Chapter II

Domains of Holomorphy

1. The Continuity Theorem

General Hartogs Figures. The subject of this chapter is the continuation of holomorphic functions. We consider domains in $\mathbb{C}^n$, for $n \geq 2$. A typical example is the *Euclidean Hartogs figure* $(\mathsf{P}^n, \mathsf{H})$, where $\mathsf{P}^n = \mathsf{P}^n(\mathbf{0}, 1)$ is the unit polydisk, and

$$\mathsf{H} = \{\mathbf{z} \in \mathsf{P}^n \,:\, |z_1| > q_1 \text{ or } |z_\nu| < q_\nu \text{ for } \nu = 2, \ldots, n\}.$$

Here $q_1, \ldots, q_n$ are real numbers with $0 < q_\nu < 1$ for $\nu = 1, \ldots, n$. Every holomorphic function f on H has a holomorphic extension $\widehat{f}$ on P^n.

> **Definition.** Let $\mathbf{g} = (g_1, \ldots, g_n) : \mathsf{P}^n \to \mathbb{C}^n$ be an injective holomorphic mapping, $\widetilde{P} := \mathbf{g}(\mathsf{P}^n)$ and $\widetilde{H} := \mathbf{g}(\mathsf{H})$. Then $\big(\widetilde{P}, \widetilde{H}\big)$ is called a *general Hartogs figure*.

We use the symbolic picture that appears as Figure II.1

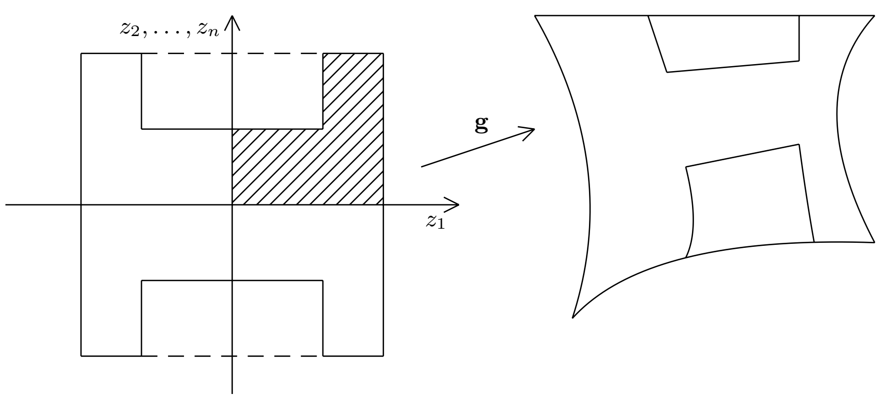


Figure II.1. General Hartogs figure

1.1 Continuity theorem. *Let $G \subset \mathbb{C}^n$ be domain, $\big(\widetilde{P}, \widetilde{H}\big)$ a general Hartogs figure with $\widetilde{H} \subset G$, f a holomorphic function on G. If $G \cap \widetilde{P}$ is connected, then f can be continued uniquely to $G \cup \widetilde{P}$.*

Proof: Let $\mathbf{g} : \mathsf{P}^n \to \mathbb{C}^n$ be an injective holomorphic mapping such that $\widetilde{P} := \mathbf{g}(\mathsf{P}^n)$ and $\widetilde{H} := \mathbf{g}(\mathsf{H})$. The function $h := f \circ \mathbf{g}$ is holomorphic in H. Therefore, there exists exactly one holomorphic function $\widehat{h}$ on P^n with $\widehat{h}|_{\mathsf{H}} = h$. Since $\mathbf{g} : \mathsf{P}^n \to \widetilde{P}$ is biholomorphic, the function $f_0 := \widehat{h} \circ \mathbf{g}^{-1}$ is defined on $\widetilde{P}$, and it is a holomorphic extension of $f|_{\widetilde{H}}$. We define

$$\widehat{f}(\mathbf{z}) := \begin{cases} f(\mathbf{z}) & \text{for } \mathbf{z} \in G, \\ f_0(\mathbf{z}) & \text{for } \mathbf{z} \in \widetilde{P}. \end{cases}$$

Since $G \cap \widetilde{P}$ is connected and $f = f_0$ on $\widetilde{H}$, it follows from the identity theorem that $\widehat{f}$ is a well-defined holomorphic function on $G \cup \widetilde{P}$. This is the desired extension of f. ∎

Example

Let $n \geq 2$ and $P' \subset\subset P$ be polydiscs around the origin in $\mathbb{C}^n$. Then every holomorphic function f on $P - \overline{P'}$ can be extended uniquely to a holomorphic function on P.

For a proof we may assume that $P = \mathsf{P}^n$ is the unit polydisk, and $P' = \mathsf{P}^n(\mathbf{0}, \mathbf{r})$, with $\mathbf{r} = (r_1, \ldots, r_n)$ and $0 < r_\nu < 1$ for $\nu = 1, \ldots, n$. It is clear that $G := P - \overline{P'}$ is a domain.

Given a point $\mathbf{z}_0 = \left(z_1^{(0)}, \ldots, z_n^{(0)}\right) \in G$ with $|z_n^{(0)}| > r_n$, we choose real numbers $q_1, \ldots, q_n$ as follows: For $\nu = 1, \ldots, n-1$, let q_ν be arbitrary numbers, with $r_\nu < q_\nu < 1$. To obtain a suitable q_n, we define an automorphism T of the unit disk D by

$$T(\zeta) := \frac{\zeta - z_n^{(0)}}{\overline{z}_n^{(0)}\zeta - 1}\,.$$

This automorphism maps $z_n^{(0)}$ onto 0 and a small disk $D \subset \{\zeta \in \mathbb{C} : r_n < |\zeta| < 1\}$ around $z_n^{(0)}$ onto a disk $K \subset \mathsf{D}$ with $0 \in K$. Notice that 0 need not be the center of K. We choose $q_n > 0$ such that $\mathsf{D}_{q_n}(0) \subset K$.

If we define $\mathsf{H} := \{\mathbf{z} \in \mathsf{P}^n : |z_1| > q_1 \text{ or } |z_\nu| < q_\nu \text{ for } \nu = 2, \ldots, n\}$, then $(\mathsf{P}^n, \mathsf{H})$ is a Euclidean Hartogs figure. The mapping $\mathbf{g} : \mathsf{P}^n \to \mathsf{P}^n$ defined by

$$\mathbf{g}(z_1, \ldots, z_n) := (z_1, \ldots, z_{n-1}, T^{-1}(z_n))$$

is biholomorphic, and $\left(\widetilde{P}, \widetilde{H}\right) = (\mathsf{P}^n, \mathbf{g}(\mathsf{H}))$ is a general Hartogs figure, with

$$\widetilde{H} \subset \{\mathbf{z} \in \mathsf{P}^n : |z_1| > r_1 \text{ or } |z_n| > r_n\} \subset G.$$

Since $\widetilde{P} \cap G = G$ is connected, the continuity theorem may be applied. The preceding example is a special case of the so-called *Kugelsatz* which we shall prove in Chapter VI.

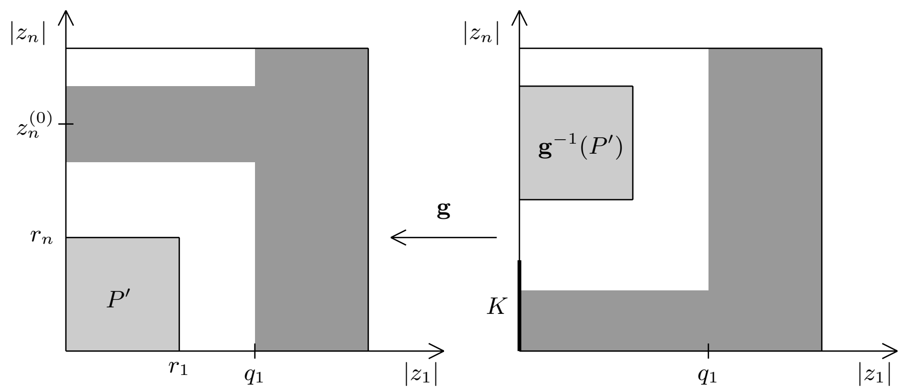


Figure II.2. A Hartogs figure for concentric polydiscs

Removable Singularities. Let $G \subset \mathbb{C}^n$ be a domain. If $A \subset G$ is an analytic set and f a holomorphic function on $G - A$ that is locally bounded along A, then by Riemann's extension theorem f has a holomorphic extension to G. If $n \geq 2$ and A is a complex linear subspace of codimension greater than or equal to 2, then **every** function holomorphic on $G - A$ has such an extension.

1.2 Theorem. *Let $\mathsf{P}^n = \mathsf{P}^n(\mathbf{0}, 1)$ be the unit polydisk in $\mathbb{C}^n$, $n \geq 2$, $k \geq 2$, and*

$$E := \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n \,:\, z_{n-k+1} = \cdots = z_n = 0\}.$$

Then every holomorphic function f on $\mathsf{P}^n - E$ can be holomorphically extended to P^n.

PROOF: Set $P' := \{\mathbf{z}' := (z_1, \ldots, z_{n-k}) \,:\, |\mathbf{z}'| < 1\}$, and for $0 < r \leq 1$ define $P''_r := \{\mathbf{z}'' = (z_{n-k+1}, \ldots, z_n) \,:\, |\mathbf{z}''| < r\}$.

Let $P'' := P''_1$ and fix an ε with $0 < \varepsilon \ll 1$. Then $\mathsf{P}^n \cap E \subset P' \times P''_\varepsilon$, and for $\mathbf{w} \in P'$ the function $f_{\mathbf{w}}(\mathbf{z}'') := f(\mathbf{w}, \mathbf{z}'')$ is holomorphic on $P'' - \overline{P''_\varepsilon}$. From the example above we know that $f_{\mathbf{w}}$ has a holomorphic extension $\widehat{f}_{\mathbf{w}}$ to P''. Now define $\widehat{f} : \mathsf{P}^n \to \mathbb{C}$ by $\widehat{f}(\mathbf{w}, \mathbf{z}'') := \widehat{f}_{\mathbf{w}}(\mathbf{z}'')$. On $\mathsf{P}^n - E$, $\widehat{f}$ is equal to f and is therefore holomorphic.

For $\mathbf{w} \in P'$ take a small open neighborhood $U = U(\mathbf{w}) \subset\subset P'$. Then $K := \overline{U} \times \partial P''_\varepsilon$ is compact. By the maximum principle we conclude that

$$\big|\,\widehat{f}(\mathbf{z}', \mathbf{z}'')\,\big| = \big|\,\widehat{f}_{\mathbf{z}'}(\mathbf{z}'')\,\big| \leq \|f_{\mathbf{z}''}\|_{\partial P''_\varepsilon} \leq \|f\|_K < \infty, \text{ for } (\mathbf{z}', \mathbf{z}'') \in U \times P''_\varepsilon - E.$$

From Riemann's extension theorem it follows that $\widehat{f}$ is holomorphic on P^n. ∎

1.3 Corollary. *For $n \geq 2$, every isolated singularity of a holomorphic function of $z_1, \ldots, z_n$ is removable.*

Riemann's extension theorem is false if we drop the condition "f bounded along the analytic set." For example, let $G \subset \mathbb{C}^n$ be a domain, $g : G \to \mathbb{C}$ a holomorphic function, and let $f : G - N(g) \to \mathbb{C}$ be defined by $f(\mathbf{z}) := 1/g(\mathbf{z})$. Then f is holomorphic on $G - N(g)$ but cannot be extended to any point of $N(g)$.

Things look quite different if there is a little hole in the hypersurface:

1.4 Proposition. *Let $n \geq 2$, $G_0 \subset \mathbb{C}^{n-1}$ a domain, $g : G_0 \to \mathbb{C}$ a* ***continuous*** *function, and $\Gamma := \{\mathbf{z} = (\mathbf{z}', z_n) \in G_0 \times \mathbb{C} : z_n = g(\mathbf{z}')\}$ the graph of g in $G := G_0 \times \mathbb{C}$. In addition, let $\mathbf{z}_0$ be a point of Γ and $U = U(\mathbf{z}_0) \subset G$ a small neighborhood.*

If f is a holomorphic function on $(G - \Gamma) \cup U$, then f has a unique holomorphic extension to G.

Proof: The uniqueness of the extension follows from the identity theorem. For the proof of existence (which is only a local problem) we may assume that $G_0 = \{\mathbf{z}' \in \mathbb{C}^{n-1} : |\mathbf{z}'| < 1\}$ and that there is a q with $0 < q < 1$ such that $|g(\mathbf{z}')| < q$ for $\mathbf{z}' \in G_0$. It also may be assumed that U is connected. Then it is clear that $G' := (G - \Gamma) \cup U \subset \mathsf{P}^n = \mathsf{P}^n(\mathbf{0}, 1)$ is connected.

Since $\mathbf{g} : \mathbf{z}' \mapsto (\mathbf{z}', g(\mathbf{z}'))$ is continuous, $U' := \mathbf{g}^{-1}(U)$ is an open neighborhood of $\mathbf{z}'_0$ with $(U' \times \mathsf{D}) \cap \Gamma \subset U$ and therefore $U' \times \mathsf{D} \subset G'$. For $\nu = 1, \ldots, n-1$ let T_ν be the automorphism of D defined by

$$T_\nu(\zeta) := \frac{\zeta - z_\nu^{(0)}}{\overline{z}_\nu^{(0)}\zeta - 1}.$$

Then $\mathbf{h} : \mathsf{P}^n \to \mathsf{P}^n$ with $\mathbf{h}(z_1, \ldots, z_n) := (T_1(z_2), \ldots, T_{n-1}(z_n), z_1)$ is holomorphic, $\mathbf{h}(\mathbf{0}) = (\mathbf{z}'_0, 0)$, and $\mathbf{h}(\{\mathbf{z} \in \mathsf{P}^n : |z_1| > q\}) \subset \{\mathbf{w} \in \mathsf{P}^n : |w_n| > q\}$.

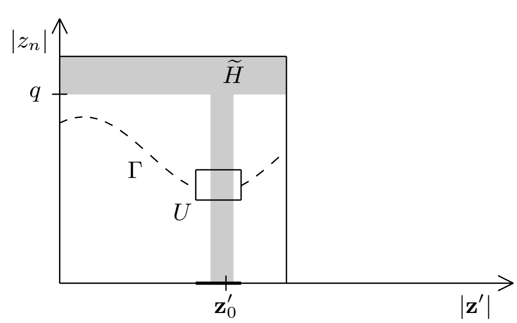


Figure II.3. Extending a holomorphic function across a hypersurface

We define $q_1 := q$, and for $\nu = 2, \dots, n$ choose q_ν such that

$$\mathbf{h}(\mathsf{D} \times \mathsf{D}_{q_2}(0) \times \cdots \times \mathsf{D}_{q_n}(0)) \subset U' \times \mathsf{D}.$$

Then $(\mathsf{P}^n, \mathsf{H})$ with $\mathsf{H} := \{\mathbf{z} \in \mathsf{P}^n : |z_1| > q_1 \text{ or } |z_\nu| < q_\nu \text{ for } \nu = 2, \dots, n\}$ is a Euclidean Hartogs figure, and $\big(\widetilde{P}, \widetilde{H}\big) = (\mathsf{P}^n, \mathbf{h}(\mathsf{H}))$ is a general Hartogs figure, with $\widetilde{H} \subset G'$ (see Figure II.3). Since $\widetilde{P} \cap G' = G'$ is connected, the proposition follows from the continuity theorem. ∎

The Continuity Principle

Sometimes we wish to use a family of analytic disks instead of a Hartogs figure.

> **Definition.** A *family of analytic disks* is given by a continuous map $\varphi : \overline{\mathsf{D}} \times [0,1] \to \mathbb{C}^n$ such that $\varphi_t(\zeta) := \varphi(\zeta, t)$ is holomorphic in D, for every $t \in [0,1]$. The set $S_t := \varphi_t(\mathsf{D})$ is called an *analytic disk*, and $bS_t := \varphi_t(\partial\mathsf{D})$ its *boundary*.

Observe that in general bS_t is not the topological boundary of S_t.

> **Definition.** A domain $G \subset \mathbb{C}^n$ is said to satisfy the *continuity principle* if for any family $\{S_t, t \in [0,1]\}$ of analytic disks in $\mathbb{C}^n$ with $\bigcup_{0 \le t \le 1} bS_t \subset G$ and $S_0 \subset G$, it follows that $\bigcup_{0 \le t \le 1} S_t \subset G$.

Example

Let P^n be the unit polydisk and $\{S_t, t \in [0,1]\}$ a family of analytic disks in $\mathbb{C}^n$ with $\bigcup_{0 \le t \le 1} bS_t \subset \mathsf{P}^n$ and $S_0 \subset \mathsf{P}^n$. Because $\overline{S}_0$ and the union of all boundaries bS_t are compact sets, there is an $\varepsilon > 0$ such that

$$\bigcup_{0 \le t \le 1} bS_t \subset \mathsf{P}^n(\mathbf{0}, 1-\varepsilon) \quad \text{and} \quad S_0 \subset \mathsf{P}^n(\mathbf{0}, 1-\varepsilon).$$

We assume that $\bigcup_{0 \le t \le 1} S_t$ is not contained in P^n, and define

$$t_0 := \inf\{t \in [0,1] : S_t \not\subset \mathsf{P}^n\}.$$

It is clear that $t_0 > 0$, $S_{t_0} \not\subset \mathsf{P}^n$, and $S_t \subset \mathsf{P}^n$ for $0 \le t < t_0$. Then S_{t_0} contains a point $\mathbf{z}_0 = \big(z_1^{(0)}, \dots, z_n^{(0)}\big) \in \partial\mathsf{P}^n$. If the family of analytic disks is given by the map $\varphi : \overline{\mathsf{D}} \times [0,1] \to \mathbb{C}^n$, and w_μ denotes the μth coordinate function, then $f_{\mu,t}(\zeta) := w_\mu \circ \varphi(\zeta, t)$ is continuous on $\overline{\mathsf{D}}$ and holomorphic in D. Choosing μ such that $|z_\mu^{(0)}| = 1$, there is a $\zeta_0 \in \mathsf{D}$ with $f_{\mu,t_0}(\zeta_0) = z_\mu^{(0)}$ and $|f_{\mu,t_0}(\zeta_0)| = 1$. But by the maximum principle we have

$$|f_{\mu,t}(\zeta_0)| \le \sup_{\partial\mathsf{D}} |f_{\mu,t}| \le 1 - \varepsilon, \text{ for } t < t_0.$$

Since $t \mapsto f_{\mu,t}(\zeta_0)$ is continuous, a contradiction is reached, and therefore P^n satisfies the continuity principle.

