_0 = \ell$ because $(m_i)_0 \leq 1$ implies $\ell \leq 1$. If $m = \sup m_i$, then we have $m_0 \leq (m_i)_0$, so $m_0 \leq n_0$, which implies $n \leq m$ by (i). Hence $m_0 = \inf(m_i)_0 = \lim(m_i)_0$. Thus we proved the equivalence: $m_i \nearrow m \iff (m_i)_0 \searrow m_0$. Finally, the equality:

$$m_\varepsilon = \frac{1}{\varepsilon}\big(1-(\varepsilon m)_0\big) \tag{24}$$

shows the remaining equivalence. Q.E.D.

Proposition 4.11. *Let φ be a faithful semi-finite normal weight on $\mathcal{M}$, and set $\mathcal{N} = \mathcal{M}_\varphi$. For each $m \in \widehat{\mathcal{N}}_+$, set*

$$\varphi_m(x) = \lim_{\varepsilon\to 0}\varphi_{m_\varepsilon}(x), \quad x \in \mathcal{M}_+. \tag{25}$$

Then the map: $m \in \widehat{\mathcal{N}}_+ \mapsto \varphi_m$ is an order preserving bijection from $\widehat{\mathcal{N}}_+$ onto the set of all $\{\sigma_t^\varphi\}$-invariant, not necessarily faithful nor semi-finite, normal weights on $\mathcal{M}$. Furthermore, we have

$$m_i \nearrow m \quad \textit{in } \widehat{\mathcal{N}}_+ \iff \varphi_{m_i} \nearrow \varphi_m \quad \textit{pointwise on } \mathcal{M}_+.$$

PROOF: For a fixed $x \in \mathcal{M}_+$, we define a normal weight φ_x on $\mathcal{N}$ by:

$$\varphi_x(a) = \varphi\big(a^{\frac{1}{2}}xa^{\frac{1}{2}}\big), \quad a \in \mathcal{N}_+. \tag{26}$$

If we prove the additivity of φ_x, then the normality follows from that of φ. Let $a, b \in \mathcal{N}_+$ and $c = a+b$. Choose $s, t \in \mathcal{N}$ as usual so that $a^{1/2} = sc^{1/2}$, $b^{1/2} = tc^{1/2}$ and $s^*s + t^*t$ is the range projection of c. If $\varphi_x(c) < +\infty$, then $y = c^{1/2}xc^{1/2} \in \mathfrak{m}_\varphi$; so sys^* and tyt^* both belong to $\mathfrak{m}_\varphi$ by Lemma VIII.2.4.(ii), and we get

$$\varphi(sys^*) + \varphi(tyt^*) = \varphi(ys^*s) + \varphi(yt^*t) = \varphi\big(y(s^*s+t^*t)\big) = \varphi(y) = \varphi_x(c);$$

$$\varphi(sys^*) = \varphi\left(a^{\frac{1}{2}}xa^{\frac{1}{2}}\right) = \varphi_x(a); \qquad \varphi(tyt^*) = \varphi\left(b^{\frac{1}{2}}xb^{\frac{1}{2}}\right) = \varphi_x(b).$$

Thus, $\varphi_x(a) + \varphi_x(b) = \varphi_x(c)$. Now, we have

$$s = \lim_{\varepsilon\to 0} a^{\frac{1}{2}}(c+\varepsilon)^{-\frac{1}{2}}, \qquad t = \lim_{\varepsilon\to 0} b^{\frac{1}{2}}(c+\varepsilon)^{-\frac{1}{2}},$$

so that

$$\begin{aligned}
c^{\frac{1}{2}}xc^{\frac{1}{2}} &= \lim_{\varepsilon\to 0}(c+\varepsilon)^{-\frac{1}{2}}cxc\,(c+\varepsilon)^{-\frac{1}{2}} \\
&= \lim_{\varepsilon\to 0}(c+\varepsilon)^{-\frac{1}{2}}(a+b)x\,(a+b)(c+\varepsilon)^{-\frac{1}{2}} \\
&\leq 2\lim_{\varepsilon\to 0}(c+\varepsilon)^{-\frac{1}{2}}\big[axa+bxb\big](c+\varepsilon)^{-\frac{1}{2}} \\
&= 2\Big(s^*a^{\frac{1}{2}}xa^{\frac{1}{2}}s + t^*b^{\frac{1}{2}}xb^{\frac{1}{2}}t\Big).
\end{aligned}$$

Therefore, if $\varphi_x(a) < +\infty$ and $\varphi_x(b) < +\infty$, then $\varphi_x(c) < +\infty$; hence $\varphi_x(a+b) = \varphi_x(a) + \varphi_x(b)$ for $a, b \in \mathcal{N}_+$.

Now, for each $m \in \widehat{\mathcal{N}}_+$, we set

$$\varphi_m(x) = \varphi_x(m), \quad x \in \mathcal{M}_+. \tag{27}$$

This makes sense by Corollary 4.9. First, by Corollary 4.9, we have

$$\varphi_x\big(\sup m_i\big) = \sup \varphi_x(m_i) \quad \text{if} \quad m_i \nearrow m.$$

Hence we have

$$\varphi_m(x) = \varphi_x(m) = \lim_{\varepsilon\to 0}\varphi_x(m_\varepsilon) = \lim_{\varepsilon\to 0}\varphi_{m_\varepsilon}(x).$$

If $\{x_i\} \nearrow x$ in $\mathcal{M}_+$, then

$$\begin{aligned}
\varphi_x(m) &= \sup_{\varepsilon>0}\ \varphi_{m_\varepsilon}(x) = \sup_{\varepsilon>0}\ \sup_i\ \varphi_{m_\varepsilon}(x_i) \\
&= \sup_i\ \sup_{\varepsilon>0}\ \varphi_{m_\varepsilon}(x_i) = \sup_i\ \varphi_m(x_i),
\end{aligned}$$

so that φ_m is normal. The additivity of φ_m follows from the convergence (25).

The invariance of φ_m under $\{\sigma_t^\varphi\}$ follows from that of φ_{m_ε}. The rest is now easy.

Q.E.D.

Definition 4.12. Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{N}$ a von Neumann subalgebra of $\mathcal{M}$. An *operator valued weight* from $\mathcal{M}$ to $\mathcal{N}$ is a map $T\colon \mathcal{M}_+ \mapsto \widehat{\mathcal{N}}_+$ which satisfies the following conditions:

$$\left.\begin{array}{lll}
\text{(a)} & T(\lambda x) = \lambda T(x), & \lambda \geq 0, \quad x \in \mathcal{M}_+ \\
\text{(b)} & T(x+y) = T(x) + T(y), & x, y \in \mathcal{M}_+ \\
\text{(c)} & T(a^*xa) = a^*T(x)a, & x \in \mathcal{M}_+, \quad a \in \mathcal{N}.
\end{array}\right\} \tag{28}$$

We say that T is *normal* if

$$\text{(d)} \quad T(x_i) \nearrow T(x) \quad \text{whenever} \quad x_i \nearrow x, \quad x_i, x \in \mathcal{M}_+.$$

As in the case of ordinary weights, we set

$$\left.\begin{aligned} \mathfrak{n}_T &= \left\{ x \in \mathcal{M} : \left\| T(x^*x) \right\| < +\infty \right\} \\ \mathfrak{m}_T &= \mathfrak{n}_T^* \mathfrak{n}_T = \left\{ \sum_{i=1}^n y_i^* x_i : x_1, \ldots, x_n,\ y_1, \ldots, y_n \in \mathfrak{n}_T \right\}. \end{aligned}\right\} \tag{29}$$

By now, the next lemma is an easy exercise:

Lemma 4.13.

(i) $\mathfrak{m}_T$ *is spanned by its positive part:*

$$\mathfrak{m}_T^+ = \left\{ x \in \mathcal{M}_+ : \left\| T(x) \right\| < +\infty \right\}.$$

(ii) $\mathfrak{m}_T$ *and* $\mathfrak{n}_T$ *are two sided modules over* $\mathcal{N}$ *.*

(iii) T *has a unique linear extension* $\dot{T} : \mathfrak{m}_T \mapsto \mathcal{N}$ *, which enjoys the module map property:*

$$\dot{T}(axb) = a\dot{T}(x)b, \quad a, b \in \mathcal{N}, \quad x \in \mathfrak{m}_T. \tag{30}$$

In particular, if $T(1) = 1$ *, then* T *is a projection of norm one from* $\mathcal{M}$ *onto* $\mathcal{N}$ *.*

In the sequel, we shall not distinguish T and $\dot{T}$ unless we need to.

Definition 4.14. We say that T is *semi-finite* if $\mathfrak{n}_T$ is σ-weakly dense in $\mathcal{M}$; *faithful* if $T(x^*x) \neq 0$ for $x \neq 0$. We denote by $\mathfrak{W}(\mathcal{M}, \mathcal{N})$ (resp. $\mathfrak{W}_0(\mathcal{M}, \mathcal{N})$) the set of (resp. faithful semi-finite) normal operator valued weights from $\mathcal{M}$ to $\mathcal{N}$. In the case that $\mathcal{N} = \mathbf{C}$, we write $\mathfrak{W}(\mathcal{M})$ (resp. $\mathfrak{W}_0(\mathcal{M})$) for $(\mathfrak{W}(\mathcal{M}, \mathbf{C})$ (resp. $\mathfrak{W}_0(\mathcal{M}, \mathbf{C})$).