Hartogs Convexity.

Definition. A domain $G \subset \mathbb{C}^n$ is called *Hartogs convex* if the following holds: If $(\widetilde{P}, \widetilde{H})$ is a general Hartogs figure with $\widetilde{H} \subset G$, then $\widetilde{P} \subset G$.

An immediate consequence of the definition is the following:

The biholomorphic image of a Hartogs convex domain is again Hartogs convex.

1.5 Theorem. *Let $G \subset \mathbb{C}^n$ be a domain that satisfies the continuity principle. Then G is Hartogs convex.*

Proof: Let $(\widetilde{P}, \widetilde{H})$ be a general Hartogs figure with $\widetilde{H} \subset G$. We assume that it is the biholomorphic image $(g(\mathsf{P}^n), g(\mathsf{H}))$ of a Euclidean Hartogs figure $(\mathsf{P}^n, \mathsf{H})$ with

$$\mathsf{H} = \{\mathbf{z} : |z_1| > q_1 \text{ or } |z_\mu| < q_\mu \text{ for } \mu = 2, \dots, n\}.$$

In order to define analytic disks we choose some r with $q_1 < r < 1$ and introduce the affine analytic disks

$$D_{\mathbf{w}} := \{\mathbf{z} = (z_1, \mathbf{z}'') \in \mathsf{P}^n = P' \times P'' : |z_1| < r \text{ and } \mathbf{z}'' = \mathbf{w}\}.$$

Since $\overline{D}_{\mathbf{w}} \subset \mathsf{P}^n$ for every $\mathbf{w} \in P''$, we can define $\varphi_{\mathbf{w}} : \overline{\mathsf{D}} \times [0,1] \to \mathbb{C}^n$ by setting $\varphi_{\mathbf{w}}(\zeta, t) := g(r\zeta, t\mathbf{w})$. Then a family $\{S_t(\mathbf{w}) : 0 \le t \le 1\}$ of analytic disks in $\widetilde{P}$ is given by

$$S_t(\mathbf{w}) := \varphi_{\mathbf{w}}(\mathsf{D} \times \{t\}) = g(D_{t\mathbf{w}}).$$

It follows that $bS_t(\mathbf{w}) \subset G$ for every $\mathbf{w} \in P''$ and every $t \in [0,1]$, and in addition, $S_0(\mathbf{w}) = g(D_{\mathbf{0}}) \subset G$. The situation is illustrated in Figure II.4.

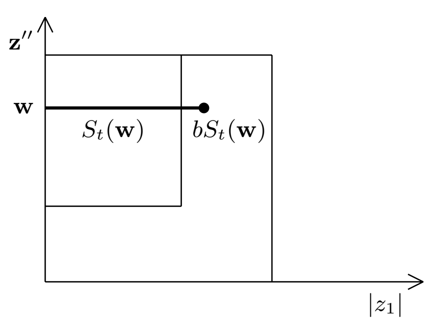


Figure II.4. Analytic disks in a Hartogs figure

Since G satisfies the continuity principle, we obtain that $g(D_{\mathbf{w}}) = S_1(\mathbf{w})$ is contained in G. This is valid for every $\mathbf{w} \in P''$. Therefore, $\widetilde{P} \subset G$, and G is Hartogs convex. ∎

1.6 Corollary. *The unit polydisk* P^n *is Hartogs convex.*

Domains of Holomorphy

> **Definition.** Let $G \subset \mathbb{C}^n$ be a domain, f holomorphic in G, and $\mathbf{z}_0 \in \partial G$ a point. The function f is called *completely singular* at $\mathbf{z}_0$ if for every connected neighborhood $U = U(\mathbf{z}_0) \subset \mathbb{C}^n$ and every connected component C of $U \cap G$ there is no holomorphic function g on U for which $g|_C = f|_C$.

Example

Let $G := \mathbb{C} - \{x \in \mathbb{R} : x \le 0\}$ and let f be a branch of the logarithm on G. Then f is completely singular at $z = 0$ but not at any point $x \in \mathbb{R}$ with $x < 0$.

> **Definition.** A domain $G \subset \mathbb{C}^n$ is called a *weak domain of holomorphy* if for every point $\mathbf{z} \in \partial G$ there is a function $f \in \mathcal{O}(G)$ that is completely singular at $\mathbf{z}$.
>
> The domain G is called a *domain of holomorphy* if there is a function $f \in \mathcal{O}(G)$ that is completely singular at **every** point $\mathbf{z} \in \partial G$.

Examples

1. Since $\mathbb{C}^n$ has no boundary point, it trivially satisfies the requirements of a domain of holomorphy.
2. It is easy to see that every domain $G \subset \mathbb{C}$ is a weak domain of holomorphy: If z_0 is a point in ∂G, then $f(z) := 1/(z - z_0)$ is holomorphic in G and completely singular at z_0.

   For $G = \mathsf{D}$ we can show even more! The function $f(z) := \sum_{\nu=0}^{\infty} z^{\nu!}$ is holomorphic in the unit disk and becomes completely singular at any boundary point. Therefore, D is a domain of holomorphy. At the end of this chapter we will see that every domain in $\mathbb{C}$ is a domain of holomorphy.
3. If $f : \mathsf{D} \to \mathbb{C}$ is a holomorphic function that becomes completely singular at every boundary point, then the same is true for $\widehat{f} : \mathsf{P}^n = \mathsf{D} \times \cdots \times \mathsf{D} \to \mathbb{C}$, defined by $\widehat{f}(z_1, \dots, z_n) := f(z_1) + \cdots + f(z_n)$. In fact, if $\mathbf{z}_0$ is a boundary point of P^n, then there exists an i such that the ith component $z_i^{(0)}$ is a boundary point of D. If $\widehat{f}$ could be extended holomorphically across $\mathbf{z}_0$,

then $\widehat{f}_i(\zeta) := \widehat{f}(z_1^{(0)}, \ldots, \zeta, \ldots, z_n^{(0)})$ would also have a holomorphic extension. But then f could not be completely singular at $z_i^{(0)}$. Therefore, the unit polydisk is a domain of holomorphy.

4. If $(\mathsf{P}^n, \mathsf{H})$ is a Euclidean Hartogs figure, then H is not a domain of holomorphy.

1.7 Proposition. *Let $G \subset \mathbb{C}^n$ be a domain. If for every point $\mathbf{z}_0 \in \partial G$ there is an open neighborhood $U = U(\mathbf{z}_0) \subset \mathbb{C}^n$ and a holomorphic function $f : G \cup U \to \mathbb{C}$ with $f(\mathbf{z}_0) = 0$ and $f(\mathbf{z}) \neq 0$ for $\mathbf{z} \in G$, then G is a weak domain of holomorphy.*

PROOF: We show that $1/f$ is completely singular at $\mathbf{z}_0$. For this assume that there is a connected open neighborhood $V = V(\mathbf{z}_0)$, a connected component $C \subset V \cap G$, and a holomorphic function F on V with $F|_C = (1/f)\ |_C$. The set $V' := V - N(f)$ is still connected and contains C. By the identity theorem the functions F and $1/f$ must coincide in V'. Then F is clearly not holomorphic at $\mathbf{z}_0$. This is a contradiction. ∎

1.8 Corollary. *Every convex domain in $\mathbb{C}^n$ is a weak domain of holomorphy.*

PROOF: If $\mathbf{z}_0 \in \partial G$, then because of the convexity there is a real linear form λ on $\mathbb{C}^n$ with $\lambda(\mathbf{z}) < \lambda(\mathbf{z}_0)$ for $\mathbf{z} \in G$. We can write λ in the form

$$\lambda(\mathbf{z}) = \sum_{\nu=1}^{n} \alpha_\nu z_\nu + \sum_{\nu=1}^{n} \overline{\alpha}_\nu \overline{z}_\nu, \text{ with } \boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \neq \mathbf{0}.$$

So $\lambda = \operatorname{Re} h(\mathbf{z})$, where $h(\mathbf{z}) := 2 \cdot \sum_{\nu=1}^{n} \alpha_\nu z_\nu$ is holomorphic on $\mathbb{C}^n$.

Since the function $f(\mathbf{z}) := h(\mathbf{z}) - h(\mathbf{z}_0)$ is holomorphic on $\mathbb{C}^n$, $f(\mathbf{z}_0) = 0$, and $f(\mathbf{z}) \neq 0$ on G, the proposition may be applied. ∎

We will show that every weak domain of holomorphy is Hartogs convex. As a tool we need the following simple geometric lemma, which will be useful in other situations as well.

1.9 Lemma (on boundary components). *Let $G \subset \mathbb{C}^n$ be a domain, $U \subset \mathbb{C}^n$ an open set with $U \cap G \neq \varnothing$ and $(\mathbb{C}^n - U) \cap G \neq \varnothing$.*

Then $G \cap \partial C \cap \partial U \neq \varnothing$ for any connected component C of $U \cap G$.

PROOF: We choose points $\mathbf{z}_1 \in C \subset U \cap G$ and $\mathbf{z}_2 \in (\mathbb{C}^n - U) \cap G$. There is a continuous path $\gamma : [0, 1] \to G$ with $\gamma(0) = \mathbf{z}_1$ and $\gamma(1) = \mathbf{z}_2$. Let $t_0 := \sup\{t \in [0, 1] : \gamma(t) \in C\}$ and $\mathbf{z}_0 := \gamma(t_0)$. Clearly, $\mathbf{z}_0 \in \partial C \cap G$, but $\mathbf{z}_0 \notin C$. Since C is a connected component of $U \cap G$, $\mathbf{z}_0$ cannot lie in $U \cap G$ and therefore even not in U. Since $\gamma(t) \in U$ for $t < t_0$, it follows that $\mathbf{z}_0 \in \partial U$. ∎

1.10 Theorem. *Let $G \subset \mathbb{C}^n$ be a weak domain of holomorphy. Then G is Hartogs convex.*

PROOF: Assume that G is not Hartogs convex. Then there is a general Hartogs figure (P, H) with $H \subset G$ but $P \cap G \neq P$. We choose an arbitrary $\mathbf{z}_0$ in H and set $C := C_{P \cap G}(\mathbf{z}_0)$.[1] Since H lies in $P \cap G$ and is connected, it follows that $H \subset C$. Furthermore, $C \subsetneqq P$.

Since $P \cap G \neq \varnothing$ and $(\mathbb{C}^n - G) \cap P \neq \varnothing$, by the lemma there is a point $\mathbf{z}_1 \in \partial C \cap \partial G \cap P$ (see Figure II.5).

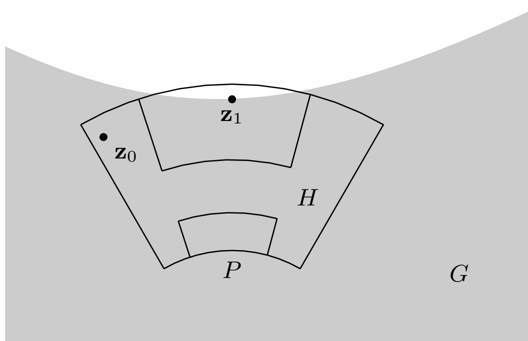


Figure II.5. G is not Hartogs convex

Let f be an arbitrary holomorphic function in G. Then $f|_C$ is also holomorphic, and by the continuity theorem it has a holomorphic extension F on P. Since P is an open connected neighborhood of $\mathbf{z}_1$, we obtain that f is not completely singular at $\mathbf{z}_1$. This completes the proof by contradiction. ∎

It follows, for example, that every convex domain is Hartogs convex. As a consequence, we see that every ball is Hartogs convex.

1.11 Theorem. *Every domain of holomorphy is Hartogs convex.*

The proof is trivial.

For the converse of this theorem one has to construct on any Hartogs convex domain a global holomorphic function that becomes completely singular at every boundary point, something that is rather difficult. It was done in 1910 by E.E. Levi in very special cases. The general case is called *Levi's problem.*

In 1942 K. Oka gave a proof for $n = 2$. At the beginning of the 1950s Oka, Bremermann, and Norguet solved Levi's problem for arbitrary n. It was gen-

[1] We denote by $C_M(\mathbf{z})$ the connected component of M containing $\mathbf{z}$.

eralized for complex manifolds (H. Grauert, 1958) and complex spaces (R. Narasimhan, 1962). Finally, in 1965 L. Hörmander published a proof that used Hilbert space methods and partial differential equations.

Exercises

1. Prove the following statements:
   (a) Finite intersections of Hartogs convex domains are Hartogs convex.
   (b) If $G_1 \subset G_2 \subset G_3 \subset \cdots$ is an ascending chain of Hartogs convex domains, then the union of all G_i is also Hartogs convex.
2. Let $G \subset \mathbb{C}^n$ be a domain, $0 \le r < R$, and $\mathbf{a} \in G$ a point. Let $U = U(\mathbf{a}) \subset G$ be an open neighborhood and define $Q := \{\mathbf{w} \in \mathbb{C}^m : r < |\mathbf{w}| < R\}$. Prove that every holomorphic function on $(G \times Q) \cup (U \times \mathsf{P}^m(\mathbf{0}, R))$ has a unique holomorphic extension to $G \times \mathsf{P}^m(\mathbf{0}, R)$.
3. Let $0 < r < R$ be given. Use Hartogs figures to prove that every holomorphic function on $\mathsf{B}_R(\mathbf{0}) - \overline{\mathsf{B}_r(\mathbf{0})}$ has a unique holomorphic extension to the whole ball $\mathsf{B}_R(\mathbf{0})$.
4. For $\varepsilon \ge 0$, consider the domain

   $$G_\varepsilon := \{(z, w) \in \mathsf{P}^2(\mathbf{0}, 1) \,:\, |z| < |w|^2 + \varepsilon\}.$$

   Prove that G_ε is Hartogs convex if and only if $\varepsilon = 0$.
5. Let $G \subset \mathbb{C}^n$ be a domain and $f : G \to \mathsf{D}_R(0) \subset \mathbb{C}$ a function, $\Gamma = \{(\mathbf{z}, w) \in G \times \mathsf{D}_R(0) \,:\, w = f(\mathbf{z})\}$ its graph. Sow that if there is a holomorphic function F in $G \times \mathsf{D}_R(0)$ that is completely singular at every point of Γ, then f is continuous. (With more effort one can show that f is holomorphic.)
6. Show that the "Hartogs triangle" $\{(z, w) \in \mathbb{C}^2 \,:\, |w| < |z| < 1\}$ is a weak domain of holomorphy.

2. Plurisubharmonic Functions

Subharmonic Functions. Recall some facts from complex analysis of one variable. A twice differentiable real-valued function h on a domain $G \subset \mathbb{C}$ is called *harmonic* if $h_{z\overline{z}}(z) \equiv 0$ on G. The real part of a holomorphic function is always harmonic, and on an open disk every harmonic function is the real part of some holomorphic function.

If $D = \mathsf{D}_r(a) \subset \mathbb{C}$ is an open disk and $\beta : \mathbb{R} \to \mathbb{R}$ a continuous periodic function with period 2π, then there is a continuous function $h : \overline{D} \to \mathbb{R}$ that is harmonic on D such that $h(re^{\mathrm{i}t}) = \beta(t)$ for every t (Dirichlet's principle).

An upper semicontinuous function $\varphi : G \to \mathbb{R} \cup \{-\infty\}$ is said to satisfy the *weak mean value property* if the following holds:

For every $a \in G$ there is an $r > 0$ with $\mathsf{D}_r(a) \subset\subset G$ and

$$\varphi(a) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+\varrho e^{\mathrm{i}t})\,dt \quad \text{for } 0<\varrho\le r.$$

Remarks

1. If $\varphi : G \to \mathbb{R}\cup\{-\infty\}$ is an upper semicontinuous function, then the sets $U_\nu := \{z \in G : \varphi(z) < \nu\}$ are open, and therefore φ is bounded from above on every compact subset $K \subset G$. It follows that the integral in the definition always exists.
2. Harmonic functions satisfy the weak mean value property (even the stronger *mean value property* with "=" instead of "≤").
3. If $f : G \to \mathbb{C}$ is a nowhere identically vanishing holomorphic function, then $\log|f|$ satisfies the weak mean value property. In fact, the function $\varphi := \log|f|$ is harmonic on $G - N(f)$, because it can be written locally as $\operatorname{Re}(\log f)$, with a suitable branch of the logarithm. And at any point $z_0 \in N(f)$ we have $\varphi(z_0) = -\infty$, so the inequality of the weak mean value property is satisfied.

2.1 Proposition. *Let $\varphi : G \to \mathbb{R}$ satisfy the weak mean value property. If φ has a global maximum in G, then φ is constant.*

PROOF: Let $a \in G$ be any point with $c := \varphi(a) \ge \varphi(z)$ for $z \in G$. We choose an $r > 0$ such that

$$\mathsf{D}_r(a) \subset\subset G \text{ and } \varphi(a) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+\varrho e^{\mathrm{i}t})\,dt \text{ for } 0<\varrho\le r.$$

Assume that there is a $b \in \mathsf{D}_r(a)$ with $\varphi(b) < \varphi(a)$. We write $b = a + \varrho e^{\mathrm{i}t_0}$ and get

$$\varphi(a) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+\varrho e^{\mathrm{i}t})\,dt < \frac{1}{2\pi}\int_0^{2\pi} \varphi(a)\,dt = \varphi(a).$$

This is a contradiction, so φ must be constant on $\mathsf{D}_r(a)$. Now we define the set $M := \{z \in G : \varphi(z) = c\}$. Obviously, M is closed in G and not empty, and we just showed that M is open. So $M = G$, and φ is constant. ∎

Definition. Let $G \subset \mathbb{C}$ be a domain. A function $s : G \to \mathbb{R}\cup\{-\infty\}$ is called *subharmonic* if the following hold:

1. s is upper semicontinuous on G.
2. If $D \subset\subset G$ is a disk, $h : \overline{D} \to \mathbb{R}$ continuous, $h|_D$ harmonic, and $h \ge s$ on ∂D, then $h \ge s$ on D.

2.2 Proposition. *Let $s_\nu : G \to \mathbb{R}\cup\{-\infty\}$ be a monotonically decreasing sequence of subharmonic functions. Then $s := \lim_{\nu\to\infty} s_\nu$ is subharmonic.*

PROOF: The limit $s = \lim_{\nu\to\infty} s_\nu = \inf\{s_\nu\}$ is upper semicontinuous. Let $D \subset\subset G$ be a disk, $h : \overline{D} \to \mathbb{R}$ continuous and harmonic on D, with $s \le h$ on ∂D. For fixed ε we consider the compact sets

$$K_\nu := \{z \in \partial D \,:\, s_\nu(z) \ge h(z) + \varepsilon\}.$$

Then $K_{\nu+1} \subset K_\nu$ and $\bigcap_{\nu=1}^{\infty} K_\nu = \varnothing$. Therefore, there is a $\nu_0 \in \mathbb{N}$ with $K_\nu = \varnothing$ for $\nu \ge \nu_0$. This means that for $\nu \ge \nu_0$, $s_\nu < h + \varepsilon$ on ∂D, and therefore the same is true on D. Since the s_ν are decreasing, $s < h + \varepsilon$ on D. This holds for every $\varepsilon > 0$, and consequently $s \le h$ on D. ∎

2.3 Proposition. *Let $(s_\alpha)_{\alpha\in A}$ be a family of subharmonic functions on G. If $s := \sup s_\alpha$ ist upper semicontinuous and finite everywhere, then s is subharmonic.*

PROOF: If $s \le h$ on ∂D, where $D \subset\subset G$ and $h : \overline{D} \to \mathbb{R}$ is continuous and harmonic on D, then $s_\alpha \le h$ on ∂D for every $\alpha \in A$. Since the s_α are subharmonic, it follows that $s_\alpha \le h$ on D for every $\alpha \in A$. But then $s \le h$ on D as well. ∎

Examples

1. Clearly, every harmonic function is subharmonic.
2. Let $s : G \to \mathbb{R}$ be a continuous subharmonic function such that $-s$ is also subharmonic. Then s is harmonic. To show this, we look at an arbitrary point $a \in G$ and choose an $r > 0$ such that $D := \mathsf{D}_r(a) \subset\subset G$. Then there is a continuous function $h : \overline{D} \to \mathbb{R}$ with $h|_{\partial D} = s|_{\partial D}$ that is harmonic on D (Dirichlet's principle). It follows that $s \le h$ on D. But because $-h$ is also harmonic, we have $-s \le -h$ on D as well. Together this gives $s = h$ on D.
3. Let $f : G \to \mathbb{C}$ be a holomorphic function. Then $s := \log|f|$ is subharmonic. In fact, if $f(z) \equiv 0$ on G, then we have $s(z) \equiv -\infty$, and there is nothing to prove. Otherwise, s is harmonic on $G - N(f)$, and we have only to look at an isolated zero a of f. We choose $D = \mathsf{D}_r(a) \subset\subset G$ and a function h that is continuous on $\overline{D}$ and harmonic on D, with $s \le h$ on ∂D. We know that s, and therefore also $s - h$, has the weak mean value property on D, and it is certainly not constant. So it must take its maximum on the boundary ∂D. This means that $s \le h$ on D.
4. Let $G \subset \mathbb{C}$ be an arbitrary domain. The *boundary distance* $\delta_G : G \to \mathbb{R}_+ \cup \{+\infty\}$ is defined by

$$\delta_G(z) := \sup\{r \in \mathbb{R} \,:\, \mathsf{D}_r(z) \subset G\}.$$

Claim: $s := -\log \delta_G$ is subharmonic on G.

PROOF: If $G = \mathbb{C}$, then $s(z) \equiv -\infty$ and there is nothing to prove. If $G \ne \mathbb{C}$, then s is real-valued and continuous. For $w \in \partial G$ we define

$s_w : G \to \mathbb{R}$ by setting $s_w(z) := -\log|z - w|$. Then $s(z) = \sup\{s_w(z) : w \in \partial G\}$. By Proposition 2.3 the claim follows. ∎

The Maximum Principle

2.4 Theorem. *Let $s : G \to \mathbb{R} \cup \{-\infty\}$ be a subharmonic function on a domain $G \subset \mathbb{C}$. If s takes its maximum on G, then it must be constant.*

PROOF: Assume that $c := s(a) \geq s(z)$ for every $z \in G$. As in the case of functions that have the weak mean value property it suffices to show that s is constant in a neighborhood of a. If this is not the case, there is a small disk $D = \mathsf{D}_r(a) \subset\subset G$ and $b \in \partial D$ with $s(a) > s(b)$. Since s is upper semicontinuous, there is a continuous function h on ∂D with $s \leq h \leq c$ and $h(b) < c$. Solving Dirichlet's problem we can construct a harmonic continuation of h on D. Now

$$h(a) = \frac{1}{2\pi} \int_0^{2\pi} h(a + re^{\mathrm{i}t})\,dt < c = s(a).$$

This is a contradiction. ∎

For later use we give the following criterion for a function to be subharmonic:

2.5 Theorem. *Let $G \subset \mathbb{C}$ be a domain and $s : G \to \mathbb{R} \cup \{-\infty\}$ an upper semicontinuous function. Suppose that for every disk $D \subset\subset G$ and every function $f \in \mathcal{O}(\overline{D})$ with $s < \mathrm{Re}(f)$ on ∂D it follows that $s < \mathrm{Re}(f)$ on D. Then s is subharmonic.*

PROOF: Let $D = \mathsf{D}_r(a) \subset\subset G$, $h : \overline{D} \to \mathbb{R}$ continuous and harmonic on D, and $s \leq h$ on ∂D. For simplicity we assume $a = 0$.

For $\nu \in \mathbb{N}$, a harmonic function h_ν on $D_\nu := \mathsf{D}_{(\nu/(\nu-1))r}(0) \supset D$ is given by

$$h_\nu(z) := h\Big(\Big(1 - \frac{1}{\nu}\Big)z\Big).$$

Then (h_ν) converges on $\overline{D}$ uniformly, increasing monotonically to h. Furthermore, for every ν there is a holomorphic function f_ν on D_ν with $\mathrm{Re}(f_\nu) = h_\nu$.

Let $\varepsilon > 0$ be given. Then there is a ν_0 such that $|h - h_\nu| < \varepsilon$ on $\overline{D}$ for $\nu \geq \nu_0$. Therefore, $s < h_\nu + \varepsilon = \mathrm{Re}(f_\nu + \varepsilon)$ on ∂D for $\nu \geq \nu_0$. By definition it follows that $s < h_\nu + \varepsilon$ on D. Since (h_ν) is increasing, it follows that $s < h + \varepsilon$ and therefore $s \leq h$ on D. ∎

Differentiable Subharmonic Functions

2.6 Lemma. *Let $s : G \to \mathbb{R}$ be a $\mathscr{C}^2$ function such that $s_{z\overline{z}} > 0$ on G. Then s is subharmonic.*

PROOF: Let $D = \mathsf{D}_r(a) \subset\subset G$ and let a continuous function $h : \overline{D} \to \mathbb{R}$ be given such that h is harmonic on D and $s \le h$ on ∂D. We define $\varphi := s - h$.

Assume that φ takes its maximum at some interior point z_0 of D. Then we look at the Taylor expansion of φ at z_0 in a small neighborhood about z_0:

$$\varphi(z_0 + z) = \varphi(z_0) + 2\,\mathrm{Re}\,Q(z) + \varphi_{z\overline{z}}(z_0)z\overline{z} + R(z),$$

where $Q(z) := \varphi_z(z_0)z + \frac{1}{2}\varphi_{zz}(z_0)z^2$ is holomorphic and $R(z)/|z|^2 \to 0$ for $z \to 0$. The function $\psi(z) := 2\,\mathrm{Re}\,Q(z)$ is harmonic, with $\psi(0) = 0$. Since it cannot assume a maximum or a minimum, it must have zeros arbitrarily close to but not equal to 0. On the other hand, $\varphi(z_0 + z) - \varphi(z_0) \le 0$ and $\varphi_{z\overline{z}}(z_0)z\overline{z} > 0$ outside $z = 0$. This is a contradiction. Thus φ must assume its maximum on the boundary of D, and $s \le h$ on D. ∎

2.7 Theorem. *Let $s : G \to \mathbb{R}$ be a $\mathscr{C}^2$ function. Then s is subharmonic if and only if $s_{z\overline{z}} \ge 0$ on G.*

PROOF: (a) Let $s_{z\overline{z}}(z) \ge 0$ for every $z \in G$. Then we define s_ν on G by setting $s_\nu := s + (1/\nu)z\overline{z}$. Obviously, $(s_\nu)_{z\overline{z}} = s_{z\overline{z}} + (1/\nu) > 0$. Then s_ν is subharmonic by the above lemma. Since (s_ν) converges, monotonically decreasing, to s, it follows that s is subharmonic.

(b) Let s be subharmonic on G. We assume that $s_{z\overline{z}}(a) < 0$ for some $a \in G$. Then there is a connected open neighborhood $U = U(a) \subset G$ such that $s_{z\overline{z}} < 0$ on U. By the lemma it follows that $-s$ is subharmonic on U. Then s must be harmonic on U. So $s_{z\overline{z}}(a) = 0$, contrary to assumption. ∎

Plurisubharmonic Functions. We return to the study of domains in arbitrary dimensions. Let $G \subset \mathbb{C}^n$ be a domain and $(\mathbf{a}, \mathbf{w})$ a tangent vector at $\mathbf{a} \in G$. We use the holomorphic mapping $\alpha_{\mathbf{a},\mathbf{w}} : \mathbb{C} \to \mathbb{C}^n$ defined by $\alpha_{\mathbf{a},\mathbf{w}}(\zeta) := \mathbf{a} + \zeta\mathbf{w}$.

> **Definition.** Let $G \subset \mathbb{C}^n$ be a domain. An upper semicontinuous function $p : G \to \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic* on G if for every tangent vector $(\mathbf{a}, \mathbf{w})$ in G the function
>
> $$p_{\mathbf{a},\mathbf{w}}(\zeta) := p \circ \alpha_{\mathbf{a},\mathbf{w}}(\zeta) = p(\mathbf{a} + \zeta\mathbf{w})$$
>
> is subharmonic on the connected component $G(\mathbf{a}, \mathbf{w})$ of the set $\alpha_{\mathbf{a},\mathbf{w}}^{-1}(G) \subset \mathbb{C}$ containing 0.

Remarks

1. Plurisubharmonicity is a local property.

2. If $f \in \mathcal{O}(G)$, then $\log|f|$ is plurisubharmonic.
3. If p_1, p_2 are plurisubharmonic, then so is $p_1 + p_2$.
4. If p is plurisubharmonic and $c > 0$, then $c \cdot p$ is plurisubharmonic.
5. If (p_ν) is a monotonically decreasing sequence of plurisubharmonic functions, then $p := \lim_{\nu\to\infty} p_\nu$ is also plurisubharmonic.
6. Let $(p_\alpha)_{\alpha\in A}$ be a family of plurisubharmonic functions. If $p := \sup(p_\alpha)$ is upper semicontinuous and finite, then it is also plurisubharmonic.
7. If a plurisubharmonic function p takes its maximum at a point of the domain G, then p is constant on G.

The Levi Form

Definition. Let $U \subset \mathbb{C}^n$ be an open set, $f \in \mathscr{C}^2(U;\mathbb{R})$, and $\mathbf{a} \in U$. The quadratic form[2] $\mathrm{Lev}(f) : T_\mathbf{a} \to \mathbb{R}$ with

$$\mathrm{Lev}(f)(\mathbf{a},\mathbf{w}) := \sum_{\nu,\mu} f_{z_\nu \overline{z}_\mu}(\mathbf{a}) w_\nu \overline{w}_\mu$$

is called the *Levi form* of f at $\mathbf{a}$.

Obviously, $\mathrm{Lev}(f)$ is linear in f.

Examples

1. In the case $n = 1$ we have $\mathrm{Lev}(s)(a,w) = s_{z\overline{z}}(a)w\overline{w}$. So s is subharmonic if and only if $\mathrm{Lev}(s)(a,w) \geq 0$ for every $a \in G$ and $w \in \mathbb{C}$.
2. Let $f(\mathbf{z}) := \|\mathbf{z}\|^2 = \sum_{i=1}^n z_i \overline{z}_i$. Then $\mathrm{Lev}(f)(\mathbf{a},\mathbf{w}) = \|\mathbf{w}\|^2$ for every $\mathbf{a}$.
3. If $f \in \mathscr{C}^2(U;\mathbb{R})$ and $\varrho : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, then
$$\mathrm{Lev}(\varrho \circ f)(\mathbf{a},\mathbf{w}) = \varrho''(f(\mathbf{a})) \cdot |(\partial f)_\mathbf{a}(\mathbf{w})|^2 + \varrho'(f(\mathbf{a})) \cdot \mathrm{Lev}(f)(\mathbf{a},\mathbf{w}).$$
4. If $\mathbf{F} : U \to V \subset \mathbb{C}^m$ is a holomorphic map and $g \in \mathscr{C}^2(V;\mathbb{R})$, then
$$\mathrm{Lev}(g \circ \mathbf{F})(\mathbf{a},\mathbf{w}) = \mathrm{Lev}(g)(\mathbf{F}(\mathbf{a}), \mathbf{F}'(\mathbf{a})(\mathbf{w})).$$
5. For $f \in \mathscr{C}^2(U;\mathbb{R})$ the Taylor expansion at $\mathbf{a} \in U$ gives
$$f(\mathbf{z}) = f(\mathbf{a}) + 2\,\mathrm{Re}(Q_f(\mathbf{z}-\mathbf{a})) + \mathrm{Lev}(f)(\mathbf{a},\mathbf{z}-\mathbf{a}) + R(\mathbf{z}-\mathbf{a}),$$
where $Q_f(\mathbf{w}) = \sum_{\nu=1}^n f_{z_\nu}(\mathbf{a})w_\nu + \frac{1}{2}\sum_{\nu,\mu} f_{z_\nu z_\mu}(\mathbf{a})w_\nu w_\mu$ is a holomorphic quadratic polynomial, and
$$\lim_{\mathbf{z}\to\mathbf{a}} \frac{R(\mathbf{z}-\mathbf{a})}{\|\mathbf{z}-\mathbf{a}\|^2} = 0.$$

[2] If $H : T \times T \to \mathbb{C}$ is a Hermitian form on a complex vextor space, then the associated *quadratic form* $Q : V \to \mathbb{R}$ is given by $Q(v) := H(v,v)$.

2.8 Theorem. *A function $f \in \mathscr{C}^2(U;\mathbb{R})$ is plurisubharmonic if and only if* $\text{Lev}(f)(\mathbf{a},\mathbf{w}) \geq 0$ *for every* $\mathbf{a} \in U$ *and every* $\mathbf{w} \in T_{\mathbf{a}}$.

PROOF: Let $(\mathbf{a},\mathbf{w})$ be a tangent vector in G and $\alpha := \alpha_{\mathbf{a},\mathbf{w}}$. Then $f \circ \alpha(0) = f(\mathbf{a})$ and

$$(f \circ \alpha)_{\zeta\overline{\zeta}}(0) = \text{Lev}(f \circ \alpha)(0,1) = \text{Lev}(f)(\mathbf{a},\mathbf{w}).$$

Now, f is plurisubharmonic if and only if $f \circ \alpha$ is subharmonic near 0 for any $\alpha = \alpha_{\mathbf{a},\mathbf{w}}$. Equivalently, $(f \circ \alpha)_{\zeta\overline{\zeta}}(0) \geq 0$ for any such α. But this is true if and only if $\text{Lev}(f)(\mathbf{a},\mathbf{w}) \geq 0$ for any tangent vector $(\mathbf{a},\mathbf{w})$ in G. ∎

2.9 Corollary. *Let $G_1 \subset \mathbb{C}^n$ and $G_2 \subset \mathbb{C}^m$ be domains, $\mathbf{F} : G_1 \to G_2$ a holomorphic map, and $g \in \mathscr{C}^2(G_1;\mathbb{R})$ plurisubharmonic. Then $g \circ \mathbf{F}$ is plurisubharmonic on G_1.*

PROOF: This is trivial, because of the formula in Example 4 above. ∎

Exhaustion Functions

For every domain $G \subset \mathbb{C}$ the function $-\log \delta_G$ is subharmonic. In higher dimensions it is in general not true that this function is plurisubharmonic for every domain G.

> **Definition.** Let $G \subset \mathbb{C}^n$ be a domain. A nonconstant continuous function $f : G \to \mathbb{R}$ is called an *exhaustion function* for G if for $c < \sup_G(f)$ all sublevel sets
>
> $$G_c(f) := \{\mathbf{z} \in G \,:\, f(\mathbf{z}) < c\}$$
>
> are relatively compact in G.