Remark 4.15. If $T : \mathcal{M}_+ \mapsto \widehat{\mathcal{N}}_+$ is a normal operator valued weight, it can be naturally extended to a normal "linear" map from $\widehat{\mathcal{M}}_+ \mapsto \widehat{\mathcal{N}}_+$. Therefore, if $\mathcal{P} \subset \mathcal{N} \subset \mathcal{M}$ and if $T \in \mathfrak{W}(\mathcal{M}, \mathcal{N})$ and $S \in \mathfrak{W}(\mathcal{N}, \mathcal{P})$ then we have $S \circ T \in \mathfrak{W}(\mathcal{M}, \mathcal{P})$.

Proposition 4.16. *If* $\mathcal{P} \subset \mathcal{N} \subset \mathcal{M}$ *are von Neumann subalgebras and if* $T \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N})$ *and* $S \in \mathfrak{W}_0(\mathcal{N}, \mathcal{P})$ *, then* $S \circ T \in \mathfrak{W}_0(\mathcal{M}, \mathcal{P})$ *.*

PROOF: The only non-trivial part is the semi-finiteness of $S \circ T$. If $x \in \mathfrak{n}_T$, then $T(x^*x) \in \mathcal{N}_+$. Choose a net $\{a_i\}$ in $\mathfrak{n}_S$ such that $a_i \to 1$ σ-strongly. Then we have

$$S \circ T\big(a_i^* x^* x a_i\big) = S\big(a_i^* T(x^*x) a_i\big) \leq \left\| T(x^*x) \right\| S(a_i^* a_i),$$

so that $xa_i \in \mathfrak{n}_{S \circ T}$. Hence $\mathfrak{n}_{S \circ T}$ is σ-strongly dense in $\mathcal{M}$ because $\mathfrak{n}_T$ is. Q.E.D.

Proposition 4.17. *If $\mathcal{N} \subset \mathcal{M}$ are von Neumann algebras and if $T \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N})$, then*

(i) $\dot{T}(\mathfrak{m}_T)$ *is a σ-weakly dense ideal of $\mathcal{N}$;*
(ii) *After extending T to the map of $\hat{\mathcal{M}}_+$ to $\hat{\mathcal{N}}_+$, as in Remark* 4.15,

$$T(\hat{\mathcal{M}}_+) = \hat{\mathcal{N}}_+ .$$

PROOF:
(i) From the module map property of T:

$$\dot{T}(axb) = a\dot{T}(x)b, \quad a, b \in \mathcal{N}, \quad x \in \mathfrak{m}_T,$$

it follows that $\dot{T}(\mathfrak{m}_T)$ is an ideal of $\mathcal{N}$. Let z denote the greatest projection of the σ-weak closure of $\dot{T}(\mathfrak{m}_T)$, which is central in $\mathcal{N}$. Assume $z \neq 1$. As $\mathfrak{n}_T$ is σ-weakly dense in $\mathcal{M}$, there exists $x \in \mathfrak{n}_T$ with $x(1-z) \neq 0$, so that $(1-z)x^*x(1-z) \in \mathfrak{m}_T \setminus \{0\}$ and as T is faithful

$$0 \neq T\big((1-z)x^*x(1-z)\big) = (1-z)T(x^*x)(1-z) = 0,$$

which is a contradiction. Hence $z = 1$, which means that $\dot{T}(\mathfrak{m}_T)$ is σ-weakly dense in $\mathcal{N}$.

(ii) Let $b \in \dot{T}(\mathfrak{m}_T)_+$. Then b is of the form $b = \dot{T}(h)$, $h \in \mathfrak{m}_T$. Replacing h by $\frac{1}{2}(h + h^*)$, h can be chosen to be self-adjoint. As $b \leq T\big(|h|\big)$, we can find $s \in \mathcal{N}$ such that $b = sT\big(|h|\big)s^*$. With $a = s|h|s^*$, we have $b = T(a)$, $a \in \mathfrak{m}_T^+$. Hence $\dot{T}(\mathfrak{m}_T)_+ = \dot{T}(\mathfrak{m}_T^+)$.

Let $\{b_i\}_{i\in I}$ be a maximal family in the ideal $\dot{T}(\mathfrak{m}_T)_+$ such that $\sum_{i\in I} b_i \leq 1$. The maximality and Proposition II.3.13 entail $1 = \sum_{i\in I} b_i$ in the σ-strong topology. Every $y \in \mathcal{N}_+$ is then of the form: $y = \sum_{i\in I} y^{1/2} b_i y^{1/2}$, so that it is of the form: $y = \sum_{i\in I} Tx_i$ with $\{x_i\} \subset \mathfrak{m}_T^+$.

Let $z \in \hat{\mathcal{N}}_+$. From Theorem 4.8, it follows that there exists a sequence $y_n \in \mathcal{N}_+$ such that $y_n \nearrow z$. Set $z_1 = y_1$ and $z_n = y_n - y_{n-1}$, $n \geq 2$. Then we have $z = \sum_{n=1}^{\infty} y_n$. Each y_n is of the form: $y_n = \sum Tx_{n,i}$ with $\{x_{n,i}\} \subset \mathfrak{m}_T^+$. Hence we have

$$z = T\bigg(\sum_{n=1}^{\infty} \sum_i x_{n,i}\bigg),$$

where T means the extended one to $\hat{\mathcal{M}}_+$. Thus T maps $\hat{\mathcal{M}}_+$ onto $\hat{\mathcal{N}}_+$. Q.E.D.

The next theorem generalizes Theorem 4.2.

Theorem 4.18. *Let $\mathcal{M} \supset \mathcal{N}$ be von Neumann algebras. There exists a faithful semi-finite normal operator valued weight $T : \mathcal{M}_+ \mapsto \hat{\mathcal{N}}_+$ if and only if there exist faithful semi-finite normal weights $\tilde{\varphi}$ on $\mathcal{M}$ and φ on $\mathcal{N}$ such that*

$$\sigma_t^{\varphi}(x) = \sigma_t^{\tilde{\varphi}}(x), \quad x \in \mathcal{N}. \tag{31}$$

If this condition is satisfied, then T can be chosen such a way that $\tilde{\varphi} = \varphi \circ T$, and T is uniquely determined by this identity.

Lemma 4.19. *Let* $m\colon \mathcal{M}_*^+ \mapsto [0, +\infty]$ *be an extended positive real valued function such that*

$$m(\lambda\varphi) = \lambda m(\varphi), \qquad \lambda \geq 0, \quad \varphi \in \mathcal{M}_*^+;$$

$$m(\varphi + \psi) = m(\varphi) + m(\psi), \qquad \varphi, \psi \in \mathcal{M}_*^+$$

is lower semi-continuous; hence a member of the extended positive cone $\hat{\mathcal{M}}_+$ *if and only if* m *is countably additive in the sense that*

$$m\left(\sum_{n=1}^{\infty} \varphi_n\right) = \sum_{n=1}^{\infty} m(\varphi_n)$$

for every $\{\varphi_n\} \subset \mathcal{M}_*^+$ *with* $\sum_{n=1}^{\infty} \|\varphi_n\| < +\infty$.

PROOF: The "only part" is trivial, so we prove the only "if" part. Suppose m is countably additive. Represent $\mathcal{M}$ on a Hilbert space $\mathfrak{H}$ and consider $\bar{m}\colon \omega \in \mathcal{L}(\mathfrak{H})_*^+ \mapsto m(\omega|_{\mathcal{M}}) \in [0, +\infty]$. We claim first that m is lower semi continuous if $\bar{m}$ is. Suppose $\bar{m}$ is lower semi-continuous. Then by Lemma 4.7, there exists a unique pair $\{A, \mathfrak{K}\}$ of a closed subspace $\mathfrak{K}$ of $\mathfrak{H}$ and a positive self-adjoint operator A on $\mathfrak{K}$ such that (18) holds. As $u\omega u^*\big|_{\mathcal{M}} = \omega|_{\mathcal{M}}$ for every $u \in \mathcal{U}(\mathcal{M}')$, $u^*\bar{m}u = \bar{m}$, so that $\{A, \mathfrak{K}\}$ is affiliated to $\mathcal{M}$, and $m = m_A$. Thus, m is a member of $\hat{\mathcal{M}}_+$. Therefore, it suffices to prove the lemma for $\mathcal{M} = \mathcal{L}(\mathfrak{H})$. Replacing m by m' defined by $m'(\varphi) = m(\varphi) + \varphi(1)$, we may assume $m(\varphi) \geq \|\varphi\|$, $\varphi \in \mathcal{M}_*^+$. As $\mathcal{M}_* = L^1(\mathcal{M}, \mathrm{Tr})$ is an ideal of $\mathcal{M}$, we can define a map $\varphi\colon x \in \mathcal{M}_+ \mapsto [0, +\infty]$ by

$$\varphi(x) = \begin{cases} m(\omega_x), & x \in L^1(\mathcal{M}, \mathrm{Tr})_+; \\ +\infty & \text{otherwise}, \end{cases}$$

where $\omega_x(a) = \mathrm{Tr}(ax)$, $a \in \mathcal{M}$, $x \in L^1(\mathcal{M}, \mathrm{Tr})$. Then φ is a weight on $\mathcal{M}$, and $\omega \geq \mathrm{Tr}$. Let $\{x_i\}_{i\in I}$ be a family of positive operators with $x = \sum x_i \in \mathcal{M}_+$. If $\mathrm{Tr}(x) = +\infty$, then $\sum_i \mathrm{Tr}(x_i) = \infty$ and

$$\infty = \sum \mathrm{Tr}(x_i) \leq \sum \omega(x_i) \leq \omega(x),$$

so that $\omega(x) = \infty = \sum \omega(x_i)$. If $\mathrm{Tr}(x) < +\infty$, then $\sum \mathrm{Tr}(x_i) < +\infty$, so that $x_i \neq 0$ for at most countably infinite i's. Hence we have

$$\sum_{i\in I} \varphi(x_i) = \sum_i m(\omega_{x_i}) = m\left(\sum \omega_{x_i}\right) = m(\omega_x) = \varphi(x).$$

Thus, φ is a completely additive weight on $\mathcal{M}$. By Theorem VII.1.11, φ is normal. If $\{x_n\}$ is a sequence in $L^1(\mathcal{M}, \mathrm{Tr})_+$ with $\lim_{n\to\infty} \left\| x - x_n \right\|_1 = 0$, then $\{x_n\}$ converges to x σ-weakly, so that

$$m(\omega_x) = \varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n) = \liminf_{n\to\infty} m(\omega_{x_n}).$$

Hence m is lower semi-continuous. Q.E.D.