Example

For $G = \mathbb{C}^n$, the function $f(\mathbf{z}) := \|\mathbf{z}\|^2$ is an exhaustion function. For $G \neq \mathbb{C}^n$, we define the *boundary distance* δ_G by

$$\delta_G(\mathbf{z}) := \text{dist}(\mathbf{z}, \mathbb{C}^n - G).$$

Then $-\delta_G$ is a bounded, and $-\log \delta_G$ an unbounded, exhaustion function. We only have to show that δ_G is continuous:

For every point $\mathbf{z} \in G$ there is a point $\mathbf{r}(\mathbf{z}) \in \mathbb{C}^n - G$ such that

$$\delta_G(\mathbf{z}) = \text{dist}(\mathbf{z}, \mathbf{r}(\mathbf{z})) \leq \text{dist}(\mathbf{z}, \mathbf{w}) \text{ for every } \mathbf{w} \in \mathbb{C}^n - G.$$

Then for two arbitrary points $\mathbf{u}, \mathbf{v} \in G$ we have

$$\begin{aligned}\delta_G(\mathbf{u}) \,=\, \|\mathbf{u}-\mathbf{r}(\mathbf{u})\| &\le \|\mathbf{u}-\mathbf{r}(\mathbf{v})\| \le \|\mathbf{u}-\mathbf{v}\| + \delta_G(\mathbf{v}),\\ \text{and in the same way} \quad \delta_G(\mathbf{v}) &\le \|\mathbf{u}-\mathbf{v}\| + \delta_G(\mathbf{u}).\end{aligned}$$

Therefore, $|\delta_G(\mathbf{u}) - \delta_G(\mathbf{v})| \le \|\mathbf{u}-\mathbf{v}\|$.

Definition. A function $f \in \mathscr{C}^2(G;\mathbb{R})$ is called *strictly plurisubharmonic* if $\mathrm{Lev}(f)(\mathbf{a},\mathbf{w}) > 0$ for $\mathbf{a} \in G$, $\mathbf{w} \in T_{\mathbf{a}}$, and $\mathbf{w} \neq \mathbf{0}$.

For a proof of the following result we refer to [Ra86], Chapter II, Proposition 4.14.

2.10 Smoothing lemma. *Let $G \subset \mathbb{C}^n$ be a domain, $f : G \to \mathbb{R}$ a continuous plurisubharmonic exhaustion function, $K \subset G$ compact, and $\varepsilon > 0$. Then there exists a $\mathscr{C}^\infty$ exhaustion function $g : G \to \mathbb{R}$ such that:*

1. *$g \ge f$ on G.*
2. *g is strictly plurisubharmonic.*
3. *$|g(\mathbf{z}) - f(\mathbf{z})| < \varepsilon$ on K.*

Exercises

1. Let $G \subset \mathbb{C}$ be a domain. Prove the following statements:
   (a) If $f : G \to \mathbb{C}$ is a holomorphic function, then $|f|^\alpha$ is subharmonic for $\alpha > 0$.
   (b) If u is subharmonic on G, then u^p is subharmonic for $p \in \mathbb{N}$.
   (c) Let $u \not\equiv -\infty$ be subharmonic on G. Then $\{z \in G \,:\, u(z) = -\infty\}$ does not contain any open subset.
2. Let $G \subset \mathbb{C}$ be a domain, $s \not\equiv -\infty$ a subharmonic function on G, $P := \{z \in G \,:\, s(z) = -\infty\}$. Show that if u is a continuous function on G and subharmonic on $G - A$, then u is subharmonic on G.
3. Let $U \subset \mathbb{C}^n$ be open, $\mathbf{f} : U \to \mathbb{C}^k$ a holomorphic map, and $\mathbf{A} \in M_k(\mathbb{R})$ a positive semidefinite matrix. Show that $\varphi(\mathbf{z}) := \mathbf{f}(\mathbf{z}) \cdot \mathbf{A} \cdot \mathbf{f}(\mathbf{z})^{\,t}$ is plurisubharmonic.
4. Let $G = \{(z,w) \in \mathbb{C}^2 \,:\, |w| < |z| < 1\}$ be the Hartogs triangle. Prove that there does not exist any bounded plurisubharmonic exhaustion function on G.
5. Are the following functions plurisubharmonic (respectively strictly plurisubharmonic)?

$$\begin{aligned} p_1(\mathbf{z}) &:= \log(1+\|\mathbf{z}\|^2), \text{ for } \mathbf{z} \in \mathbb{C}^n,\\ p_2(\mathbf{z}) &:= -\log(1-\|\mathbf{z}\|^2), \text{ for } \|\mathbf{z}\| < 1,\\ p_3(\mathbf{z}) &:= \|\mathbf{z}\|^2 e^{-\operatorname{Re}(z_n)}, \text{ for } \mathbf{z} \in \mathbb{C}^n.\end{aligned}$$

6. Consider a domain $G \subset \mathbb{C}^n$ and a function $f \in \mathscr{C}^2(G)$. Prove that f is strictly plurisubharmonic if and only if for every open set $U \subset\subset G$ there is an $\varepsilon > 0$ such that $f(\mathbf{z}) - \varepsilon\|\mathbf{z}\|^2$ is plurisubharmonic on U.

3. Pseudoconvexity

Pseudoconvexity

Definition. A domain $G \subset \mathbb{C}^n$ is called *pseudoconvex* if there is a strictly plurisubharmonic $\mathscr{C}^\infty$ exhaustion function on G.

Remarks

1. By the smoothing lemma the following is clear: If $-\log \delta_G$ is plurisubharmonic, then G is pseudoconvex.
2. Pseudoconvexity is invariant under biholomorphic transformations.

3.1 Theorem. *If $G \subset \mathbb{C}^n$ is a pseudoconvex domain, then G satisfies the continuity principle.*

PROOF: Let $p : G \to \mathbb{R}$ be a strictly plurisubharmonic exhaustion function. Suppose that there exists a family $\{S_t : 0 \le t \le 1\}$ of analytic disks given by a continuous mapping $\varphi : \overline{\mathsf{D}} \times [0,1] \to \mathbb{C}^n$ such that $S_0 \subset G$ and $bS_t \subset G$ for every $t \in [0,1]$, but not all S_t are contained in G.

The functions $p \circ \varphi_t : \mathsf{D} \to G$ are subharmonic for every t with $S_t \subset G$. It follows by the maximum principle that $p|S_t \le \max_{bS_t} p$ for all those t.

We define $t_0 := \inf\{t \in [0,1] : S_t \not\subset G\}$. Then $t_0 > 0$, $S_{t_0} \subset \overline{G}$, and S_{t_0} meets ∂G in at least one point $\mathbf{z}_0$. We can find an increasing sequence (t_ν) converging to t_0 and a sequence of points $\mathbf{z}_\nu \in S_{t_\nu}$ converging to $\mathbf{z}_0$. So $p(\mathbf{z}_\nu) \to c_0 := \sup_G(p)$, but there is a $c < c_0$ such that $p|_{bS_t} \le c$ for every $t \in [0,1]$. This is a contradiction. ∎

3.2 Corollary. *If G is pseudoconvex, then G is Hartogs convex.*

The Boundary Distance

3.3 Theorem. *If $G \subset \mathbb{C}^n$ is a Hartogs convex domain, then $-\log \delta_G$ is plurisubharmonic on G.*

PROOF: For $\mathbf{z} \in G$ and $\mathbf{u} \in \mathbb{C}^n$ with $\|\mathbf{u}\| = 1$ we define

$$\delta_{G,\mathbf{u}}(\mathbf{z}) := \sup\{t > 0 : \mathbf{z} + \tau\mathbf{u} \in G \text{ for } |\tau| \le t\}.$$

Then $\delta_G(\mathbf{z}) = \inf\{\delta_{G,\mathbf{u}}(\mathbf{z}) : \|\mathbf{u}\| = 1\}$, and it is sufficient to show that $-\log \delta_{G,\mathbf{u}}$ is plurisubharmonic for fixed $\mathbf{u}$.

(a) Unfortunately, $\delta_{G,\mathbf{u}}$ does not need to be continuous, but it is lower semicontinuous:

Let $\mathbf{z}_0 \in G$ be an arbitrary point and $c < \delta_{G,\mathbf{u}}(\mathbf{z}_0)$. Then the compact set $K := \{\mathbf{z} = \mathbf{z}_0 + \tau\mathbf{u} \,:\, |\tau| \le c\}$ is contained in G, and there is a $\delta > 0$ such that $\{\mathbf{z} \,:\, \operatorname{dist}(K, \mathbf{z}) < \delta\} \subset G$.

For $\mathbf{z} \in B_\delta(\mathbf{z}_0)$ and $|\tau| \le c$ we have

$$\|(\mathbf{z} + \tau\mathbf{u}) - (\mathbf{z}_0 + \tau\mathbf{u})\| = \|\mathbf{z} - \mathbf{z}_0\| < \delta, \text{ and therefore } \delta_{G,\mathbf{u}}(\mathbf{z}) \ge c.$$

(b) The function $-\log \delta_{G,\mathbf{u}}$ is upper semicontinuous, and we have to show that

$$s(\zeta) := -\log \delta_{G,\mathbf{u}}(\mathbf{z}_0 + \zeta\mathbf{b})$$

is subharmonic for fixed $\mathbf{u}, \mathbf{z}_0, \mathbf{b}$. First consider the case that $\mathbf{u}$ and $\mathbf{b}$ are linearly dependent: $\mathbf{b} = \lambda\mathbf{u}$, $\lambda \ne 0$.

Let G_0 be the connected component of 0 in $\{\zeta \in \mathbb{C} \,:\, \mathbf{z}_0 + \zeta\mathbf{b} \in G\}$. Then

$$\begin{aligned}
\delta_{G,\mathbf{u}}(\mathbf{z}_0 + \zeta\mathbf{b}) &= \sup\{t > 0 \,:\, \mathbf{z}_0 + \zeta\mathbf{b} + \tau\mathbf{u} \in G \text{ for } |\tau| \le t\} \\
&= \sup\{t > 0 \,:\, \zeta + \tau/\lambda \in G_0 \text{ for } |\tau| \le t\} \\
&= |\lambda| \cdot \sup\{r > 0 \,:\, \zeta + \sigma \in G_0 \text{ for } |\sigma| \le r\} \\
&= |\lambda| \cdot \delta_{G_0}(\zeta),
\end{aligned}$$

and this function is in fact subharmonic.

(c) Now assume that $\mathbf{u}$ and $\mathbf{b}$ are linearly independent. Since these vectors are fixed, we can restrict ourselves to the following special situation:

$$n = 2, \quad \mathbf{z}_0 = \mathbf{0}, \quad \mathbf{b} = \mathbf{e}_1, \quad \text{and} \quad \mathbf{u} = \mathbf{e}_2.$$

Then $s(\zeta) = -\log \sup\{t > 0 \,:\, (\zeta, \tau) \in G \text{ for } |\tau| \le t\}$. We use holomorphic functions to show that s is subharmonic. Let $R > r > 0$ be real numbers such that $(\zeta, 0) \in G$ for $|\zeta| < R$, and let $f : \mathsf{D}_R(0) \to \mathbb{C}$ be a holomorphic function such that $s < h := \operatorname{Re} f$ on $\partial\mathsf{D}_r(0)$. We have to show that $s < h$ on $\mathsf{D}_r(0)$.

We have the following equivalences:

$$\begin{aligned}
s(\zeta) < h(\zeta) \quad &\iff \quad \sup\{t > 0 \,:\, (\zeta, \tau) \in G \text{ for } |\tau| \le t\} > e^{-h(\zeta)} \\
&\iff \quad \big(\zeta, c \cdot e^{-f(\zeta)}\big) \in G \text{ for } c \in \overline{\mathsf{D}}.
\end{aligned}$$

(d) Define a holomorphic map $\mathbf{F}$ by

$$\mathbf{F}(z_1, z_2) := \big(rz_1, z_2 e^{-f(rz_1)}\big).$$

Then $\mathbf{F}$ is well defined on a neighborhood of the unit polydisk $\mathsf{P}^2 = \mathsf{P}^2(\mathbf{0}, 1)$. It must be shown that $\mathbf{F}(\mathsf{P}^2) \subset G$. We already know the following:

1. $\mathbf{F}(z_1, z_2) \in G$ for $|z_1| = 1$ and $|z_2| \le 1$, because $s(t) < h(t)$ on $\partial\mathsf{D}_r(0)$.
2. $\mathbf{F}(z_1, 0) \in G$ for $|z_1| \le 1$, because $(\zeta, 0) \in G$ for $|\zeta| \le r$.

These facts will be used to construct an appropriate Hartogs figure. First, note that

$$J_{\mathbf{F}}(z_1, z_2) = \begin{pmatrix} r & 0 \\ * & e^{-f(rz_1)} \end{pmatrix}, \quad \text{so } \det J_{\mathbf{F}}(z_1, z_2) \neq 0.$$

By the inverse function theorem it follows that $\mathbf{F}$ is biholomorphic.

For $0 < \delta < 1$ we define $\mathbf{h}_\delta : \mathbb{C}^2 \to \mathbb{C}^2$ by $\mathbf{h}_\delta(z_1, z_2) := (z_1, \delta z_2)$ and apply $\mathbf{h}_\delta$ to the compact set

$$C := \{(z_1, z_2) \in \mathbb{C}^2 : (|z_1| \le 1,\ z_2 = 0) \text{ or } (|z_1| = 1,\ |z_2| \le 1)\} \subset \overline{\mathsf{P}^2}\,.$$

Consequently,

$$C_\delta := \mathbf{h}_\delta(C) = \{(z_1, z_2) \in \mathbb{C}^2 : (|z_1| \le 1,\ z_2 = 0) \text{ or } (|z_1| = 1,\ |z_2| \le \delta)\}.$$

Then $\mathbf{F}(C_\delta) \subset G$, as we saw above, and therefore $C_\delta \subset \mathbf{F}^{-1}(G)$.

For $0 < \varepsilon < \min(\delta, 1 - \delta)$ we define a neighborhood U_ε of C_δ by $U_\varepsilon :=$

$\{(z_1, z_2) \in \mathbb{C}^2 : (|z_1| < 1+\varepsilon,\ |z_2| < \varepsilon) \text{ or } (1-\varepsilon < |z_1| < 1+\varepsilon,\ |z_2| < \delta+\varepsilon)\}$.

If we choose ε small enough, then $U_\varepsilon \subset \mathbf{F}^{-1}(G)$.

Finally, we define $\mathsf{H}_\varepsilon := \mathbf{h}_\delta^{-1}(U_\varepsilon \cap \mathsf{P}^2) \cap \mathsf{P}^2$ (see Figure II.6). Then

$$\begin{aligned} \mathsf{H}_\varepsilon &= \{(z_1, z_2) \in \mathsf{P}^2 : (z_1, \delta z_2) \in U_\varepsilon \cap \mathsf{P}^2\} \\ &= \Big\{(z_1, z_2) \in \mathbb{C}^2 : \big(|z_1| < 1,\ |z_2| < \frac{\varepsilon}{\delta}\big) \text{ or } (1 - \varepsilon < |z_1| < 1,\ |z_2| < 1)\Big\}. \end{aligned}$$

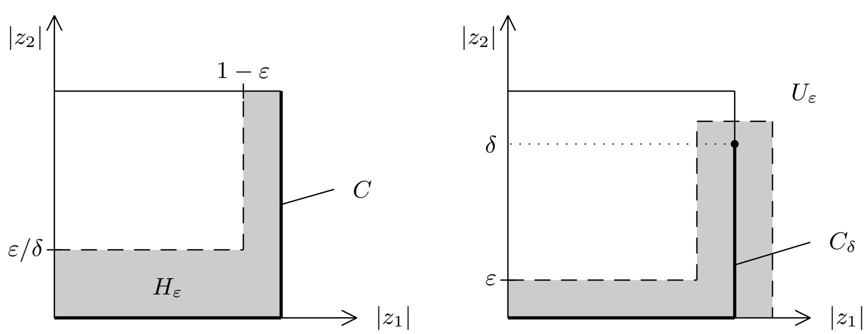


Figure II.6. Construction of the Hartogs figure

Since $(\mathsf{P}^2, \mathsf{H}_\varepsilon)$ is a Euclidean Hartogs figure, $(\mathbf{F} \circ \mathbf{h}_\delta(\mathsf{P}^2), \mathbf{F} \circ \mathbf{h}_\delta(\mathsf{H}_\varepsilon))$ is a general Hartogs figure with $\mathbf{F} \circ \mathbf{h}_\delta(\mathsf{H}_\varepsilon) \subset \mathbf{F}(U_\varepsilon \cap \mathsf{P}^2) \subset G$. Since G is Hartogs pseudoconvex, it follows that $\mathbf{F} \circ h_\delta(\mathsf{P}^2) \subset G$. This is valid for every $\delta < 1$. But $\mathsf{P}^2 = \bigcup_{0<\delta<1} \mathbf{h}_\delta(\mathsf{P}^2)$. Therefore, $\mathbf{F}(\mathsf{P}^2) \subset G$, which was to be shown. ∎

3.4 Theorem. *The following properties of a domain $G \subset \mathbb{C}^n$ are equivalent:*

1. *G satisfies the continuity principle.*
2. *G is Hartogs pseudoconvex.*
3. *$-\log \delta_G$ is plurisubharmonic on G.*
4. *G is pseudoconvex.*

PROOF:

(1) $\Longrightarrow$ (2) is Theorem 1.5,
(2) $\Longrightarrow$ (3) is Theorem 3.3,
(3) $\Longrightarrow$ (4) follows from the smoothing lemma,
(4) $\Longrightarrow$ (1) was proved in Theorem 3.1. ∎

Properties of Pseudoconvex Domains

3.5 Theorem. *If $G_1, G_2 \subset \mathbb{C}^n$ are pseudoconvex domains, then $G_1 \cap G_2$ is pseudoconvex.*

PROOF: The statement is trivial if one uses Hartogs pseudoconvexity. ∎

3.6 Theorem. *Let $G_1 \subset G_2 \subset \ldots \subset \mathbb{C}^n$ be an ascending chain of pseudoconvex domains. Then $G := \bigcup_{\nu=1}^{\infty} G_\nu$ is again pseudoconvex.*

PROOF: This follows immediately from the continuity principle. ∎

3.7 Theorem. *A domain $G \subset \mathbb{C}^n$ is pseudoconvex if and only if there is an open covering $(U_\iota)_{\iota \in I}$ of $\overline{G}$ such that $U_\iota \cap G$ is pseudoconvex for every $\iota \in I$.*

PROOF:

"$\Longrightarrow$" is trivial. The other direction will be proved in two steps. At first, we assume that G is bounded.

For any point $\mathbf{z}_0 \in \partial G$ there is an open set U_ι such that $\mathbf{z}_0 \in U_\iota$ and $G \cap U_\iota$ is pseudoconvex. If we choose a neighborhood $W = W(\mathbf{z}_0) \subset U_\iota$ so small that $\operatorname{dist}(\mathbf{z}, \partial U_\iota) > \operatorname{dist}(\mathbf{z}, \mathbf{z}_0)$ for every $\mathbf{z} \in W \cap G$, then $\delta_G(\mathbf{z}) = \delta_{G \cap U_\iota}(\mathbf{z})$ on $W \cap G$. This shows that there is an open neighborhood $U = U(\partial G)$ such that $-\log \delta_G$ is plurisubharmonic on $U \cap G$ (we use the fact that ∂G is compact). Now, $G - U \subset\subset G$. We define

$$c := \sup\{-\log \delta_G(\mathbf{z}) \,:\, \mathbf{z} \in G - U\},$$

and

$$p(\mathbf{z}) := \max\bigl(-\log \delta_G(\mathbf{z}), \|z\|^2 + c + 1\bigr).$$

Then p is a plurisubharmonic exhaustion function, and by the smoothing lemma, G is pseudoconvex.

If G is unbounded, we write it as an ascending union of the domains $G_\nu := \mathsf{B}_\nu(\mathbf{0}) \cap G$. Each G_ν is bounded and satisfies the hypothesis, so is pseudoconvex. Then G is also a pseudoconvex domain. ∎

Exercises

1. Suppose that $G_1 \subset \mathbb{C}^n$ and $G_2 \subset \mathbb{C}^m$ are domains.
   (a) Show that if G_1 and G_2 are pseudoconvex, then $G_1 \times G_2$ is a pseudoconvex domain in $\mathbb{C}^{n+m}$
   (b) Show that if there is a proper holomorphic map $\mathbf{f} : G_1 \to G_2$ and G_2 is pseudoconvex, then G_1 is also pseudoconvex.
2. Let $G \subset \mathbb{C}^n$ be a domain and $\varrho : G \to \mathbb{R}$ a lower semicontinuous positive function. Prove that

$$\widehat{G} := \{(\mathbf{z}', w) \in G \times \mathbb{C} \,:\, |w| < \varrho(\mathbf{z}')\}$$

   is pseudoconvex if and only if $-\log \varrho$ is plurisubharmonic.
3. A domain $G \subset \mathbb{C}^n$ is pseudoconvex if and only if for every compact set $K \subset G$ the set

$$\widehat{K}_{\text{pl}} := \Bigl\{\mathbf{z} \in G \,:\, p(\mathbf{z}) \le \sup_K p \text{ for all plurisubharmonic functions } p \text{ on } G\Bigr\}$$

   is relatively compact in G.

4. Levi Convex Boundaries

Boundary Functions

Definition. Let $G \subset \mathbb{C}^n$ be a domain. The boundary of G is called *smooth* at $\mathbf{z}_0 \in \partial G$ if there is an open neighborhood $U = U(\mathbf{z}_0) \subset \mathbb{C}^n$ and a function $\varrho \in \mathscr{C}^\infty(U; \mathbb{R})$ such that:
1. $U \cap G = \{\mathbf{z} \in U \,:\, \varrho(\mathbf{z}) < 0\}$.
2. $(d\varrho)_{\mathbf{z}} \ne 0$ for $\mathbf{z} \in U$.

The function ϱ is called a local *defining function* (or *boundary function*).

Remark. Without loss of generality we may assume that $\varrho_{y_n} \ne 0$. Then by the implicit function theorem there are neighborhoods

$$U' \text{ of } (\mathbf{z}_0', x_n^{(0)}) = \big(z_1^{(0)}, \ldots, z_{n-1}^{(0)}, x_n^{(0)}\big) \in \mathbb{C}^{n-1} \times \mathbb{R}, \quad U'' \text{ of } y_n^{(0)} \in \mathbb{R},$$

and a $\mathscr{C}^\infty$ function $\gamma : U' \to U''$ such that $\{(\mathbf{z}', x_n, y_n) \in U' \times U'' \,:\, \varrho(\mathbf{z}', x_n + \mathrm{i}y_n) = 0\} = \{(\mathbf{z}', x_n, \gamma(\mathbf{z}', x_n)) \,:\, (\mathbf{z}', x_n) \in U'\}$.

Making the neighborhood $U := \{(\mathbf{z}', x_n + \mathrm{i}y_n) \,:\, (\mathbf{z}', x_n) \in U' \text{ and } y_n \in U''\}$ small enough and correcting the sign if necessary, one can achieve that

$$U \cap G = \{(\mathbf{z}', x_n + \mathrm{i}y_n) \in U \,:\, y_n < \gamma(\mathbf{z}', x_n)\}.$$

In particular, $U \cap \partial G = \{\mathbf{z} \in U \,:\, \varrho(\mathbf{z}) = 0\}$ is a $(2n-1)$-dimensional differentiable submanifold of U.

4.1 Lemma. *Let ∂G be smooth at $\mathbf{z}_0$, and let ϱ_1, ϱ_2 be two local defining functions on $U = U(\mathbf{z}_0)$. Then there is a $\mathscr{C}^\infty$ function h on U such that:*

1. *$h > 0$ on U.*
2. *$\varrho_1 = h \cdot \varrho_2$ on U.*
3. *$(d\varrho_1)_{\mathbf{z}} = h(\mathbf{z}) \cdot (d\varrho_2)_{\mathbf{z}}$ for $\mathbf{z} \in U \cap \partial G$.*

Proof: Define $h := \varrho_1/\varrho_2$ on $U - \partial G$. After a change of coordinates, we have $\mathbf{z}_0 = \mathbf{0}$ and $\varrho_2 = y_n$. Then $g(t) := \varrho_1(\mathbf{z}', x_n + \mathrm{i}t)$ is a smooth function that vanishes at $t = 0$. Therefore,

$$\begin{aligned} \varrho_1(\mathbf{z}', z_n) &= g(y_n) - g(0) \\ &= \int_0^{y_n} g'(s)\,ds \;=\; y_n \cdot \int_0^1 g'(ty_n)\,dt \\ &= \varrho_2(\mathbf{z}', x_n + \mathrm{i}y_n) \cdot h(\mathbf{z}', z_n), \end{aligned}$$

where

$$h(\mathbf{z}', x_n + \mathrm{i}y_n) = \int_0^1 \frac{\partial \varrho_1}{\partial y_n}(\mathbf{z}', x_n + \mathrm{i}ty_n)\,dt$$

is smooth.

For $\mathbf{z} \in \partial G$ we have $(d\varrho_1)_{\mathbf{z}} = h(\mathbf{z}) \cdot (d\varrho_2)_{\mathbf{z}}$. Therefore, $h(\mathbf{z}) \neq 0$, and even greater than 0, since $h(\mathbf{z}) \geq 0$ by continuity. ∎

4.2 Theorem. *Let $G \subset\subset \mathbb{C}^n$ be a bounded domain with smooth boundary. Then ∂G is a differentiable submanifold, and there exists a global defining function.*

Proof: We can find open sets $V_i \subset\subset U_i \subset \mathbb{C}^n$, $i = 1, \ldots, N$, such that:

1. $\{V_1, \ldots, V_N\}$ is an open covering of ∂G.
2. For each i there exists a local defining function ϱ_i for G on U_i.

3. For each i there is a smooth function $\varphi_i : U_i \to \mathbb{R}$ with $\varphi_i|_{V_i} \equiv 1$, $\varphi_i|_{\mathbb{C}^n - U_i} \equiv 0$, and $\varphi_i \geq 0$ in general.

Define $\varphi := \sum_i \varphi_i$ (so $\varphi > 0$ on ∂G) and $\psi_i := \varphi_i/\varphi$. Then $\sum_i \psi_i \equiv 1$ on ∂G. The system of the functions ψ_i is called a *partition of unity* on ∂G.

The function $\varrho := \sum_{i=1}^N \psi_i \varrho_i$ is now a global defining function for G. We leave it to the reader to check the details. ∎

The Levi Condition. For the remainder of this section let $G \subset\subset \mathbb{C}^n$ be a bounded domain with smooth boundary, and $\varrho : U = U(\partial G) \to \mathbb{R}$ a global defining function. Then at any $\mathbf{z}_0 \in \partial G$ the real tangent space of the boundary

$$T_{\mathbf{z}_0}(\partial G) := \{\mathbf{v} \in T_{\mathbf{z}_0} \,:\, (d\varrho)_{\mathbf{z}_0}(\mathbf{v}) = 0\}$$

is a $(2n-1)$-dimensional real subspace of $T_{\mathbf{z}_0}$. The space

$$H_{\mathbf{z}_0}(\partial G) := T_{\mathbf{z}_0}(\partial G) \cap \mathrm{i}T_{\mathbf{z}_0}(\partial G) = \{\mathbf{v} \in T_{\mathbf{z}_0} \,:\, (\partial\varrho)_{\mathbf{z}_0}(\mathbf{v}) = 0\}$$

is called the *complex* (or *holomorphic*) *tangent space* of the boundary at $\mathbf{z}_0$. It is a $(2n-2)$-dimensional real subspace of $T_{\mathbf{z}_0}$, with a natural complex structure, so an $(n-1)$-dimensional complex subspace[3].

Definition. The domain G is said to satisfy the *Levi condition* (respectively the *strict Levi condition*) at $\mathbf{z}_0 \in \partial G$ if $\mathrm{Lev}(\varrho)$ is positive semidefinite (respectively positive definite) on $H_{\mathbf{z}_0}(\partial G)$. The domain G is called *Levi convex* (respectively *strictly Levi convex*) if G satisfies the Levi condition (respectively the strict Levi condition) at every point $\mathbf{z} \in \partial G$.

Remark. The Levi conditions do not depend on the choice of the boundary function, and they are invariant under biholomorphic transformations.

If $\varrho_1 = h \cdot \varrho_2$, with $h > 0$, then for $\mathbf{z} \in \partial G$,

$$\mathrm{Lev}(\varrho_1)(\mathbf{z}, \mathbf{w}) = h(\mathbf{z}) \cdot \mathrm{Lev}(\varrho_2)(\mathbf{z}, \mathbf{w}) + 2\,\mathrm{Re}\{(\overline{\partial}h)_{\mathbf{z}}(\mathbf{w}) \cdot (\partial\varrho_2)_{\mathbf{z}}(\mathbf{w})\}.$$

So on $H_{\mathbf{z}}(\partial G)$ the Levi forms of ϱ_1 and ϱ_2 differ only by a positive constant.

Affine Convexity. Recall some facts from real analysis:

A set $M \subset \mathbb{R}^n$ is *convex* if for every two points $\mathbf{x}, \mathbf{y} \in M$, the closed line segment from $\mathbf{x}$ to $\mathbf{y}$ is contained in M. In that case, for each point $\mathbf{x}_0 \in \mathbb{R}^n - M$ there is a real hyperplane $H \subset \mathbb{R}^n$ with $\mathbf{x}_0 \in H$ and $M \cap H = \varnothing$. This property was already used in Section 1.

[3] $H_{\mathbf{z}}(\partial G)$ is often denoted by $T^{1,0}_{\mathbf{z}}(\partial G)$.

If $\mathbf{a} \in \mathbb{R}^n$, $U = U(\mathbf{a})$ is an open neighborhood and $\varphi : U \to \mathbb{R}$ is at least $\mathscr{C}^2$, then the quadratic form

$$\mathrm{Hess}(\varphi)(\mathbf{a}, \mathbf{w}) := \sum_{\nu,\mu} \varphi_{x_\nu x_\mu}(\mathbf{a}) w_\nu w_\mu$$

is known as the *Hessian* of φ at $\mathbf{a}$.

4.3 Proposition. *Let $G \subset\subset \mathbb{R}^n$ be a domain with smooth boundary, and ϱ a global defining function with $(d\varrho)_{\mathbf{x}} \neq 0$ for $\mathbf{x} \in \partial G$. Then G is convex if and only if* $\mathrm{Hess}(\varrho)$ *is positive semidefinite on every tangent space $T_{\mathbf{x}}(\partial G)$.*

PROOF: Let G be convex, and $\mathbf{x}_0 \in \partial G$ an arbitrary point. Then $T := T_{\mathbf{x}_0}(\partial G)$ is a real hyperplane with $T \cap G = \varnothing$. For $\mathbf{w} \in T$ and $\alpha(t) := \mathbf{x}_0 + t\mathbf{w}$ we have

$$(\varrho \circ \alpha)''(0) = \mathrm{Hess}(\varrho)(\mathbf{x}_0, \mathbf{w}).$$

Since $\varrho(\mathbf{x}_0) = 0$ and $\varrho \circ \alpha(t) \geq 0$, it follows that $\varrho \circ \alpha$ has a minimum at $t = 0$. Then $(\varrho \circ \alpha)''(0) \geq 0$, and $\mathrm{Hess}(\varrho)$ is positive semidefinite on T.

Now let the criterion be fulfilled, assume that $\mathbf{0} \in G$, and define ϱ_ε by

$$\varrho_\varepsilon(\mathbf{x}) := \varrho(\mathbf{x}) + \frac{\varepsilon}{N}\|\mathbf{x}\|^N.$$

For small ε and large N the set $G_\varepsilon := \{\mathbf{x} : \varrho_\varepsilon(\mathbf{x}) < 0\}$ is a domain. We have $G_\varepsilon \subset G_{\varepsilon'} \subset G$ for $\varepsilon' < \varepsilon$, and $\bigcup_{\varepsilon>0} G_\varepsilon = G$. Therefore, it is sufficient to show that G_ε is convex.

The Hessian of ϱ_ε is positive definite on $T_{\mathbf{x}}(\partial G)$ for every $\mathbf{x} \in \partial G$. Thus this also holds in a neighborhood U of ∂G. If ε is small enough, then $\partial G_\varepsilon \subset U$. We consider

$$S := \{(\mathbf{x}, \mathbf{y}) \in G_\varepsilon \times G_\varepsilon \,:\, t\mathbf{x} + (1-t)\mathbf{y} \in G_\varepsilon,\ \text{for } 0 < t < 1\}.$$

Then S is an open subset of the connected set $G_\varepsilon \times G_\varepsilon$. Suppose that S is not a closed subset. Then there exist points $\mathbf{x}_0, \mathbf{y}_0 \in G_\varepsilon$ and a $t_0 \in (0,1)$ with $t_0\mathbf{x}_0 + (1-t_0)\mathbf{y}_0 \in \partial G_\varepsilon$. So the function $t \mapsto \varrho_\varepsilon \circ \alpha(t)$, with $\alpha(t) := t\mathbf{x}_0 + (1-t)\mathbf{y}_0$, has a maximum at t_0. Then $(\varrho_\varepsilon \circ \alpha)''(t_0) \leq 0$ and $\mathrm{Hess}(\varrho_\varepsilon)(\alpha(t_0), \mathbf{x}_0 - \mathbf{y}_0) \leq 0$. This is a contradiction. ∎

A domain $G = \{\varrho < 0\}$ is called *strictly convex* at $\mathbf{x}_0 \in \partial G$ if $\mathrm{Hess}(\varrho)$ is positive definite at $\mathbf{x}_0$. This property is independent of ϱ and invariant under affine transformations.

Now we return to Levi convexity.