Lemma 4.20. *Let $\varphi \in \mathfrak{W}(\mathcal{M})$ and $\{a_n\}$ be a sequence in $\mathcal{M}_{s(\varphi)}$. The following two conditions are equivalent:*

(i) $\varphi(x) = \sum_{n=1}^{\infty} \varphi(a_n x a_n^*)\,, \quad x \in \mathcal{M}_+\,;$ (32)

(ii) $\{a_n\} \subset \mathcal{D}\big(\sigma_{-\mathrm{i}/2}^{\varphi}\big)$ *and* $\sum_{n=1}^{\infty} \sigma_{-\mathrm{i}/2}^{\varphi}(a_n)^* \, \sigma_{-\mathrm{i}/2}^{\varphi}(a_n) = s(\varphi)\,.$ (33)

PROOF: As we can view the problem in the reduced algebra $\mathcal{M}_{s(\varphi)}$, we may assume that φ is faithful.

(i) $\Longrightarrow$ (ii): By Theorem VIII.3.17, each a_n belongs to $\mathcal{D}\big(\sigma_{-\mathrm{i}/2}^{\varphi}\big)$ and satisfies the inequality $\big\|\sigma_{-\mathrm{i}/2}^{\varphi}(a_n)\big\| \le 1$. We then have, for each $x \in \mathfrak{n}_{\varphi}$,

$$\begin{aligned} \big\|\eta_{\varphi}(x)\big\|^2 = \varphi(x^*x) &= \sum_{n=1}^{\infty} \varphi\big(a_n x^* x a_n^*\big) = \sum_{n=1}^{\infty} \big\|\eta_{\varphi}(x a_n^*)\big\|^2 \\ &= \sum_{n=1}^{\infty} \big\| J \sigma_{-\mathrm{i}/2}^{\varphi}(a_n) J \eta_{\varphi}(x)\big\|^2 \qquad \text{by (VIII.3.49)} \\ &= \left(J \sum_{n=1}^{\infty} \sigma_{-\mathrm{i}/2}^{\varphi}(a_n)^* \sigma_{-\mathrm{i}/2}^{\varphi}(a_n) J \eta_{\varphi}(x) \;\middle|\; \eta_{\varphi}(x)\right). \end{aligned}$$

Hence we have

$$1 = \sum_{n=1}^{\infty} \sigma_{-\mathrm{i}/2}^{\varphi}(a_n)^* \sigma_{-\mathrm{i}/2}^{\varphi}(a_n).$$

(ii) $\Longrightarrow$ (i): This implication follows from the above computation traced backward. Q.E.D.

PROOF OF THEOREM 4.18: Suppose that $\mathcal{M} \supset \mathcal{N}$ are von Neumann algebras, and

$$\sigma_t^{\tilde{\varphi}}(x) = \sigma_t^{\varphi}(x), \quad x \in \mathcal{N},$$

for some $\tilde{\varphi} \in \mathfrak{W}_0(\mathcal{M})$ and $\varphi \in \mathfrak{W}_0(\mathcal{N})$. We fix $\tilde{\varphi}$ and φ. For each $\psi \in \mathfrak{W}(\mathcal{N})$, we have the cocycle derivative $u_t = (\mathrm{D}\psi : \mathrm{D}\varphi)_t \in \mathcal{N},\ t \in \mathbf{R}$. By Theorem VIII.3.21, there corresponds $\tilde{\psi} \in \mathfrak{W}(\mathcal{M})$ such that $s(\tilde{\psi}) = s(\psi)$ and $(\mathrm{D}\tilde{\psi} : \mathrm{D}\tilde{\varphi})_t = u_t = (\mathrm{D}\psi : \mathrm{D}\varphi)_t$. As we have

$$\big(\mathrm{D}(\lambda\psi) : \mathrm{D}\varphi\big)_t = \lambda^{\mathrm{i}t}\big(\mathrm{D}\psi : \mathrm{D}\varphi\big)_t = \lambda^{\mathrm{i}t}\big(\mathrm{D}\tilde{\psi} : \mathrm{D}\tilde{\varphi}\big)_t = \big(\mathrm{D}(\lambda\tilde{\psi}) : \mathrm{D}\tilde{\varphi}\big)_t,$$

we obtain $\widetilde{\lambda\psi} = \lambda\tilde{\psi},\ \lambda > 0$.

By the chain rule for cocycle derivatives, we have $(\mathrm{D}\tilde{\psi}_1 : \mathrm{D}\tilde{\psi}_2)_t = (\mathrm{D}\psi_1 : \mathrm{D}\psi_2)_t$ for any $\psi_1 \in \mathfrak{W}(\mathcal{N})$ and $\psi_2 \in \mathfrak{W}_0(\mathcal{N})$. Let $\{\psi_n\}$ be a sequence in $\mathcal{N}_*^+$ such that $\psi = \sum_{n=1}^{\infty} \psi_n \in \mathcal{N}_*^+$. Let $u_t^n = (\mathrm{D}\psi_n : \mathrm{D}\psi)_t$ by restricting our consideration to $\mathcal{N}_{s(\psi)}$ and $\mathcal{M}_{s(\tilde{\psi})}$. Then by Theorem VIII.3.17, $u_{-\mathrm{i}/2}^n$ is defined and

$$\psi_n(x) = \psi\Big(\big(u_{-\mathrm{i}/2}^n\big)^* x \,\big(u_{-\mathrm{i}/2}^n\big)\Big), \quad x \in \mathcal{N}.$$

Therefore, Lemma 4.20 entails that

$$\begin{aligned} s(\psi) &= \sum_{n=1}^{\infty} \sigma_{-i/2}^{\psi}\Big(\big((u_{-i/2}^{n})^*\big)^*\Big) \sigma_{-i/2}^{\psi}\Big(\big((u_{-i/2}^{n})^*\big)\Big) \\ &= \sum_{n=1}^{\infty} \sigma_{i/2}^{\psi}\big(u_{-i/2}^{n}\big)\sigma_{i/2}^{\psi}\big(u_{-i/2}^{n}\big)^* \\ &= \sum_{n=1}^{\infty} \sigma_{i/2}^{\tilde\psi}\big((\mathrm{D}\tilde\psi_n : \mathrm{D}\tilde\psi)_{-i/2}\big)\, \sigma_{i/2}^{\tilde\psi}\, \big((\mathrm{D}\tilde\psi_n : \mathrm{D}\tilde\psi)_{-i/2}\big)^*. \end{aligned}$$

As $s(\psi) = s(\tilde\psi)$, the above calculation shows that

$$\tilde\psi(x) = \sum_{n=1}^{\infty} \tilde\psi_n(x), \quad x \in \mathcal{M}_+.$$

Hence the map: $\psi \in \mathcal{N}_*^+ \mapsto \tilde\psi \in \mathfrak{W}(\mathcal{M})$ is homogeneous and countably additive. Hence, the map: $\psi \in \mathcal{N}_*^+ \mapsto \tilde\psi(x) \in [0, +\infty]$, $x \in \mathcal{M}_+$, gives rise to a map $T : \mathcal{M}_+ \mapsto \hat{\mathcal{N}}_+$ by Lemma 4.19.

For every $u \in \mathcal{U}(\mathcal{N})$, we have

$$\begin{aligned} \big(\mathrm{D}(u\psi u^*)^\sim : \mathrm{D}\tilde\varphi\big)_t &= \big(\mathrm{D}(u\psi u^*) : \mathrm{D}\varphi\big)_t = u\,(\mathrm{D}\psi : \mathrm{D}\varphi)_t\, \sigma_t^{\varphi}(u^*) \\ &= u\big(\mathrm{D}\tilde\psi : \mathrm{D}\tilde\varphi\big)_t \sigma_t^{\tilde\varphi}(u^*), \end{aligned}$$

so that $(u\psi u^*)^\sim = u\tilde\psi u^*$, $u \in \mathcal{U}(\mathcal{N})$. Thus, we have $T(uxu^*) = uT(x)u^*$, $x \in \mathcal{M}_+$, $u \in \mathcal{U}(\mathcal{N})$. Hence $\mathfrak{n}_T$ is invariant by the right multiplication of $\mathcal{U}(\mathcal{N})$, and $T(ux) = uT(x)$, $T(xu) = T(x)u$ for every $x \in \mathfrak{m}_T = \mathfrak{n}_T^*\mathfrak{n}_T$ and $u \in \mathcal{U}(\mathcal{N})$. As $\mathcal{U}(\mathcal{N})$ spans $\mathcal{N}$ linearly, $\mathfrak{m}_T$ is a two sided module over $\mathcal{N}$ and $T(axb) = aT(x)b$ for $a, b \in \mathcal{N}$ and $x \in \mathfrak{m}_T$.

We now want to prove

$$T(axa^*) = aT(x)a^*, \quad a \in \mathcal{N}, \quad x \in \mathcal{M}_+. \tag{34}$$

First, we observe that if $a \in \mathcal{N}_\psi$ with ψ faithful, then $\big(\mathrm{D}a^*\psi a : \mathrm{D}\psi\big)_t = \big(a^*a\big)^{it}$, where $\big(a^*a\big)^{it}$ should be considered in the reduced algebra $\mathcal{N}_{s_r(a)}$ by the right support $s_r(a)$ of a, so that

$$\big(\mathrm{D}(a^*\psi a)^\sim : \mathrm{D}\tilde\psi\big)_t = \big(\mathrm{D}a^*\psi a : \mathrm{D}\psi\big)_t = \big(a^*a\big)^{it};$$

hence $(a^*\psi a)^\sim = a^*\psi a$. If ψ is not faithful, then we consider $\psi'' = \psi + \psi'$ with $\psi' \in \mathfrak{W}(\mathcal{N})$ such that $s(\psi') = 1 - s(\psi)$; and apply the above argument to ψ to conclude that $(a^*\psi a)^\sim = a^*\tilde\psi a$, $a \in \mathcal{N}_\psi$. Now, we apply the above whole argument to $\mathcal{N} \otimes M_2(\mathbf{C})$ and $\mathcal{M} \otimes M_2(\mathbf{C})$ with $\psi \otimes \mathrm{Tr}$ and $\tilde\psi \otimes \mathrm{Tr}$ and observe that

$\sigma_t^{\psi\otimes\mathrm{Tr}}(x) = \sigma_t^{\tilde{\psi}\otimes\mathrm{Tr}}(x)$ for every $x \in \mathcal{N} \otimes M_2(\mathbf{C})$. This implies that there exists a map $S\colon (\mathcal{M} \otimes M_2)_+ \mapsto (\mathcal{N} \otimes M_2)\widehat{_+}$ such that

$$\big(\psi \otimes \mathrm{Tr}\big) \circ S(x) = \big(\tilde{\psi} \otimes \mathrm{Tr}\big)(x), \quad x \in (\mathcal{M} \otimes M_2)_+.$$

As $\mathbf{C} \otimes M_2 \subset \mathcal{N}_\psi$ for any faithful ψ, we have $(\psi \otimes \mathrm{Tr})^\sim = \tilde{\psi} \otimes \mathrm{Tr}$. If $a \in \mathcal{N}$ has $\|a\| \le 1$, then set

$$u = \begin{pmatrix} a & \big(1 - aa^*\big)^{\frac{1}{2}} \\ -\big(1 - a^*a\big)^{\frac{1}{2}} & a^* \end{pmatrix} \in \mathcal{N} \otimes M_2.$$

Then u is unitary. Therefore, we have

$$\big(u^*(\psi \otimes \mathrm{Tr})u\big)^\sim = u^*\big(\tilde{\psi} \otimes \mathrm{Tr}\big)u.$$

With $\{e_{i,j}\}$ the standard matrix unit of M_2, we have

$$\Big[\big(1 \otimes e_{ij}\big)^*\big(\psi \otimes \mathrm{Tr}\big)\big(1 \otimes e_{ij}\big)\Big]^\sim = \big(1 \otimes e_{ij}\big)^*\big(\tilde{\psi} \otimes \mathrm{Tr}\big)\big(1 \otimes e_{ij}\big)$$

as $(1 \otimes e_{ij}) \in (\mathcal{N} \otimes M_2)_{\psi\otimes\mathrm{Tr}}$. Hence we conclude that

$$\begin{aligned}&\Big[\big(1 \otimes e_{11}\big)u^*\big(1 \otimes e_{11}\big)\big(\psi \otimes \mathrm{Tr}\big)\big(1 \otimes e_{11}\big)u\,\big(1 \otimes e_{11}\big)\Big]^\sim \\ &\quad= \big(a \otimes e_{11}\big)^*\big(\tilde{\psi} \otimes \mathrm{Tr}\big)\big(a \otimes e_{11}\big),\end{aligned}$$

i.e.

$$\Big[\big(a^* \otimes e_{11}\big)\big(\psi \otimes \mathrm{Tr}\big)\big(a \otimes e_{11}\big)\Big]^\sim = \big(a^* \otimes e_{11}\big)\big(\tilde{\psi} \otimes \mathrm{Tr}\big)\big(a \otimes e_{11}\big).$$

This means that

$$(a^*\psi a)^\sim = a^*\tilde{\psi}a$$

for every $a \in \mathcal{N}$ with $\|a\| \le 1$, which entails (34). Therefore T is an operator valued weight of $\mathcal{M}_+$ onto $\hat{\mathcal{N}}_+$ such that $\tilde{\psi} = \psi \circ T$, $\psi \in \mathfrak{W}_0(\mathcal{N})$. As $\tilde{\psi}$ is faithful for any $\psi \in \mathfrak{W}_0(\mathcal{N})$, T is faithful.

The semi-finiteness and the uniqueness of T together with the converse follow from the following lemma. Q.E.D.

Lemma 4.21. *Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann algebras and $T\colon \mathcal{M}_+ \to \hat{\mathcal{N}}_+$ be a normal operator valued weight.*

(i) *If $\psi \circ T$ is semi-finite for some $\psi \in \mathfrak{W}_0(\mathcal{N})$, then T is semi-finite;*

(ii) *If $\psi \in \mathfrak{W}_0(\mathcal{N})$ and $S \in \mathfrak{W}(\mathcal{M}, \mathcal{N})$ satisfy $\psi \circ T = \psi \circ S$, then $T = S$;*

(iii) *If $T \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N})$, then*

$$\sigma_t^{\psi\circ T}(x) = \sigma_t^{\psi}(x), \quad x \in \mathcal{N}. \tag{35}$$

PROOF:

(i) Set $\tilde{\psi} = \psi \circ T$. Let $h \in \mathfrak{m}_{\tilde{\psi}}^{+}$ and

$$T(h) = \int_0^\infty \lambda \, \mathrm{d}e(\lambda) + \infty p$$

be the spectral decomposition. As $\psi \circ T(h) < +\infty$, and ψ is faithful, we have $p = 0$. Hence $e(\lambda) \nearrow 1$ as $\lambda \nearrow \infty$, so that $e(\lambda)he(\lambda) \to h$ strongly as $\lambda \nearrow \infty$, and $T\big(e(\lambda)he(\lambda)\big) = e(\lambda)T(h)e(\lambda) \in \mathcal{N}_+$; hence $e(\lambda)he(\lambda) \in \mathfrak{m}_T^+$. Therefore $\mathfrak{m}_T^+$ is σ-strongly dense in $\mathcal{M}_+$.

(ii) For any $x \in \mathcal{M}_+$ and $a \in \mathcal{N}$, we have

$$\psi\big(a^*T(x)a\big) = \psi\big(T(a^*xa)\big) = \psi\big(S(a^*xa)\big) = \psi\big(a^*S(x)a\big),$$

so that $a\psi a^* \circ T = a\psi a^* \circ S$ for any $a \in \mathcal{N}$. Therefore if $x \in \mathfrak{m}_T^+ \cup \mathfrak{m}_S^+$, then $T(x) = S(x)$ since $\big\{a\psi a^* : a \in \mathfrak{n}_\psi\big\}$ is a dense subset of $\mathcal{N}_*^+$. For a general $x \in \mathcal{M}_+$, consider the spectral decomposition:

$$m = S(x) = \int_0^\infty \lambda \, \mathrm{d}e(\lambda) + \infty p \in \hat{\mathcal{N}}_+ \, ;$$

$$n = T(x) = \int_0^\infty \lambda \, \mathrm{d}f(\lambda) + \infty q \in \hat{\mathcal{N}}_+ .$$

Then we have $e(\lambda)xe(\lambda) \in \mathfrak{m}_S \cap \mathfrak{m}_T$ and

$$\begin{aligned} me(\lambda) &= e(\lambda)S(x)e(\lambda) = S\big(e(\lambda)xe(\lambda)\big) = T\big(e(\lambda)xe(\lambda)\big) \\ &= e(\lambda)T(x)e(\lambda) = e(\lambda)ne(\lambda); \end{aligned}$$

similarly

$$nf(\lambda) = f(\lambda)mf(\lambda), \quad \lambda \geq 0.$$

Hence we have, for every $\xi \in \bigcup_{\lambda \geq 0} e(\lambda)\mathfrak{H}_\psi$,

$$\big\|m^{\frac{1}{2}}\xi\big\|^2 = m(\omega_\xi) = n(\omega_\xi) = \big\|n^{\frac{1}{2}}\xi\big\|^2;$$

similarly

$$\big\|m^{\frac{1}{2}}\xi\big\|^2 = \big\|n^{\frac{1}{2}}\xi\big\|^2, \quad \xi \in \bigcup_{\lambda \geq 0} f(\lambda)\mathfrak{H}_\psi .$$

But $m^{1/2}$ (resp. $n^{1/2}$) is essentially self-adjoint on $\bigcup e(\lambda)\mathfrak{H}_\psi$ (resp. $\bigcup f(\lambda)\mathfrak{H}_\psi$), so we get

$$\big\|m^{\frac{1}{2}}\xi\big\|^2 = \big\|n^{\frac{1}{2}}\xi\big\|^2, \quad \xi \in \mathfrak{D}\big(m^{\frac{1}{2}}\big) \cup \mathfrak{D}\big(n^{\frac{1}{2}}\big).$$

Hence there exists a partial isometry $u \in \mathcal{N}$ such that $m^{1/2} = un^{1/2}$ and $n^{1/2} = u^*m^{1/2}$, so that the uniqueness of the polar decomposition implies $m^{1/2} = n^{1/2}$ and $(1 - p) = (1 - q)$. Thus $m = n$ as elements of $\hat{\mathcal{N}}_+$. Hence $S = T$.