4.4 Lemma. *Let $U \subset \mathbb{C}^n$ be open and $\varphi \in \mathscr{C}^2(U;\mathbb{R})$. Then*

$$\mathrm{Lev}(\varphi)(\mathbf{z}, \mathbf{w}) = \frac{1}{4}\left(\mathrm{Hess}(\varphi)(\mathbf{z}, \mathbf{w}) + \mathrm{Hess}(\varphi)(\mathbf{z}, \mathrm{i}\mathbf{w})\right).$$

Proof: This is a simple calculation! ∎

4.5 Theorem. *Let $G \subset\subset \mathbb{C}^n$ be a domain with smooth boundary. Then the following statements are equivalent:*

1. *G is strictly Levi convex.*
2. *There is an open neighborhood $U = U(\partial G)$ and a strictly plurisubharmonic function $\varrho \in \mathscr{C}^\infty(U;\mathbb{R})$ such that $U \cap G = \{\mathbf{z} \in U \,:\, \varrho(\mathbf{z}) < 0\}$ and $(d\varrho)_\mathbf{z} \neq 0$ for $\mathbf{z} \in U$.*
3. *For every $\mathbf{z} \in \partial G$ there is an open neighborhood $W = W(\mathbf{z}) \subset \mathbb{C}^n$, an open set $V \subset \mathbb{C}^n$, and a biholomorphic map $\mathbf{F} : W \to V$ such that $\mathbf{F}(W \cap G)$ is convex and even strictly convex at every point of $\mathbf{F}(W \cap \partial G)$.*

Proof:

(1) $\Longrightarrow$ (2) : We choose a global defining function ϱ for G, and an open neighborhood $U = U(\partial G)$ such that ϱ is defined on U with $(d\varrho)_\mathbf{z} \neq 0$ for $\mathbf{z} \in U$. Let $A > 0$ be a real constant, and $\varrho_A := e^{A\varrho} - 1$. Then ϱ_A is also a global defining function, and

$$\operatorname{Lev}(\varrho_A)(\mathbf{z},\mathbf{w}) = Ae^{A\varrho(\mathbf{z})}\left[\operatorname{Lev}(\varrho)(\mathbf{z},\mathbf{w}) + A|(\partial\varrho)_\mathbf{z}(\mathbf{w})|^2\right].$$

The set $K := \partial G \times S^{2n-1}$ is compact, and

$$K_0 := \{(\mathbf{z},\mathbf{w}) \in K \,:\, \operatorname{Lev}(\varrho)(\mathbf{z},\mathbf{w}) \le 0\}$$

is a closed subset. Since $\operatorname{Lev}(\varrho)$ is positive definite on $H_\mathbf{z}(\partial G)$, we have $(\partial\varrho)_\mathbf{z}(\mathbf{w}) \neq 0$ for $(\mathbf{z},\mathbf{w}) \in K_0$. Therefore,

$$\begin{aligned} M &:= \min_K \operatorname{Lev}(\varrho)(\mathbf{z},\mathbf{w}) \;>\; -\infty, \\ C &:= \min_{K_0} |(\partial\varrho)_\mathbf{z}(\mathbf{w})|^2 \;>\; 0. \end{aligned}$$

We choose A so large that $A \cdot C + M > 0$. Then

$$\operatorname{Lev}(\varrho_A)(\mathbf{z},\mathbf{w}) = A \cdot [\operatorname{Lev}(\varrho)(\mathbf{z},\mathbf{w}) + A|(\partial\varrho)_\mathbf{z}(\mathbf{w})|^2] \ge A \cdot (M + AC) > 0$$

for $(\mathbf{z},\mathbf{w}) \in K_0$, and

$$\operatorname{Lev}(\varrho_A)(\mathbf{z},\mathbf{w}) > A^2 \cdot |(\partial\varrho)_\mathbf{z}(\mathbf{w})|^2 \ge 0$$

for $(\mathbf{z},\mathbf{w}) \in K - K_0$.

So $\operatorname{Lev}(\varrho_A)(\mathbf{z},\mathbf{w}) > 0$ for every $\mathbf{z} \in \partial G$ and every $\mathbf{w} \in \mathbb{C}^n - \{\mathbf{0}\}$. By continuity, ϱ_A is strictly plurisubharmonic in a neighborhood of ∂G.

(2) $\Longrightarrow$ (3) : We consider a point $\mathbf{z}_0 \in \partial G$ and make some simple coordinate transformations:

By the translation $\mathbf{z} \mapsto \mathbf{w} = \mathbf{z} - \mathbf{z}_0$ we replace $\mathbf{z}_0$ by the origin, and a permutation of coordinates ensures that $\varrho_{w_1}(\mathbf{0}) \neq 0$.

The linear transformation

$$\mathbf{w} \mapsto \mathbf{u} = \big(\varrho_{w_1}(\mathbf{0})w_1 + \cdots + \varrho_{w_n}(\mathbf{0})w_n, w_2, \ldots, w_n\big)$$

gives $u_1 = \mathbf{w} \cdot \nabla\varrho(\mathbf{0})^{\,t}$, and therefore

$$\begin{aligned}
\varrho(\mathbf{u}) &= 2\,\mathrm{Re}\,\big(\mathbf{u} \cdot \nabla(\varrho \circ \mathbf{w})(\mathbf{0})^{\,t}\big) + \text{ terms of degree } \geq 2 \\
&= 2\,\mathrm{Re}\,\big(\mathbf{u} \cdot J_{\mathbf{w}}(\mathbf{0})^{\,t} \cdot \nabla\varrho(\mathbf{0})^{\,t}\big) + \text{ terms of degree } \geq 2 \\
&= 2\,\mathrm{Re}\,\big(\mathbf{w} \cdot \nabla\varrho(\mathbf{0})^{\,t}\big) + \text{ terms of degree } \geq 2 \\
&= 2\,\mathrm{Re}(u_1) + \text{ terms of degree } \geq 2.
\end{aligned}$$

Finally, we write $\varrho(\mathbf{u}) = 2\,\mathrm{Re}(u_1 + Q(\mathbf{u})) + \mathrm{Lev}(\varrho)(\mathbf{0}, \mathbf{u}) + \cdots$, where Q is a quadratic holomorphic polynomial, and make the biholomorphic transformation

$$\mathbf{u} \mapsto \mathbf{v} = (u_1 + Q(\mathbf{u}), u_2, \ldots, u_n).$$

It follows that

$$\varrho(\mathbf{v}) = 2\,\mathrm{Re}(v_1) + \mathrm{Lev}(\varrho)(\mathbf{0}, \mathbf{v}) + \text{ terms of order } \geq 3.$$

By the uniqueness of the Taylor expansion

$$\varrho(\mathbf{v}) = D\varrho(\mathbf{0})(\mathbf{v}) + \frac{1}{2}\mathrm{Hess}(\varrho)(\mathbf{0}, \mathbf{v}) + \text{ terms of order } \geq 3\,,$$

and therefore $\mathrm{Hess}(\varrho)(\mathbf{0}, \mathbf{v}) = 2 \cdot \mathrm{Lev}(\varrho)(\mathbf{0}, \mathbf{v}) > 0$ for $\mathbf{v} \neq \mathbf{0}$ (in the new coordinates). Everything works in a neighborhood that may be chosen to be convex.

(3) $\Longrightarrow$ (1) : This follows from Lemma 4.4:

$$\mathrm{Hess}(\varrho) > 0 \text{ on } T_{\mathbf{z}}(\partial G) \implies \mathrm{Lev}(\varrho) > 0 \text{ on } H_{\mathbf{z}}(\partial G).$$

The latter property is invariant under biholomorphic transformations. ■

A Theorem of Levi. Let $G \subset\subset \mathbb{C}^n$ be a domain with smooth boundary. If G is strictly Levi convex, then it is easy to see that G is pseudoconvex. We wish to demonstrate that even the weaker Levi convexity is equivalent to pseudoconvexity. For that purpose we extend the boundary distance to a function on $\mathbb{C}^n$.

$$d_G(\mathbf{z}) := \begin{cases} \delta_G(\mathbf{z}) & \text{for } \mathbf{z} \in G, \\ 0 & \text{for } \mathbf{z} \in \partial G, \\ -\delta_{\mathbb{C}^n - \overline{G}}(\mathbf{z}) & \text{for } \mathbf{z} \notin \overline{G}. \end{cases}$$

4.6 Lemma. *$-d_G$ is a smooth defining function for G.*

PROOF: We use real coordinates $\mathbf{x} = (x_1, \dots, x_N)$ with $N = 2n$. It is clear that $G = \{\mathbf{x} : -d_G(\mathbf{x}) < 0\}$.

Let $\mathbf{x}_0 \in \partial G$ be an arbitrary point and $\varrho : U(\mathbf{x}_0) \to \mathbb{R}$ a local defining function. We may assume that $\varrho_{x_N}(\mathbf{x}_0) \neq 0$. Then by the implicit function theorem there is a product neighborhood $U' \times U''$ of $\mathbf{x}_0$ in U and a smooth function $h : U' \to \mathbb{R}$ such that

$$\{(\mathbf{x}', x_N) \in U' \times U'' : \varrho(\mathbf{x}', x_N) = 0\} = \{(\mathbf{x}', h(\mathbf{x}')) : \mathbf{x}' \in U'\}.$$

It follows that $\mathbf{0} = \nabla_{\mathbf{x}'}\varrho(\mathbf{x}', h(\mathbf{x}')) + \varrho_{x_N}(\mathbf{x}', h(\mathbf{x}')) \cdot \nabla h(\mathbf{x}')$.

At the point $(\mathbf{x}', h(\mathbf{x}')) \in \partial G$ the gradient $\nabla\varrho(\mathbf{x}', h(\mathbf{x}'))$ is normal to ∂G and directed outward from G. Every point $\mathbf{y}$ in a small neighborhood of the boundary has a unique representation $\mathbf{y} = \mathbf{x} + t \cdot \nabla\varrho(\mathbf{x})$, where $t = -d_G(\mathbf{y})$ and $\mathbf{x}$ is the point where the perpendicular from $\mathbf{y}$ to ∂G meets the boundary. Therefore, we define the smooth map $\mathbf{F} : U' \times \mathbb{R} \to \mathbb{R}^N$ by

$$\mathbf{y} = \mathbf{F}(\mathbf{x}', t) := (\mathbf{x}', h(\mathbf{x}')) + t \cdot \nabla\varrho(\mathbf{x}', h(\mathbf{x}')).$$

Then there are smooth functions $\mathbf{A}$ and $\mathbf{b}$ such that

$$J_{\mathbb{R},\mathbf{F}}(\mathbf{x}', t) = \begin{pmatrix} \mathbf{E}_{N-1} + t \cdot \mathbf{A}(\mathbf{x}') & \nabla_{\mathbf{x}'}\varrho(\mathbf{x}', h(\mathbf{x}'))^{\,t} \\ \nabla h(\mathbf{x}') + t \cdot \mathbf{b}(\mathbf{x}') & \varrho_{x_N}(\mathbf{x}', h(\mathbf{x}')) \end{pmatrix},$$

and therefore

$$\begin{aligned} \det J_{\mathbb{R},\mathbf{F}}(\mathbf{x}', 0) &= \det \begin{pmatrix} \mathbf{E}_{N-1} & -\varrho_{x_N}(\mathbf{x}', h(\mathbf{x}')) \cdot \nabla h(\mathbf{x}')^{\,t} \\ \nabla h(\mathbf{x}') & \varrho_{x_N}(\mathbf{x}', h(\mathbf{x}')) \end{pmatrix} \\ &= \varrho_{x_N}(\mathbf{x}', h(\mathbf{x}')) \cdot \det \begin{pmatrix} \mathbf{E}_{N-1} & -\nabla h(\mathbf{x}')^{\,t} \\ \mathbf{0}' & 1 + \|\nabla h(\mathbf{x}')\|^2 \end{pmatrix} \\ &= \varrho_{x_N}(\mathbf{x}', h(\mathbf{x}'))(1 + \|\nabla h(\mathbf{x}')\|^2) \neq 0. \end{aligned}$$

It follows that there exists an $\varepsilon > 0$ such that $\mathbf{F}$ maps $U' \times (-\varepsilon, \varepsilon)$ diffeomorphically onto a neighborhood $W = W(\mathbf{x}_0)$, and $U' \times \{0\}$ onto $\partial G \cap W$. Moreover, since $d_G(\mathbf{x} + t \cdot \nabla\varrho(\mathbf{x})) = -t$ for $|t| < \varepsilon$ and ε small enough, it follows that $d_G = (-t) \circ \mathbf{F}^{-1}$ is a smooth function near ∂G. If $\mathbf{p}'$ is defined by $\mathbf{p}'(\mathbf{x}', t) := (\mathbf{x}', 0)$, then the projection

$$\mathbf{p} = \mathbf{p}' \circ \mathbf{F}^{-1} : \mathbf{x} + t \cdot \nabla\varrho(\mathbf{x}) \mapsto \mathbf{x}, \text{ for } \mathbf{x} \in \partial G,$$

is a smooth map, and d_G is given by $d_G(\mathbf{y}) = \sigma \cdot \|\mathbf{y} - \mathbf{p}(\mathbf{y})\|$, where $\sigma = 1$ for $\mathbf{y} \in G$ and $\sigma = -1$ elsewhere.

For $\mathbf{y} \notin \partial G$ we have

$$\begin{aligned}(d_G)_{y_\nu}(\mathbf{y}) &= \frac{\sigma}{\|\mathbf{y}-\mathbf{p}(\mathbf{y})\|}\cdot\sum_{k=1}^{N}(y_k - p_k(\mathbf{y}))(\delta_{k\nu}-(p_k)_{y_\nu}(\mathbf{y}))\\ &= \frac{\sigma}{\|\mathbf{y}-\mathbf{p}(\mathbf{y})\|}\cdot\left[y_\nu - p_\nu(\mathbf{y}) - \big(\mathbf{y}-\mathbf{p}(\mathbf{y})\,|\,\mathbf{p}_{y_\nu}(\mathbf{y})\big)_N\right],\end{aligned}$$

and therefore

$$\nabla d_G(\mathbf{y}) = \frac{\sigma}{\|\mathbf{y}-\mathbf{p}(\mathbf{y})\|}\cdot[\mathbf{y}-\mathbf{p}(\mathbf{y}) - D\mathbf{p}(\mathbf{y})(\mathbf{y}-\mathbf{p}(\mathbf{y}))].$$

Since $\varrho(\mathbf{p}(\mathbf{y}))\equiv 0$, it follows that $D\mathbf{p}(\mathbf{y})(\nabla\varrho(\mathbf{p}(\mathbf{y}))) = 0$. But $\mathbf{y}-\mathbf{p}(\mathbf{y})$ is a multiple of $\nabla\varrho(\mathbf{p}(\mathbf{y}))$. Together this gives

$$\nabla d_G(\mathbf{y}) = \sigma\cdot\frac{\mathbf{y}-\mathbf{p}(\mathbf{y})}{\|\mathbf{y}-\mathbf{p}(\mathbf{y})\|} = \pm\frac{\nabla\varrho(\mathbf{p}(\mathbf{y}))}{\|\nabla\varrho(\mathbf{p}(\mathbf{y}))\|}.$$

If $\mathbf{y}$ tends to ∂G, we obtain that $\nabla d_G(\mathbf{y}) \ne \mathbf{0}$. ∎

E.E. Levi showed that every domain of holomorphy with smooth boundary is Levi convex, and locally the boundary of a strictly Levi convex domain G is the "natural boundary" for some holomorphic function in G. Here we prove the following result, which is sometimes called "Levi's theorem".

4.7 Theorem. *A domain G with smooth boundary is pseudoconvex if and only if it is Levi convex.*

PROOF:

(1) Let G be pseudoconvex. The function $-d_G$ is a smooth boundary function for G, and $-\log d_G = -\log\delta_G$ is plurisubharmonic on G, because of the pseudoconvexity. We calculate

$$\operatorname{Lev}(-\log d_G)(\mathbf{z},\mathbf{w}) = \frac{1}{d_G(\mathbf{z})}\cdot\operatorname{Lev}(-d_G)(\mathbf{z},\mathbf{w}) + \frac{1}{d_G(\mathbf{z})^2}\cdot|(\partial(d_G))_{\mathbf{z}}(\mathbf{w})|^2.$$

This is nonnegative in G. If $\mathbf{z}\in G$, $\mathbf{w}\in T_{\mathbf{z}}$, and $(\partial(d_G))_{\mathbf{z}}(\mathbf{w}) = 0$, it follows that $\operatorname{Lev}(-d_G)(\mathbf{z},\mathbf{w}) \ge 0$. This remains true for $\mathbf{z}\to\partial G$, so $-d_G$ satisfies the Levi condition.

(2) Let G be Levi convex, and suppose that G is not pseudoconvex. Then in any neighborhood U of the boundary there exists a point $\mathbf{z}_0$ where the Levi form of $-\log\delta_G$ has a negative eigenvalue. This means that there is a vector $\mathbf{w}_0$ such that

$$\varphi_{\zeta\overline{\zeta}}(0) = \operatorname{Lev}(\log\delta_G)(\mathbf{z}_0,\mathbf{w}_0) > 0,\ \text{ for } \varphi(\zeta) := \log\delta_G(\mathbf{z}_0+\zeta\mathbf{w}_0).$$

Consider the Taylor expansion

$$\begin{aligned}\varphi(\zeta) &= \varphi(0) + 2\operatorname{Re}(\varphi_\zeta(0)\zeta + \frac{1}{2}\varphi_{\zeta\zeta}(0)\zeta^2) + \varphi_{\zeta\overline{\zeta}}(0)|\zeta|^2 + \cdots\\ &= \varphi(0) + \operatorname{Re}(A\zeta + B\zeta^2) + \lambda|\zeta|^2 + \cdots,\end{aligned}$$

with complex constants A, B and a real constant $\lambda > 0$.

We choose a point $\mathbf{p}_0 \in \partial G$ with $\delta_G(\mathbf{z}_0) = \|\mathbf{p}_0 - \mathbf{z}_0\|$, and an arbitrary $\varepsilon > 0$. Then an analytic disk $\psi : \mathsf{D}_\varepsilon(0) \to \mathbb{C}^n$ can be defined by

$$\psi(\zeta) := \mathbf{z}_0 + \zeta\mathbf{w}_0 + \exp(A\zeta + B\zeta^2)(\mathbf{p}_0 - \mathbf{z}_0).$$

We have $\psi(0) = \mathbf{p}_0$, and we wish to show that $\psi(\zeta) \in G$, for $0 < |\zeta| < \varepsilon$ and ε sufficiently small.

Since $\varphi(\zeta) \ge \varphi(0) + \operatorname{Re}(A\zeta + B\zeta^2) + (\lambda/2)|\zeta|^2$ near $\zeta = 0$, it follows that

$$\begin{aligned} \delta_G(\mathbf{z}_0 + \zeta\mathbf{w}_0) &= \exp(\varphi(\zeta)) \\ &\ge \exp(\varphi(0)) \cdot \left|\exp\left(A\zeta + B\zeta^2\right)\right| \cdot \exp\Big(\frac{\lambda}{2}|\zeta|^2\Big) \\ &> \delta_G(\mathbf{z}_0) \cdot \left|\exp\left(A\zeta + B\zeta^2\right)\right| \\ &= \left\|\exp\left(A\zeta + B\zeta^2\right)(\mathbf{p}_0 - \mathbf{z}_0)\right\|, \end{aligned}$$

for ζ small and $\neq 0$. This means that we can choose the ε in such a way that $\psi(\zeta) \in G$, for $0 < |\zeta| < \varepsilon$. The analytic disc is tangent to ∂G from the interior of G.