(iii) We prove $\mathcal{G}(\sigma_{-\mathrm{i}}^{\psi}) \subset \mathcal{G}(\sigma_{-\mathrm{i}}^{\psi\circ T})$. Let $\tilde{\psi} = \psi \circ T$, and $(a, b) \in \mathcal{G}(\sigma_{-\mathrm{i}}^{\psi})$. As $a \in \mathcal{D}(\sigma_{-\mathrm{i}/2}^{\psi})$ and $b^* \in \mathcal{D}(\sigma_{-\mathrm{i}/2}^{\psi}) = \mathcal{D}(\sigma_{\mathrm{i}/2}^{\psi})^*$, by Lemma VIII.3.18, there exists $M \geq 0$ such that for every $x \in \mathcal{N}_+$

$$\psi(axa^*) \leq M^2\psi(x), \qquad \psi(b^*xb) \leq M^2\psi(x).$$

Taking increasing limit, we see that the above inequalities are valid for every $x \in \hat{\mathcal{N}}_+$, so that

$$\tilde{\psi}(axa^*) \leq M^2\tilde{\psi}(x), \qquad \tilde{\psi}(b^*xb) \leq M^2\tilde{\psi}(x) \tag{36}$$

for every $x \in \mathcal{M}_+$; therefore $\mathfrak{n}_{\tilde{\psi}}a^* \subset \mathfrak{n}_{\tilde{\psi}}$, $\mathfrak{n}_{\tilde{\psi}}b \subset \mathfrak{n}_{\tilde{\psi}}$ and

$$\left.\begin{aligned} &\|\eta_{\tilde{\psi}}(xa^*)\| \leq M\|\eta_{\tilde{\psi}}(x)\|;\\ &\|\eta_{\tilde{\psi}}(xb)\| \leq M\|\eta_{\tilde{\psi}}(x)\|, \quad x \in \mathfrak{n}_{\tilde{\psi}}. \end{aligned}\right\} \tag{36$'$}$$

We will prove that $(a, b) \in \mathcal{G}(\sigma_{-\mathrm{i}}^{\tilde{\psi}})$. By Theorem VIII.3.25, it suffices to show

$$\tilde{\psi}(ax) = \tilde{\psi}(xb), \quad x \in \mathfrak{m}_{\tilde{\psi}}. \tag{37}$$

Fix $x_0 = y_0^*z_0$ with $y_0, z_0 \in \mathfrak{n}_{\tilde{\psi}} \cap \mathfrak{n}_T$. As $\mathfrak{n}_{\tilde{\psi}}a^* \subset \mathfrak{n}_{\tilde{\psi}}$ and $\mathfrak{n}_T$ is a right $\mathcal{N}$-module, we have

$$ax_0 = \left(y_0a^*\right)^*z_0 \in \left(\mathfrak{n}_{\tilde{\psi}} \cap \mathfrak{n}_T\right)^*\left(\mathfrak{n}_{\tilde{\psi}} \cap \mathfrak{n}_T\right) \subset \mathfrak{m}_{\tilde{\psi}} \cap \mathfrak{m}_T.$$

Similarly, we get

$$x_0b = y_0^*(z_0b) \in \mathfrak{m}_{\tilde{\psi}} \cap \mathfrak{m}_T.$$

Since $(a, b) \in \mathcal{G}(\sigma_{-\mathrm{i}}^{\psi})$, we have

$$\psi \circ \dot{T}(ax_0) = \psi\left(a\dot{T}(x_0)\right) = \psi\left(\dot{T}(x_0)b\right) = \psi \circ \dot{T}(x_0b),$$

so that

$$\tilde{\psi}(ax_0) = \tilde{\psi}(x_0b). \tag{38}$$

Now suppose $x = y^*z$ with $y, z \in \mathfrak{n}_{\tilde{\psi}}$. Since $\psi \circ T(y^*y) < +\infty$, we have the spectral decomposition of $T(y^*y)$: $\quad T(y^*y) = \int_0^\infty \lambda \,\mathrm{d}e(\lambda)$. For any $\lambda > 0$, we have $ye(\lambda) \in \mathfrak{n}_T$ and

$$\psi \circ T\left(e(\lambda)y^*ye(\lambda)\right) = \psi\left(e(\lambda)T(y^*y)e(\lambda)\right) \leq \psi \circ T(y^*y) < +\infty,$$

so that $ye(\lambda) \in \mathfrak{n}_{\tilde{\psi}}$. Furthermore,

$$\begin{aligned} \left\|\eta_{\tilde{\psi}}\left(ye(\lambda) - y\right)\right\|^2 &= \psi \circ T\Big(\left(ye(\lambda) - y\right)^*\left(ye(\lambda) - y\right)\Big)\\ &= \psi\Big(\left(1 - e(\lambda)\right)T(y^*y)\Big)\left(1 - e(\lambda)\right)\\ &= \psi\left(\int_\lambda^\infty \mu \,\mathrm{d}e(\mu)\right) \to 0 \quad \text{as} \quad \lambda \to \infty. \end{aligned}$$

Similarly, with the spectral decomposition of $T(z^*z)$:

$$T(z^*z) = \int_0^\infty \lambda \, \mathrm{d} f(\lambda),$$

we have $zf(\lambda) \in \mathfrak{n}_{\tilde{\psi}} \cap \mathfrak{n}_T$ and

$$\lim_{\lambda\to\infty} \left\| \eta_{\tilde{\psi}}\big(zf(\lambda) - z\big) \right\| = 0.$$

By (36′), we get

$$\lim_{\lambda\to\infty} \left\| \eta_{\tilde{\psi}}\big(ye(\lambda)a^*\big) - \eta_{\tilde{\psi}}\big(ya^*\big) \right\| = 0; \qquad \lim_{\lambda\to\infty} \left\| \eta_{\tilde{\psi}}\big(zf(\lambda)b\big) - \eta_{\tilde{\psi}}(zb) \right\| = 0.$$

Therefore, we have, by the previous arguments for x_0,

$$\begin{aligned}\tilde{\psi}(ax) &= \big(\eta_{\tilde{\psi}}(z) \mid \eta_{\tilde{\psi}}(ya^*)\big) = \lim_{\lambda\to\infty} \Big(\eta_{\tilde{\psi}}\big(zf(\lambda)\big) \Bigm| \eta_{\tilde{\psi}}\big(ye(\lambda)a^*\big)\Big) \\ &= \lim_{\lambda\to\infty} \tilde{\psi}\Big(a\big(ye(\lambda)\big)^*\big(zf(\lambda)\big)\Big) = \lim_{\lambda\to\infty} \tilde{\psi}\Big(\big(ye(\lambda)\big)^*\big(zf(\lambda)\big)b\Big) \\ &= \lim_{\lambda\to\infty} \Big(\eta_{\tilde{\psi}}\big(zf(\lambda)b\big) \Bigm| \eta_{\tilde{\psi}}\big(ye(\lambda)\big)\Big) = \big(\eta_{\psi}(zb) \mid \eta_{\psi}(y)\big) = \tilde{\psi}(xb).\end{aligned}$$

Therefore, we conclude that $(a, b) \in \mathcal{G}\big(\sigma_{-\mathrm{i}}^{\tilde{\psi}}\big)$, i.e. $\mathcal{G}\big(\sigma_{-\mathrm{i}}^{\psi}\big) \subset \mathcal{G}\big(\sigma_{-\mathrm{i}}^{\tilde{\psi}}\big)$.

As seen in the proof of Proposition VIII.3.24, $x \in \mathcal{M}$ is of exponential type relative to $\{\sigma_t^{\tilde{\psi}}\}$ if and only if

$$x \in \bigcap_{n\in\mathbf{Z}} \mathcal{D}\big(\sigma_{-n\mathrm{i}}^{\tilde{\psi}}\big), \qquad \sup_{n\in\mathbf{Z}} \left\|\sigma_{-n\mathrm{i}}^{\tilde{\psi}}(x)\right\| \mathrm{e}^{-c|n|} < +\infty \tag{39}$$

for some $c > 0$. Hence $\mathcal{N}_{\exp}^{\psi} \subset \mathcal{M}_{\exp}^{\tilde{\psi}}$. For each $x \in \mathcal{N}_{\exp}^{\psi}$, we consider

$$y(\alpha) = \sigma_\alpha^{\psi}(x) - \sigma_\alpha^{\tilde{\psi}}(x), \quad \alpha \in \mathbf{C}.$$

From the above arguments, it follows that the function $f : \alpha \in \mathbf{C} \mapsto \omega\big(y(\alpha)\big) \in \mathbf{C}$ for any $\omega \in \mathcal{M}_*$ is an entire function of exponential type and that $f(-\mathrm{i}n) = 0$, $n \in \mathbf{Z}$. Hence $f(\alpha) = 0$ for every $\alpha \in \mathbf{C}$, thus $y(\alpha) = 0$. This means that

$$\sigma_t^{\psi}(x) = \sigma_t^{\tilde{\psi}}(x), \quad x \in \mathcal{N}_{\exp}^{\psi}.$$

As $\mathcal{N}_{\exp}^{\psi}$ is σ-weakly dense in $\mathcal{N}$, we conclude (35). Q.E.D.

Corollary 4.22. *Suppose $\mathcal{N} \subset \mathcal{M}$ is a pair of von Neumann algebras, and $T \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N})$.*

(i) *For every $\varphi \in \mathfrak{W}_0(\mathcal{N})$,*

$$\sigma_t^{\varphi\circ T}(x) = \sigma_t^{\varphi}(x), \quad x \in \mathcal{N}; \tag{35}$$

(ii) *For every pair* $\varphi, \psi \in \mathfrak{W}_0(\mathcal{N})$ *,*

$$\big(\mathrm{D}\psi \circ T : \mathrm{D}\varphi \circ T\big)_t = \big(\mathrm{D}\psi : \mathrm{D}\varphi\big)_t, \quad t \in \mathbf{R}. \tag{40}$$

(iii) *If* S *is another element of* $\mathfrak{W}_0(\mathcal{M}, \mathcal{N})$ *, then for any* $\varphi \in \mathfrak{W}_0(\mathcal{N})$ *,* $\big(\mathrm{D}\varphi \circ T : \mathrm{D}\varphi \circ S\big)_t$ *,* $t \in \mathbf{R}$*, belongs to* $\mathcal{N}^c = \mathcal{N}' \cap \mathcal{M}$ *and does not depends on the choice of* $\varphi \in \mathfrak{W}_0(\mathcal{N})$ *, i.e.*

$$\big(\mathrm{D}\varphi \circ T : \mathrm{D}\varphi \circ S\big)_t = \big(\mathrm{D}\psi \circ T : \mathrm{D}\psi \circ S\big)_t, \quad t \in \mathbf{R}$$

for any other $\psi \in \mathfrak{W}_0(\mathcal{N})$ *.*

PROOF: We have proved (i) in Lemma 4.21.

(ii) Consider the 2×2-matrix algebras $\mathcal{N} \otimes M_2 \subset \mathcal{M} \otimes M_2$. As $\sigma_t^{\varphi \otimes \mathrm{Tr}} = \sigma_t^{\varphi} \otimes \mathrm{id}$ and $\sigma_t^{(\varphi \circ T) \otimes \mathrm{Tr}} = \sigma_t^{\varphi \circ T} \otimes \mathrm{id}$ agree on $\mathcal{N} \otimes M_2$, there exists $T' \in \mathfrak{W}_0\big(\mathcal{M} \otimes M_2, \mathcal{N} \otimes M_2\big)$ by Theorem 4.18 such that

$$(\varphi \circ T) \otimes \mathrm{Tr} = (\varphi \otimes \mathrm{Tr}) \circ T'.$$

We denote this T' by $T \otimes \mathrm{id}$. Now consider the balanced weight $\rho = \varphi \oplus \psi$ on $\mathcal{N} \otimes M_2$ and $\tilde{\rho} = \rho \circ (T \otimes \mathrm{id})$, and observe that $\tilde{\rho} = \tilde{\varphi} \oplus \tilde{\psi}$, where $\tilde{\varphi} = \varphi \circ T$ and $\tilde{\psi} = \psi \circ T$. As $(1 \otimes e_{21}) \in \mathcal{N} \otimes M_2$, we have

$$\big(\mathrm{D}\psi \circ T : \mathrm{D}\varphi \circ T\big)_t = \sigma_t^{\tilde{\rho}}(1 \otimes e_{21}) = \sigma_t^{\rho}(1 \otimes e_{21}) = (\mathrm{D}\psi : \mathrm{D}\varphi)_t$$

by (i).

(iii) For $x = \sum x_{ij} \otimes e_{ij} \in \mathcal{M} \otimes M_2$, set

$$R\Big(\sum_{i,j=1}^{2} x_{ij} \otimes e_{ij}\Big) = \big(T(x_{11}) + S(x_{22})\big) \otimes 1.$$

Then it follows that $R \in \mathfrak{W}_0\big(\mathcal{M} \otimes M_2, \mathcal{N} \otimes \mathbf{C}\big)$. Identifying $\mathcal{N}$ with $\mathcal{N} \otimes \mathbf{C}$, we consider $\varphi \circ R$ and observe that $\varphi \circ R = \varphi \circ T \oplus \varphi \circ S$. Hence

$$\big(\mathrm{D}\varphi \circ T : \mathrm{D}\varphi \circ S\big)_t \otimes e_{12} = \sigma_t^{\varphi \circ R}(1 \otimes e_{12}).$$

As $1 \otimes e_{12} \in \big(\mathcal{N} \otimes \mathbf{C}\big)' \cap \mathcal{M} \otimes M_2$ and $\sigma_t^{\varphi \circ R}$ leaves $\mathcal{N} \otimes \mathbf{C}$ invariant, $\sigma_t^{\varphi \circ R}(1 \otimes e_{12})$ belongs to $\big(\mathcal{N} \otimes \mathbf{C}\big)' \cap \mathcal{M} \otimes M_2 = \mathcal{N}^c \otimes M_2$, which means that

$$\big(\mathrm{D}\varphi \circ T : \mathrm{D}\varphi \circ S\big)_t \in \mathcal{N}^c.$$

For another $\psi \in \mathfrak{W}_0(\mathcal{N})$, we have

$$\begin{aligned}
&\big(\mathrm{D}\psi \circ T : \mathrm{D}\psi \circ S\big)_t \\
&\quad = \big(\mathrm{D}\psi \circ T : \mathrm{D}\varphi \circ T\big)_t \big(\mathrm{D}\varphi \circ T : \mathrm{D}\varphi \circ S\big)_t \big(\mathrm{D}\varphi \circ S : \mathrm{D}\psi \circ S\big)_t \\
&\quad = (\mathrm{D}\psi : \mathrm{D}\varphi)_t \big(\mathrm{D}\varphi \circ T : \mathrm{D}\psi \circ S\big)_t (\mathrm{D}\varphi : \mathrm{D}\psi)_t \\
&\quad = (\mathrm{D}\psi : \mathrm{D}\varphi)_t (\mathrm{D}\varphi : \mathrm{D}\psi)_t \big(\mathrm{D}\varphi \circ T : \mathrm{D}\varphi \circ S\big)_t \\
&\quad = \big(\mathrm{D}\varphi \circ T : \mathrm{D}\varphi \circ S\big)_t
\end{aligned}$$

because $\big(\mathrm{D}\varphi \circ T : \mathrm{D}\varphi \circ S\big)_t \in \mathcal{N}^c$. Q.E.D.

Definition 4.23. Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann algebras.

(i) For $T \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N})$, let σ_t^T be the restriction of $\sigma_t^{\varphi\circ T}$, $\varphi \in \mathfrak{W}_0(\mathcal{M})$, to the relative commutant $\mathcal{N}^c = \mathcal{N}' \cap \mathcal{M}$, which is independent of the choice of φ. We call it the *modular automorphism group* of T.

(ii) For $T_1, T_2 \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N})$, we set

$$\big(\mathrm{D}T_1 : \mathrm{D}T_2\big)_t = \big(\mathrm{D}\varphi \circ T_1 : \mathrm{D}\varphi \circ T_2\big)_t, \quad t \in \mathbf{R},$$

with $\varphi \in \mathfrak{W}_0(\mathcal{N})$ which does not depend on the choice of φ, and call it the *cocycle derivative* of T_1 relative to T_2.

Theorem 4.24. *Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann algebras. There exists a bijection: $T \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N}) \longleftrightarrow T' \in \mathfrak{W}_0(\mathcal{N}', \mathcal{M}')$ such that*

$$\sigma_t^T = \sigma_{-t}^{T'}, \quad T \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N});$$

$$\big(\mathrm{D}T_1 : \mathrm{D}T_2\big)_t = \big(\mathrm{D}T_2' : \mathrm{D}T_1'\big)_{-t}, \quad T_1, T_2 \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N}).$$

In particular, $\mathfrak{W}_0(\mathcal{M}, \mathcal{N}) \neq \emptyset$ if and only if $\mathfrak{W}_0(\mathcal{N}', \mathcal{M}') \neq \emptyset$.