Now $f(\zeta) = d_G(\psi(\zeta))$ is a smooth function with a local minimum at $\zeta = 0$. Therefore $(\partial d_G)_{\mathbf{p}_0}(\psi'(0)) = (\partial f)_0(1) = 0$, and

$$f(\zeta) = \operatorname{Re}\left(f_{\zeta\zeta}(0)\zeta^2\right) + f_{\zeta\overline{\zeta}}|\zeta|^2 + \text{terms of order } \ge 3.$$

Since $\operatorname{Re}\left(f_{\zeta\zeta}(0)e^{2it}\right) + f_{\zeta\overline{\zeta}} \ge 0$ for every t, it follows that

$$\operatorname{Lev}(d_G)(\mathbf{p}_0, \psi'(0)) = f_{\zeta\overline{\zeta}}(0) > 0.$$

This is a contradiction to the Levi condition at $\mathbf{p}_0$, because $-d_G$ is a defining function for G. ∎

Exercises

1. Prove Lemma 4.4.
2. Assume that $G \subset\subset \mathbb{C}^2$ has a smooth boundary that is Levi convex outside a point $\mathbf{a}$ that is not isolated in ∂G. Show that G is pseudoconvex.
3. Assume that $G \subset \mathbb{C}^2$ is an arbitrary domain and that $S \subset G$ is a smooth real surface with the following property: In every point of S the tangent to S is not a complex line. Prove that for every compact set $K \subset G$ there are arbitrarily small pseudoconvex neighborhoods of $S \cap K$.
4. Assume that $G \subset\subset \mathbb{C}^2$ is a domain with smooth boundary. Then G is strictly Levi convex at a point $\mathbf{z}_0 \in \partial G$ if and only if the following condition is satisfied:

   There is a neighborhood $U = U(\mathbf{z}_0)$, a holomorphic function $\varphi : \mathsf{D} \to U$ with $\varphi(0) = \mathbf{z}_0$ and $\varphi'(0) \neq 0$, and a local defining function ϱ on U such that $(\varrho \circ \varphi)(\zeta) > 0$ on $\mathsf{D} - \{0\}$ and $(\varrho \circ \varphi)_{\zeta\overline{\zeta}}(0) > 0$.

5. Let $G \subset\subset \mathbb{C}^n$ be a domain with smooth boundary. If G satisfies the strict Levi condition at $\mathbf{z}_0 \in \partial G$, then prove that the following hold:
   (a) There is no analytic disk $\varphi : \mathsf{D} \to \mathbb{C}^n$ with
   $$\varphi(0) = \mathbf{z}_0 \quad \text{and} \quad \lim_{\zeta \to 0} \frac{\delta_G(\varphi(\zeta))}{\|\varphi(\zeta) - \varphi(0)\|^2} = 0\,.$$
   (b) There are a neighborhood $U = U(\mathbf{z}_0)$ and a holomorphic function f in U with $\overline{G} \cap \{\mathbf{z} \in U \,:\, f(\mathbf{z}) = 0\} = \{\mathbf{z}_0\}$.
6. A bounded domain $G \subset \mathbb{C}^n$ is called *strongly pseudoconvex* if there are a neighborhood $U = U(\partial G)$ and a strictly plurisubharmonic function $\varrho \in \mathscr{C}^2(U)$ such that $G \cap U = \{\mathbf{z} \in U \,:\, \varrho(\mathbf{z}) < 0\}$. Notice that a strongly pseudoconvex domain does not necessarily have a smooth boundary!

   Prove the following results about a strongly pseudoconvex bounded domain G:
   (a) G is pseudoconvex.
   (b) If G has a smooth boundary, then G is strictly Levi convex.
   (c) For every $\mathbf{z} \in \partial G$ there is a neighborhood $U = U(\mathbf{z})$ such that $U \cap G$ is a weak domain of holomorphy.
7. Let $G \subset \mathbb{C}^n$ be a pseudoconvex domain. Then prove that there is a family of domains $G_\nu \subset G$ such that the following hold:
   (a) $G_\nu \subset\subset G_{\nu+1}$ for every ν.
   (b) $\bigcup_{\nu=1}^{\infty} G_\nu = G$.
   (c) For every ν there is a strictly plurisubharmonic function $f_\nu \in \mathscr{C}^\infty(G_{\nu+1})$ such that G_ν is a connected component of the set
   $$\{\mathbf{z} \in G_{\nu+1} \,:\, f_\nu(\mathbf{z}) < 0\}\,.$$

5. Holomorphic Convexity

Affine Convexity We will investigate relationships between pseudoconvexity and affine convexity. Let us begin with some observations about convex domains in $\mathbb{R}^N$.

Let $\mathscr{L}$ be the set of affine linear functions $f : \mathbb{R}^N \to \mathbb{R}$ with
$$f(\mathbf{x}) = a_1 x_1 + \cdots + a_N x_N + b, \quad a_1, \ldots, a_N, b \in \mathbb{R}.$$

If M is a convex set and $\mathbf{x}_0$ a point not contained in M, then there exists a function $f \in \mathscr{L}$ with $f(\mathbf{x}_0) = 0$ and $f|_M < 0$. For any $c \in \mathbb{R}$, the set $\{\mathbf{x} \in \mathbb{R}^N \,:\, f(\mathbf{x}) < c\}$ is a convex half-space.

Definition. Let $M \subset \mathbb{R}^N$ be an arbitrary subset. Then the set
$$H(M) := \Big\{\mathbf{x} \in \mathbb{R}^N \,:\, f(\mathbf{x}) \le \sup_M f, \text{ for all } f \in \mathscr{L}\Big\}$$
is called the *affine convex hull* of M.

5.1 Proposition. *Let $M, M_1, M_2 \subset \mathbb{R}^N$ be arbitrary subsets. Then*

1. *$M \subset H(M)$.*
2. *$H(M)$ is closed and convex.*
3. *$H(H(M)) = H(M)$.*
4. *If $M_1 \subset M_2$, then $H(M_1) \subset H(M_2)$.*
5. *If M is closed and convex, then $H(M) = M$.*
6. *If M is bounded, then $H(M)$ is also bounded.*

PROOF: (1) is trivial.

(2) If $\mathbf{x}_0 \notin H(M)$, then there is an $f \in \mathscr{L}$ with $f(\mathbf{x}_0) > \sup_M f$. By continuity, $f(\mathbf{x}) > \sup_M f$ in a neighborhood of $\mathbf{x}_0$. Therefore, $H(M)$ is closed.

If $\mathbf{x}_0, \mathbf{y}_0$ are two points in $H(M)$, then they are contained in every convex half-space $E = \{\mathbf{x} : f(\mathbf{x}) < \sup_M f\}$, and also the closed line segment from $\mathbf{x}_0$ to $\mathbf{y}_0$ is contained in each of these half-spaces. This shows that $H(M)$ is convex.

(3) We have to show that $H(H(M)) \subset H(M)$. If $\mathbf{x} \in H(H(M))$ is an arbitrary point and f an element of $\mathscr{L}$, then $f(\mathbf{x}) \le \sup_{H(M)} f \le \sup_M f$, by the definition of $H(M)$.

(4) is trivial.

(5) Let M be closed and convex. If $\mathbf{x}_0 \notin M$, then there is a point $\mathbf{y}_0 \in M$ such that $\operatorname{dist}(\mathbf{x}_0, M) = \operatorname{dist}(\mathbf{x}_0, \mathbf{y}_0)$ (because M is closed). Let $\mathbf{z}_0$ be a point in the open line segment from $\mathbf{x}_0$ to $\mathbf{y}_0$. Then $\mathbf{z}_0 \notin M$, and there is a function $f \in \mathscr{L}$ with $f(\mathbf{z}_0) = 0$ and $f|_M < 0$. Since $t \mapsto f(t\mathbf{x}_0 + (1-t)\mathbf{y}_0)$ is a monotone function, it follows that $f(\mathbf{x}_0) > 0$ and therefore $\mathbf{x}_0 \notin H(M)$.

(6) If M is bounded, there is an $R > 0$ such that M is contained in the closed convex set $\overline{B_R(\mathbf{0})}$. Thus $H(M) \subset \overline{B_R(\mathbf{0})}$. ∎

Remark. $H(M)$ is the smallest closed convex set that contains M.

5.2 Theorem. *A domain $G \subset \mathbb{R}^N$ is convex if and only if $K \subset\subset G$ implies that $H(K) \subset\subset G$.*

PROOF: Let G be a convex domain, and $M \subset\subset G$ a subset. Then $H(M)$ is closed and contained in the bounded set $H(\overline{M})$. Therefore, $\overline{H(M)}$ is compact, and it remains to show that $H(M) \subset G$. If there is a point $\mathbf{x}_0 \in H(M) - G$, then there is a function $f \in \mathscr{L}$ with $f(\mathbf{x}_0) = 0$ and $f|_G < 0$. It follows that $\sup_{\overline{M}} f < 0$, and $f(\mathbf{x}_0) > \sup_M f$. This is a contradiction to $\mathbf{x}_0 \in H(M)$.

On the other hand, let the criterion be fulfilled. If $\mathbf{x}_0, \mathbf{y}_0$ are two points of G, then $K := \{\mathbf{x}_0, \mathbf{y}_0\}$ is a relatively compact subset of G. It follows that $H(K)$ is contained in G. Since $H(K)$ is closed and convex, the closed line segment from $\mathbf{x}_0$ to $\mathbf{y}_0$ is also contained in G. Therefore G is convex. ∎

Holomorphic Convexity. Now we replace affine linear functions by holomorphic functions.

Definition. Let $G \subset \mathbb{C}^n$ be a domain and $K \subset G$ a subset. The set

$$\widehat{K} = \widehat{K}_G := \Big\{\mathbf{z} \in G : |f(\mathbf{z})| \le \sup_K |f|, \text{ for all } f \in \mathcal{O}(G)\Big\}$$

is called the *holomorphically convex hull* of K in G.

5.3 Proposition. *Let $G \subset \mathbb{C}^n$ be a domain, and K, K_1, K_2 subsets of G. Then*

1. $K \subset \widehat{K}$.
2. $\widehat{K}$ *is closed in* G.
3. $\widehat{\widehat{K}} = \widehat{K}$.
4. *If* $K_1 \subset K_2$, *then* $\widehat{K_1} \subset \widehat{K_2}$.
5. *If* K *is bounded, then* $\widehat{K}$ *is also bounded.*

PROOF: (1) is trivial.

(2) Let $\mathbf{z}_0$ be a point of $G - \widehat{K}$. Then there exists a holomorphic function f on G with $|f(\mathbf{z}_0)| > \sup_K |f|$. By continuity, this inequality holds on an entire neighborhood $U = U(\mathbf{z}_0) \subset G$. So $G - \widehat{K}$ is open.

(3) $\sup_{\widehat{K}} |f| \le \sup_K |f|$.

(4) is trivial.

(5) If K is bounded, it is contained in a closed polydisk $\overline{\mathsf{P}^n(\mathbf{0}, r)}$. The coordinate functions z_ν are holomorphic in G. For $\mathbf{z} \in \widehat{K}$ we have $|z_\nu| \le \sup_K |z_\nu| \le r$. Hence $\widehat{K}$ is also bounded. ∎

Definition. A domain $G \subset \mathbb{C}^n$ is called *holomorphically convex* if $K \subset\subset G$ implies that $\widehat{K} \subset\subset G$.

Example

In $\mathbb{C}$ every domain is holomorphically convex:

Let $K \subset\subset G$ be an arbitrary subset. Then $\widehat{K}$ is bounded, and it remains to show that the closure of $\widehat{K}$ is contained in G. If there is a point $z_0 \in \overline{\widehat{K}} - G$, then z_0 lies in $\partial\widehat{K} \cap \partial G$. We consider the holomorphic function $f(z) := 1/(z - z_0)$ in G. If (z_ν) is a sequence in $\widehat{K}$ converging to z_0, then $|f(z_\nu)| \le \sup_K |f| \le \sup_{\overline{K}} |f| < \infty$. This is a contradiction. For $n \ge 2$, we will show that there are domains that are not holomorphically convex. But we have the following result.

5.4 Proposition. *If $G \subset \mathbb{C}^n$ is an affine convex domain, then it is holomorphically convex.*

PROOF: Let K be relatively compact in G. Then $H(K) \subset\subset G$. If $\mathbf{z}_0$ is a point of $G - H(K)$, then there exists an affine linear function $\lambda \in \mathcal{L}$ with $\lambda(\mathbf{z}_0) > \sup_K \lambda$. Replacing λ by $\lambda - \lambda(\mathbf{0})$ we may assume that λ is a homogeneous linear function of the form

$$\lambda(\mathbf{z}) = 2\,\mathrm{Re}(\alpha_1 z_1 + \cdots + \alpha_n z_n).$$

Then $f(\mathbf{z}) := \exp(2 \cdot (\alpha_1 z_1 + \cdots + \alpha_n z_n))$ is holomorphic in G, and $|f(\mathbf{z})| = \exp(\lambda(\mathbf{z}))$. Therefore, $|f(\mathbf{z}_0)| > \sup_K |f|$, and $\mathbf{z}_0 \in G - \widehat{K}$. This proves $\widehat{K} \subset\subset G$. ∎

In general, holomorphic convexity is a much weaker property than affine convexity.

The Cartan–Thullen Theorem. Let $G \subset \mathbb{C}^n$ be a domain, and $\varepsilon > 0$ a small real number. We define

$$G_\varepsilon := \{\mathbf{z} \in G \,:\, \delta_G(\mathbf{z}) \ge \varepsilon\}.$$

Here are some properties of the set G_ε:

1. If $\mathbf{z} \in G$, then there is an $\varepsilon > 0$ such that $\delta_G(\mathbf{z}) \ge \varepsilon$. Therefore, $G = \bigcup_{\varepsilon > 0} G_\varepsilon$.
2. If $\varepsilon_1 \le \varepsilon_2$, then $G_{\varepsilon_1} \supset G_{\varepsilon_2}$.
3. G_ε is a closed subset of $\mathbb{C}^n$. In fact, if $\mathbf{z}_0 \in \mathbb{C}^n - G_\varepsilon$, then $\delta_G(\mathbf{z}_0) < \varepsilon$ or $\mathbf{z}_0 \notin G$. In the latter case, the ball $\mathsf{B}_\varepsilon(\mathbf{z}_0)$ is contained in $\mathbb{C}^n - G_\varepsilon$. If $\mathbf{z}_0 \in G - G_\varepsilon$ and $\delta := \delta_G(\mathbf{z}_0)$, then $\mathsf{B}_{\varepsilon-\delta}(\mathbf{z}_0) \subset \mathbb{C}^n - G_\varepsilon$. So $\mathbb{C}^n - G_\varepsilon$ is open.

5.5 Lemma. *Let $G \subset \mathbb{C}^n$ be a domain, $K \subset G$ a compact subset, and f a holomorphic function in G. If $K \subset G_\varepsilon$, then for any δ with $0 < \delta < \varepsilon$ there exists a constant $C > 0$ such that the following inequality holds:*

$$\sup_K |D^\alpha f(\mathbf{z})| \le \frac{\alpha!}{\delta^{|\alpha|}} \cdot C.$$

PROOF: For $0 < \delta < \varepsilon$, $G' := \{\mathbf{z} \in G : \operatorname{dist}(K, \mathbf{z}) < \delta\}$ is open and relatively compact in G, and for any $\mathbf{z} \in K$ the closed polydisk $\overline{\mathsf{P}^n(\mathbf{z}, \delta)}$ is contained in $\overline{G'} \subset G$. If T is the distinguished boundary of the polydisk and $|f| \le C$ on $\overline{G'}$, then the Cauchy inequalities yield

$$|D^\alpha f(\mathbf{z})| \le \frac{\alpha!}{\delta^{|\alpha|}} \cdot \sup_T |f| \le \frac{\alpha!}{\delta^{|\alpha|}} \cdot C.$$

∎

5.6 Theorem (Cartan–Thullen). *If G is a weak domain of holomorphy, then G is holomorphically convex.*

PROOF: Let $K \subset\subset G$. We want to show that $\widehat{K} \subset\subset G$. Let $\varepsilon := \operatorname{dist}(K, \mathbb{C}^n - G) \ge \operatorname{dist}(\overline{K}, \mathbb{C}^n - G) > 0$. Clearly, K lies in G_ε.