PROOF: Suppose $\mathfrak{W}_0(\mathcal{M}, \mathcal{N}) \neq \emptyset$. Let $T \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N})$. Fix $\varphi \in \mathfrak{W}_0(\mathcal{N})$, and $\psi \in \mathfrak{W}_0(\mathcal{M}')$. We then have, by Theorem 3.8, with $H = \mathrm{d}\varphi \circ T / \mathrm{d}\psi$, that

$$H^{\mathrm{i}t} x H^{-\mathrm{i}t} = \sigma_t^{\varphi\circ T}(x), \quad x \in \mathcal{M}, \qquad H^{\mathrm{i}t} y H^{-\mathrm{i}t} = \sigma_{-t}^{\psi}(y), \quad y \in \mathcal{M}'.$$

Choose $\omega \in \mathfrak{W}_0(\mathcal{N}')$ and let $K = \mathrm{d}\varphi / \mathrm{d}\omega$. Then

$$K^{\mathrm{i}t} x K^{-\mathrm{i}t} = \sigma_t^{\varphi}(x) = H^{\mathrm{i}t} x H^{-\mathrm{i}t}, \quad x \in \mathcal{N}; \qquad K^{\mathrm{i}t} y K^{-\mathrm{i}t} = \sigma_{-t}^{\omega}(y), \quad y \in \mathcal{N}'.$$

Hence $u_t = H^{-\mathrm{i}t} K^{\mathrm{i}t}$ belongs $\mathcal{N}'$ and

$$\begin{aligned} u_{s+t} &= H^{-\mathrm{i}s} H^{-\mathrm{i}t} K^{\mathrm{i}s} K^{\mathrm{i}t} = H^{-\mathrm{i}s} K^{\mathrm{i}s} K^{-\mathrm{i}s} H^{-\mathrm{i}t} K^{\mathrm{i}t} K^{\mathrm{i}s} \\ &= u_s \sigma_s^{\omega}(u_t), \quad s, t \in \mathbf{R}. \end{aligned}$$

Hence $\{u_s\}$ is a one cocycle for σ^{ω}, so that there exists $\tilde{\psi} \in \mathfrak{W}_0(\mathcal{N}')$ such that $\big(\mathrm{D}\tilde{\psi} : \mathrm{D}\omega\big)_t = u_t$. Then we have for every $y \in \mathcal{N}'$

$$\begin{aligned} \sigma_t^{\tilde{\psi}}(y) &= \big(\mathrm{D}\tilde{\psi} : \mathrm{D}\omega\big)_t^* \sigma_t^{\omega}(y) \big(\mathrm{D}\tilde{\psi} : \mathrm{D}\omega\big)_t^* \\ &= H^{-\mathrm{i}t} K^{\mathrm{i}t} K^{-\mathrm{i}t} y K^{\mathrm{i}t} K^{-\mathrm{i}t} H^{\mathrm{i}t} = H^{-\mathrm{i}t} y H^{\mathrm{i}t}. \end{aligned}$$

Therefore, we obtain $\sigma_t^{\tilde{\psi}}(y) = \sigma_t^{\psi}(y)$ for every $y \in \mathcal{M}'$. Theorem 4.18 entails the existence of $T' \in \mathfrak{W}_0(\mathcal{N}', \mathcal{M}')$ such that $\tilde{\psi} = \psi \circ T'$.

Now for $x \in \mathcal{N}^c = \mathcal{N}' \cap \mathcal{M} = (\mathcal{M}')' \cap \mathcal{N}'$,

$$\sigma_t^T(x) = \sigma_t^{\varphi\circ T}(x) = H^{\mathrm{i}t} x H^{-\mathrm{i}t} = \sigma_{-t}^{\psi\circ T'}(x) = \sigma_{-t}^{T'}(x).$$

Choose another $S \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N})$ and observe that $u_t = (\mathrm{D}S : \mathrm{D}T)_t = \big(\mathrm{D}\varphi \circ S : \mathrm{D}\varphi \circ T\big)_t$ belongs to $\mathcal{N}^c$, and it is a $\sigma^{\varphi\circ T}$-cocycle. Namely,

$$u_{s+t} = u_s \sigma_s^{\varphi\circ T}(u_t) = u_s H^{\mathrm{is}} u_t H^{-\mathrm{is}}.$$

Set $v_t = u_{-t}^* = (\mathrm{D}T : \mathrm{D}S)_{-t}$. Then we have

$$\begin{aligned} v_{s+t} &= u_{-(s+t)}^* = u_{-t}^* \sigma_{-t}^{\varphi\circ T}(u_{-s}^*) = v_t H^{-\mathrm{i}t} u_{-s}^* H^{\mathrm{i}t} \\ &= v_t \sigma_t^{\psi\circ T}(u_{-s}^*) = v_t \sigma_t^{\psi\circ T}(v_t). \end{aligned}$$

Hence there exists $\omega \in \mathfrak{W}_0(\mathcal{N}')$ with $v_t = (\mathrm{D}\omega : \mathrm{D}\psi \circ T)_t$. Then we have, for every $y \in \mathcal{M}'$,

$$\sigma_t^{\omega}(y) = v_t \sigma_t^{\psi\circ T'}(y) v_t^* = v_t \sigma_t^{\psi}(y) v_t^* = \sigma_t^{\psi}(y)$$

because $v_t \in \mathcal{M}$. Hence there exists $S' \in \mathfrak{W}_0(\mathcal{N}', \mathcal{M}')$ by Theorem 4.18 such that $\omega = \psi \circ S'$. We therefore obtain

$$(\mathrm{D}S' : \mathrm{D}T')_t = \big(\mathrm{D}\psi \circ S' : \mathrm{D}\psi\circ!T'\big)_t = \big(\mathrm{D}\omega : \mathrm{D}(\psi\circ T)\big)_t = v_t = (\mathrm{D}T : \mathrm{D}S)_{-t}.$$

As S' is uniquely determined by the formula $\omega = \psi \circ S'$, S' is determined by $(\mathrm{D}S' : \mathrm{D}T')_t$. Therefore, the map: $S \in \mathfrak{W}_0(\mathcal{M}, \mathcal{N}) \mapsto S' \in \mathfrak{W}_0(\mathcal{N}', \mathcal{M}')$ is an injection. By symmetry, this correspondence is bijective. Q.E.D.

Corollary 4.25. *Let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra. Then there exists a bijection: $\varphi \in \mathfrak{W}_0(\mathcal{M}) \mapsto T_\varphi \in \mathfrak{W}_0\big(\mathcal{L}(\mathfrak{H}), \mathcal{M}'\big)$ such that*

$$\sigma_t^{\varphi} = \sigma_{-t}^{T_\varphi};$$

$$(\mathrm{D}\psi : \mathrm{D}\varphi)_t = (\mathrm{D}T_\varphi : \mathrm{D}T_\psi)_{-t}, \quad t \in \mathbf{R};$$

$$\psi \circ T_\varphi = \mathrm{Tr}\left(\left(\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right) \cdot\right), \quad \psi \in \mathfrak{W}_0(\mathcal{M}'),$$

where the weight $\mathrm{Tr}\big(\big(\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\cdot\big)\big)$ *means the one given in Lemma* VIII.2.8 *relative to the standard trace* Tr *on* $\mathcal{L}(\mathfrak{H})$.

This is an immediate consequence of Theorem 4.24 and its proof.

Exercise IX.4

1) Let $\mathcal{N}$ be a von Neumann subalgebra of a von Neumann algebra $\mathcal{M}$. Suppose that there exists a normal projection $\mathcal{E}$ of norm one from $\mathcal{M}$ onto $\mathcal{N}$. Set $N_{\mathcal{E}} = \big\{x \in \mathcal{M} : \mathcal{E}(x^*x) = 0\big\}$.

(**a**) Show that there exists a projection $e \in \mathcal{M} \cap \mathcal{N}'$ such that the central support of e in $\mathcal{N}'$ is the identity and $N_{\mathcal{E}} = (1-e)\mathcal{M}$.

(**b**) Prove that the map $\mathcal{E}'$ given by $\mathcal{E}'(x) = \mathcal{E}(x)e$, $x \in \mathcal{M}_e$, is a faithful projection of norm one from $\mathcal{M}_e$ onto $\mathcal{N}_e$.

(**c**) Prove that if $\mathcal{M}$ is semi-finite, then $\mathcal{N}$ must be also semi-finite. (Hint: Recall the proof of Theorem V.2.30.)

(**d**) Prove that if $\mathcal{M}$ is of type I, then $\mathcal{N}$ must be of type I. (Hint: By (a) and (b) we may assume that $\mathcal{E}$ is faithful. By (c), there exists a finite projection e in $\mathcal{N}$ with the central support in $\mathcal{N}$ equal to 1. Restricting $\mathcal{E}$ to $\mathcal{M}_e$ and $\mathcal{N}_e$, we may assume that $\mathcal{N}$ is finite. If τ is a normal trace on $\mathcal{N}$, then $\varphi = \tau \circ \mathcal{E}$ is a normal positive functional on $\mathcal{M}$ such that $s(\varphi) \in \mathcal{N}' \cap \mathcal{M}$ and $\mathcal{M}_\varphi \supset \mathcal{N}_{s(\varphi)}$. Restricting our attention to $\mathcal{M}_{s(\varphi)}$ and $\mathcal{N}_{s(\varphi)}$, we may assume that φ is faithful. Thus the situation is that τ is a faithful norm trace on $\mathcal{N}$ and $\mathcal{E}$ is a faithful conditional expectation relative to φ. Let $\tilde{\tau}$ be a faithful semi-finite normal trace on $\mathcal{M}$ and h be the element in $L^1(\mathcal{M}, \tilde{\tau})_+$ corresponding to φ. Then $\mathcal{N} \subset \{h\}' \cap \mathcal{M}$. We complete the proof by showing that $\mathcal{M}_\varphi = \{h\}' \cap \mathcal{M}$ contains only von Neumann algebras of type I based on the fact that h is integrable relative to $\tilde{\tau}$ in the von Neumann algebra of type I, which entails that $\mathcal{M}_\varphi$ is finite and of type I.)

(**e**) Prove that if $\mathcal{M}$ is atomic, then $\mathcal{N}$ is also atomic.

2) Let $\mathcal{M}$ be a von Neumann algebra and G a group of automorphisms of $\mathcal{M}$, i.e. a subgroup of $\mathrm{Aut}(\mathcal{M})$. Following the line of arguments suggested below, show that if $\mathcal{M}$ admits sufficiently many G-invariant normal states in the sense that for any non-zero $x \in \mathcal{M}$ there exists a normal state φ such that $\varphi(x^*x) > 0$ and $\varphi \circ \alpha = \varphi, \alpha \in G$, then there exists a unique faithful normal projection $\mathcal{E}$ of norm one from $\mathcal{M}$ onto the fixed point subalgebra $\mathcal{M}^G = \left\{ x \in \mathcal{M} : \alpha(x) = x,\ \alpha \in G \right\}$ such that $\mathcal{E} \circ \alpha = \mathcal{E},\ \alpha \in G$.

(**a**) Let $\mathfrak{S}_*^G$ be the set of all G-invariant normal states. Show that the support, $s(\varphi)$, of each $\varphi \in \mathfrak{S}_*^G$ is a projection in $\mathcal{M}^G$.

(**b**) Let $\mathcal{F}$ be a maximal family of elements of $\mathfrak{S}_*^G$ with orthogonal support and set $\psi = \sum_{\varphi \in \mathcal{F}} \varphi$. Show that ψ is a faithful semi-finite normal weight on $\mathcal{M}$ such that ψ is semi-finite on $\mathcal{M}^G$ and the modular automorphism group $\left\{ \sigma_t^\psi : t \in \mathbf{R} \right\}$ leaves $\mathcal{M}^G$ globally invariant.

(**c**) Apply Theorem 4.2 to $\mathcal{M}$, $\mathcal{M}^G$ and ψ.

3) Under the same hypothesis and the notations, for each $x \in \mathcal{M}$ let $K(x)$ be the σ-weak convex closure of the orbit $\mathrm{Orb}(x) = \left\{ \alpha(x) : \alpha \in G \right\}$ of x. Following the arguments suggested below, i.e. without making use of the G-invariant faithful projection $\mathcal{E}$ of norm one, shown in the last problem, prove that $K(x) \cap \mathcal{M}^G \neq \emptyset$, in fact $K(x) \cap \mathcal{M}^G = \left\{ \mathcal{E}(x) \right\}$ as this will give alternative proof, which does make use of the modular theory, for the existence and the uniqueness of a faithful G-invariant normal projection of norm one.

(**a**) Observe first that for each $\varphi \in \mathfrak{S}_*^G$ the function: $x \in \mathcal{M} \mapsto \|x\|_\varphi = \varphi(x^*x)^{1/2}$ is lower semi-continuous.

(**b**) Observe that the set $K_F(x) = \left\{ y \in K(x) : \|y\|_\varphi = \inf\{\|z\|_\varphi : z \in K(x)\},\ \varphi \in F \right\}$ is a non-empty closed face of the σ-weakly compact convex set $K(x)$ for any finite subset F of $\mathfrak{S}_*^G$ and that $\alpha\left(K_F(x)\right) = K_F(x),\ \alpha \in G$.

(**c**) Show that $K_{\mathfrak{S}_*^G}(x) = \{ y \in K(x) : \|y\|_\varphi = \inf\{ \|z\|_\varphi : z \in K(x) \},\ \varphi \in \mathfrak{S}_*^G \} \neq \emptyset$ and in fact that $K_{\mathfrak{S}_*^G}(x)$ is the singleton set $\{\mathcal{E}(x)\}$.

4) Let $\mathfrak{H} = L^2(\mathbf{R})$ and $\mathcal{A} = L^\infty(\mathbf{R})$ act on $\mathfrak{H}$ by multiplication. Let $u(t)$ be the element of $\mathcal{A}$ corresponding to the function: $s \in \mathbf{R} \mapsto \mathrm{e}^{ist} \in \mathbf{T}$. Let φ be the faithful semi-finite normal weight on $\mathcal{A}$ obtained by the integration relative to the Lebesgue measure on $\mathbf{R}$. Show that the operator valued weight T from $\mathcal{L}(\mathfrak{H})$ to $\mathcal{A}$ such that $\mathrm{Tr} = \varphi \circ T$ is given by $T(x) = \int_{\mathbf{R}} u(t)xu(t)^* \,\mathrm{d}t,\ \ t \in \mathbf{R},\ \ x \in \mathcal{L}(\mathfrak{H})_+$.

5) Let G be an abelian locally compact group with $\hat{G}$ the dual group. Let $\mathfrak{H} = L^2(G)$ be the Hilbert space of square integrable functions relative to the Haar measure and set $\{u(p)\xi\}(s) = \langle s, p\rangle \xi(s),\ \xi \in \mathfrak{H},\ s \in G$ and $p \in \hat{G}$. Let $\mathcal{A} = L^\infty(G)$ act on $\mathfrak{H}$ by multiplication as in the last problem. Show that if φ is the faithful semi-finite normal weight on $\mathcal{A}$ given by the integration relative to the Haar measure on G, then the operator valued weight T from $\mathcal{L}(\mathfrak{H})$ to $\mathcal{A}$ such that $\mathrm{Tr} = \varphi \circ T$ is given by: $T(x) = \int_{\hat{G}} u(p)xu(p)^* \,\mathrm{d}p,\ \ x \in \mathcal{L}(\mathfrak{H})_+$.

6) Let G be a discrete countable group, $\mathfrak{H} = \ell^2(G)$ and $\mathcal{M} = \mathcal{R}_\ell(G)$. Let τ be the normalized trace given by $\tau(x) = (x\xi_0 \mid \xi_0),\ x \in \mathcal{M}$ where $\xi_0(s) = 1$ for $s = e$ and 0 for $s \neq e$. Let λ and ρ be the left and right regular representations of G on $\mathfrak{H}$ respectively.

(**a**) Show that the operator valued weight T of $\mathcal{L}(\mathfrak{H})$ to $\mathcal{M}$ such that $\mathrm{Tr} = \tau \circ T$ is given by: $T(x) = \sum_{s\in G} \rho(s)x\rho(s)^*,\ x \in \mathcal{L}(\mathfrak{H})_+$.

(**b**) Show that the operator valued weight T in (a) is also given by:
$x = \sum_{s\in G} \mathrm{Tr}\big(\lambda(s)^*x\big)\lambda(s),\ \ x \in \mathcal{L}(\mathfrak{H})_+$.

Notes on Chapter IX

The cones $\mathfrak{P}^\sharp$ and $\mathfrak{P}^\flat$ were introduced by Takesaki in [708] in the case of a cyclic and separating vector, to connect vectors in the Hilbert space and normal positive functional on the von Neumann algebra in question. The systematic study of one parameter family of convex cones in the Hilbert space $\mathfrak{H}$, on which $\mathcal{M}$ acts, with a distinguished cyclic and separating vector ξ_0 was done by Araki, [424]. Independently, Connes, [459], and Haagerup, [537], considered the natural cone $\mathfrak{P}^\natural$. Theorems 1.2 and 1.14 were proven by them. The inequality (11) is referred as Powers-Størmer inequality as it was proved first by them for $\mathcal{L}(\mathfrak{H})$, [671].

Before the Tomita-Takesaki theory was established in the late sixties and the early seventies, non-commutative integration meant the theory presented in Section 2. One can trace the history of the subject to the work of Murray and von Neumann. They proved for example that every symmetric closed operator affiliated with a factor of type II_1 is automatically self-adjoint and that the collection of densely defined operators affiliated with a factor of type II_1 forms an involutive algebra under the natural operations. It was this fact that von Neumann thought the theory of operator algebras would play a significant role in quantum physics by accommodating algebraic operations for unbounded physical quantities, a dream which was not realized in its original form. Araki and Woods in a joint work, [430], showed in 1963 that the most of factors appearing in quantum physics is either of type I or type III, a fact that shattered the long held dream of many specialists. Nevertheless, the theory presented in Section 3 really tells that the theory related to traces is indeed non-commutative analogue of the usual integration-measure theory. The theory was completed by the hands of Dixmier, [502], Dye, [99], Segal, [326], and many others. Theory presented here followed the approach given by Nelson, [647].

The theory of spatial derivatives is due to Connes, [472], inspired by the work of Woronowicz on phase system, [741], and the work of Haagerup on operator valued weights, [541]. The theory presented here follows the line set by Falcone in his thesis, [518, 519], which gives more streamlined approach to the theory of relative tensor product of Sauvageot, [688]. The theory of bimodules goes further far beyond the one presented here. It was originally developed by Connes although its full account has never been published. He called it a *correspondence.* The motivation and its applications are presented in his book, [480].

The concept of conditional expectations has played significant roles in the development of the theory of operator algebras. It was first discovered by Dixmier, [85], in the course of his theory of non-commutative integration for traces. But it was Umegaki, [392, 726], who viewed the projection of norm one determined by a normal tracial state as a non-commutative counter part of the conditional expectation in probability theory. Subsequently, he developed a theory of conditional expectations closely following the analogy with probability theory. It was, however, the remarkable bimodule property which yielded a wide range of applications as we will see in the later chapters in the structure theory of factors and subfactors. The theory of operator valued weights was developed by Haagerup, [541]. But it had appeared long in disguise before his formal treatise. For example, the theory of integrable groups actions on a von Neumann algebra which will be treated later, Chapter X, made use of operator valued weights without the full theory. Haagerup's original approach however used the structure theory of von Neumann algebras of type III, i.e. the duality theorem, Theorem XII.1.1, applied to the modular automorphism groups. We took an approach given by Falcone and Takesaki, [518, 519], which develops the theory within the framework of non-commutative integration.