We assert that the holomorphically convex hull $\widehat{K}$ lies even in G_ε. Suppose this is not so. Then there is a $\mathbf{z}_0 \in \widehat{K} - G_\varepsilon$. Now let f be any holomorphic function in G. In a neighborhood $U = U(\mathbf{z}_0) \subset G$, f has a Taylor expansion

$$f(\mathbf{z}) = \sum_{\nu \ge 0} a_\nu (\mathbf{z} - \mathbf{z}_0)^\nu, \text{ with } a_\nu = \frac{1}{\nu!} D^\nu f(\mathbf{z}_0).$$

The function $\mathbf{z} \mapsto a_\nu(\mathbf{z}) := \frac{1}{\nu!} D^\nu f(\mathbf{z})$ is holomorphic in G. Therefore, $|a_\nu(\mathbf{z}_0)| \le \sup_K |a_\nu(\mathbf{z})|$. By the lemma, for any δ with $0 < \delta < \varepsilon$ there exists a $C > 0$ such that $\sup_K |a_\nu(\mathbf{z})| \le C/\delta^{|\nu|}$, and then

$$|a_\nu(\mathbf{z} - \mathbf{z}_0)^\nu| \le C \cdot \left(\frac{|z_1 - z_1^{(0)}|}{\delta} \right)^{\nu_1} \cdots \left(\frac{|z_n - z_n^{(0)}|}{\delta} \right)^{\nu_n}.$$

On any polydisk $\mathsf{P}^n(\mathbf{z}_0, \delta)$ the Taylor series is dominated by a geometric series. Therefore, it converges on $P = \mathsf{P}^n(\mathbf{z}_0, \varepsilon)$ to a holomorphic function $\widehat{f}$. We have $f = \widehat{f}$ near $\mathbf{z}_0$, and then on the connected component Q of $\mathbf{z}_0$ in $P \cap G$. Since P meets G and $\mathbb{C}^n - G$, it follows from Lemma 1..9 that there is a point $\mathbf{z}_1 \in P \cap \partial Q \cap \partial G$. Then f cannot be completely singular at $\mathbf{z}_1$. This is a contradiction, because f is an arbitrary holomorphic function in G, and G is a weak domain of holomorphy. ∎

Exercises

1. Let $G_1 \subset G_2 \subset \mathbb{C}^n$ be domains. Assume that for every $f \in \mathcal{O}(G_1)$ there is a sequence of functions $f_\nu \in \mathcal{O}(G_2)$ converging compactly on G_1 to f. Show that for every compact set $K \subset G_1$ it follows that $\widehat{K}_{G_2} \cap G_1 = \widehat{K}_{G_1}$.

2. Let $\mathbf{F} : G_1 \to G_2$ be a proper holomorphic map between domains in $\mathbb{C}^n$, respectively $\mathbb{C}^m$. Show that if G_2 is holomorphically convex, then so is G_1.
3. Let $G \subset \mathbb{C}^n$ be a domain and $S \subset G$ be a closed analytic disk with boundary bS. Show that $S \subset \widehat{(bS)}_G$.
4. Define the domain $G \subset \mathbb{C}^2$ by $G := \mathsf{P}^2(\mathbf{0}, 1) - \overline{\mathsf{P}^2(\mathbf{0}, 1/2)}$. Construct the holomorphically convex hull $\widehat{K}_G$ for $K := \{(z_1, z_2) \,:\, z_1 = 0 \text{ and } |z_2| = 3/4\}$. Is $\widehat{K}_G$ a relatively compact subset of G?
5. Let $\mathcal{F}$ be a family of functions in the domain G. For a compact subset $K \subset G$ we define

$$\widehat{K}_{\mathcal{F}} := \left\{ \mathbf{z} \in G \,:\, |f(\mathbf{z})| \le \sup_K |f| \text{ for all } f \in \mathcal{F} \right\}.$$

   The domain G is called *convex with respect to* $\mathcal{F}$, provided that $\widehat{K}_{\mathcal{F}}$ is relatively compact in G whenever K is. Prove:
   (a) Every bounded domain is convex with respect to the family $\mathscr{C}^0(G)$ of all continuous functions.
   (b) The unit ball $\mathsf{B} = \mathsf{B}_1(\mathbf{0})$ is convex with respect to the family of holomorphic functions $z_\nu^k \cdot z_\mu^l$ with $\nu, \mu = 1, \ldots, n$ and $k, l \in \mathbb{N}_0$.

6. Singular Functions

Normal Exhaustions. Let $G \subset \mathbb{C}^n$ be a domain. If G is holomorphically convex, we want to construct a holomorphic function in G that is completely singular at every boundary point. For that we use "normal exhaustions."

Definition. A *normal exhaustion* of G is a sequence (K_ν) of compact subsets of G such that:
1. $K_\nu \subset\subset (K_{\nu+1})^\circ$, for every ν.
2. $\bigcup_{\nu=1}^\infty K_\nu = G$.

6.1 Theorem. *Any domain G in $\mathbb{C}^n$ admits a normal exhaustion. If G is holomorphically convex, then there is a normal exhaustion (K_ν) with $\widehat{K}_\nu = K_\nu$ for every ν.*

Proof: In the general case, $K_\nu := \overline{\mathsf{P}^n(\mathbf{0}, \nu)} \cap G_{1/\nu}$ gives a normal exhaustion. If G is holomorphically convex, $\widehat{K}_\nu \subset\subset G$ for every ν. We construct a new exhaustion by induction.

Let $K_1^* := \widehat{K}_1$. Suppose that compact sets $K_1^*, \ldots, K_{\nu-1}^*$ have been constructed, with $\widehat{K}_j^* = K_j^*$ for $j = 1, \ldots, \nu - 1$, and $K_j^* \subset\subset (K_{j+1}^*)^\circ$. Then there exists a $\lambda(\nu) \in \mathbb{N}$ such that $K_{\nu-1}^* \subset (K_{\lambda(\nu)})^\circ$. Let $K_\nu^* := \widehat{K}_{\lambda(\nu)}$.

It is clear that (K_ν^*) is a normal exhaustion with $\widehat{K}_\nu^* = K_\nu^*$. ∎

Unbounded Holomorphic Functions. Again let $G \subset \mathbb{C}^n$ be a domain.

6.2 Theorem. *Let (K_ν) be a normal exhaustion of G with $\widehat{K}_\nu = K_\nu$, $\lambda(\mu)$ a strictly monotonic increasing sequence of natural numbers, and $(\mathbf{z}_\mu)$ a sequence of points with $\mathbf{z}_\mu \in K_{\lambda(\mu)+1} - K_{\lambda(\mu)}$.*

Then there exists a holomorphic function f in G such that $|f(\mathbf{z}_\mu)|$ is unbounded.

PROOF: The function f is constructed as the limit function of an infinite series $f = \sum_{\mu=1}^{\infty} f_\mu$. By induction we define holomorphic functions f_μ in G such that:

1. $|f_\mu|_{K_{\lambda(\mu)}} < 2^{-\mu}$ for $\mu \geq 1$.
2. $|f_\mu(\mathbf{z}_\mu)| > \mu + 1 + \sum_{j=1}^{\mu-1} |f_j(\mathbf{z}_\mu)|$ for $\mu \geq 2$.

Let $f_1 := 0$. Now for $\mu \geq 2$ suppose that $f_1, \dots, f_{\mu-1}$ have been constructed. Since $\mathbf{z}_\mu \in K_{\lambda(\mu)+1} - K_{\lambda(\mu)}$ and $\widehat{K}_{\lambda(\mu)} = K_{\lambda(\mu)}$, there exists a function g holomorphic in G such that $|g(\mathbf{z}_\mu)| > q := \sup_{K_{\lambda(\mu)}} |g|$. By multiplication by a suitable constant we can make

$$|g(\mathbf{z}_\mu)| > 1 > q.$$

If we set $f_\mu := g^k$ with a sufficiently large k, then f_μ has the properties (1) and (2).

We assert that $\sum_\mu f_\mu$ converges compactly in G. To prove this, first note that for $K \subset G$ an arbitrary compact subset, there is a $\mu_0 \in \mathbb{N}$ such that $K \subset K_{\lambda(\mu_0)}$. By construction $\sup_K |f_\mu| < 2^{-\mu}$ for $\mu \geq \mu_0$. Since the geometric series $\sum_\mu 2^{-\mu}$ dominates $\sum_\mu f_\mu$ in K, the series of the f_μ is normally convergent on K. This shows that $f = \sum_\mu f_\mu$ is holomorphic in G. Moreover,

$$\begin{aligned}
|f(\mathbf{z}_\mu)| &\geq |f_\mu(\mathbf{z}_\mu)| - \sum_{\nu \neq \mu} |f_\nu(\mathbf{z}_\mu)| \\
&> \mu + 1 - \sum_{\nu > \mu} |f_\nu(\mathbf{z}_\mu)| \\
&> \mu + 1 - \sum_{\nu > \mu} 2^{-\nu} \qquad (\text{since } \mathbf{z}_\mu \in K_{\lambda(\nu)} \text{ for } \nu > \mu) \\
&\geq \mu \qquad (\text{since } \sum_{\nu \geq 1} 2^{-\nu} = 1).
\end{aligned}$$

It follows that $|f(\mathbf{z}_\mu)| \to \infty$ for $\mu \to \infty$. ∎

The following is an important consequence:

6.3 Theorem. *A domain G is holomorphically convex if and only if for any infinite set D that is discrete in G there exists a function f holomorphic in G such that $|f|$ is unbounded on D.*

PROOF: (1) Let G be holomorphically convex, $D \subset G$ infinite and discrete. Moreover, let (K_ν) be a normal exhaustion of G with $\widehat{K}_\nu = K_\nu$. Then $K_\nu \cap D$ is finite (or empty) for every $\nu \in \mathbb{N}$. We construct a sequence of points $\mathbf{z}_\mu \in D$ by induction.

Let $\mathbf{z}_1 \in D - K_1$ be arbitrary, and $\lambda(1) \in \mathbb{N}$ minimal with the property that $\mathbf{z}_1$ lies in $K_{\lambda(1)+1}$. Now suppose the points $\mathbf{z}_1, \ldots, \mathbf{z}_{\mu-1}$ and the numbers $\lambda(1), \ldots, \lambda(\mu-1)$ have been constructed such that

$$\mathbf{z}_\nu \in K_{\lambda(\nu)+1} - K_{\lambda(\nu)}, \text{ for } \nu = 1, \ldots, \mu - 1.$$

Then we choose $\mathbf{z}_\mu \in D - K_{\lambda(\mu-1)+1}$ and $\lambda(\mu)$ minimal with the property that $\mathbf{z}_\mu$ lies in $K_{\lambda(\mu)+1}$. By the theorem above there is a holomorphic function f in G such that $|f(\mathbf{z}_\mu)| \to \infty$ for $\mu \to \infty$. Therefore, $|f|$ is unbounded on D.

(2) Now suppose that the criterion is satisfied, and $K \subset\subset G$. Then $\widehat{K} \subset G$, and we have to show that $\widehat{K}$ is compact. Let $(\mathbf{z}_\nu)$ be any sequence of points in $\widehat{K}$. Then

$$\sup\{|f(\mathbf{z}_\nu)| \,:\, \nu \in \mathbb{N}\} \le \sup_K |f| < \infty, \text{ for every } f \in \mathcal{O}(G).$$

Therefore, $\{\mathbf{z}_\nu \,:\, \nu \in \mathbb{N}\}$ cannot be discrete in G. Thus the sequence $(\mathbf{z}_\nu)$ has a cluster point $\mathbf{z}_0$ in G. Since $\widehat{K}$ is closed, $\mathbf{z}_0$ belongs to $\widehat{K}$. So G is holomorphically convex. ∎

Sequences. For a domain $G \subset \mathbb{C}^n$ we wish to construct a sequence that accumulates at every point of its boundary.

6.4 Theorem. *Let (K_ν) be a normal exhaustion of G. Then there exists a strictly monotonic increasing sequence $\lambda(\mu)$ of natural numbers and a sequence $(\mathbf{z}_\mu)$ of points in G such that:*

1. $\mathbf{z}_\mu \in K_{\lambda(\mu)+1} - K_{\lambda(\mu)}$*, for every μ.*
2. *If $\mathbf{z}_0$ is a boundary point of G and $U = U(\mathbf{z}_0)$ an open connected neighborhood, then every connected component of $U \cap G$ contains infinitely many points of the sequence $(\mathbf{z}_\mu)$.*

PROOF: This is a purely topological result, since we make no assumption about G. The proof is carried out in several steps.

(1) Let $\mathcal{B} = \{B_\nu : \nu \in \mathbb{N}\}$ be the countable system of balls with rational center and rational radius meeting ∂G. Every intersection $B_\nu \cap G$ has at most countably many connected components. Thus we obtain a countable family

$$\mathcal{C} = \{C_\mu : \exists\, \nu \in \mathbb{N} \text{ such that } C_\mu \text{ is a connected component of } B_\nu \in \mathcal{B}\}.$$

(2) By induction, the sequences $\lambda(\mu)$ and $(\mathbf{z}_\mu)$ are constructed. Let $\mathbf{z}_1$ be arbitrary in $C_1 - K_1$. Then there is a unique number $\lambda(1)$ such that $\mathbf{z}_1 \in K_{\lambda(1)+1} - K_{\lambda(1)}$.

Now suppose $\mathbf{z}_1, \ldots, \mathbf{z}_{\mu-1}$ have been constructed such that

$$\mathbf{z}_j \in C_j \cap (K_{\lambda(j)+1} - K_{\lambda(j)}), \text{ for } j = 1, \ldots, \mu - 1.$$

We choose $\mathbf{z}_\mu \in C_\mu - K_{\lambda(\mu-1)+1}$ and $\lambda(\mu)$ as usual. That is possible, since there is a point $\mathbf{w} \in B_{\nu(\mu)} \cap \partial C_\mu \cap \partial G$ if C_μ is a connected component of $B_{\nu(\mu)} \cap G$. Then $\mathbb{C}^n - K_{\lambda(\mu-1)+1}$ is an open neighborhood of $\mathbf{w}$ and contains points of C_μ.

(3) Now we show that property (2) of the theorem is satisfied. Let $\mathbf{z}_0$ be a point of ∂G, $U = U(\mathbf{z}_0)$ an open connected neighborhood, and Q a connected component of $U \cap G$. We assume that only finitely many $\mathbf{z}_\mu$ lie in Q, say $\mathbf{z}_1, \ldots, \mathbf{z}_m$. Then

$$U^* := U - \{\mathbf{z}_1, \ldots, \mathbf{z}_m\} \quad \text{and} \quad Q^* := Q - \{\mathbf{z}_1, \ldots, \mathbf{z}_m\}$$

are open connected sets that contain no $\mathbf{z}_\mu$. Obviously, Q^* is a connected component of $G \cap U^*$.

There is a point $\mathbf{w}_0$ in $U^* \cap \partial Q^* \cap \partial G$, and a ball $B_\nu \subset U^*$ with $\mathbf{w}_0 \in B_\nu$. Then $B_\nu \cap G \subset U^* \cap G$. Moreover, $B_\nu \cap G$ must contain a point $\mathbf{w}_1 \in Q^*$. The connected component C^* of $\mathbf{w}_1$ in $B_\nu \cap G$ is a subset of the connected component of $\mathbf{w}_1$ in $U^* \cap G$. But C^* is an element C_{μ_0} of $\mathcal{C}$. By construction it contains the point $\mathbf{z}_{\mu_0}$. That is a contradiction. Infinitely many members of the sequence belong to Q. ∎

6.5 Theorem. *If G is holomorphically convex, then it is a domain of holomorphy.*

PROOF: Let (K_ν) be a normal exhaustion of G with $\widehat{K}_\nu = K_\nu$ and choose sequences $\lambda(\mu) \in \mathbb{N}$ and $(\mathbf{z}_\mu)$ in G such that $\mathbf{z}_\mu \in K_{\lambda(\mu)+1} - K_{\lambda(\mu)}$. We may assume that for every point $\mathbf{z}_0 \in \partial G$, every open connected neighborhood $U = U(\mathbf{z}_0)$, and every connected component Q of $U \cap G$ there are infinitely many $\mathbf{z}_\mu$ in Q.

Now let f be holomorphic in G and unbounded on $D := \{\mathbf{z}_\mu : \mu \in \mathbb{N}\}$. It is clear that f is completely singular at every point $\mathbf{z}_0 \in \partial G$. ∎

Remark. It is not necessary that a completely singular holomorphic function is unbounded. In 1978, D. Catlin showed in his dissertation that if $G \subset\subset \mathbb{C}^n$ is a holomorphically convex domain with smooth boundary, then there exists a function holomorphic in G and smooth in a neighborhood of $\overline{G}$ that is completely singular at every point of the boundary of G.

Exercises

1. A domain $G \subset\subset \mathbb{C}^n$ is holomorphically convex if and only if for every $\mathbf{z} \in \partial G$ there is a neighborhood $U(\mathbf{z})$ such that $U \cap G$ is a domain of holomorphy.
2. Let $G_1 \subset \mathbb{C}^n$ and $G_2 \subset \mathbb{C}^m$ be domains of holomorphy. If $\mathbf{f} : G_1 \to \mathbb{C}^m$ is a holomorphic mapping, then $f^{-1}(G_2) \cap G_1$ is a domain of holomorphy.
3. Find a bounded holomorphic function on the unit disk D that is singular at every boundary point.

7. Examples and Applications

Domains of Holomorphy

7.1 Proposition. *Every domain in the complex plane $\mathbb{C}$ is a domain of holomorphy.*

PROOF: We have already shown that every domain in $\mathbb{C}$ is holomorphically convex. Therefore, such a domain is also a domain of holomorphy. ∎

7.2 Theorem. *The following statements about domains $G \in \mathbb{C}^n$ are equivalent:*

1. *G is a weak domain of holomorphy.*
2. *G is holomorphically convex.*
3. *For every infinite discrete subset $D \subset G$ there exists a holomorphic function f in G such that $|f|$ is unbounded on D.*
4. *G is a domain of holomorphy.*

The equivalences have all been proved in the preceding paragraphs. Furthermore, we know that every domain of holomorphy is pseudoconvex. Still missing here is the proof of the Levi problem: Every pseudoconvex domain is holomorphically convex. We say more about this in Chapter V.

Every affine convex open subset of $\mathbb{C}^n$ is a domain of holomorphy. The n-fold Cartesian product of plane domains is a further example.

7.3 Proposition. *If $G_1, \ldots, G_n \subset \mathbb{C}$ are arbitrary domains, then $G := G_1 \times \cdots \times G_n$ is a domain of holomorphy.*

Proof: Let $D = \{\mathbf{z}_\mu = (z_1^\mu, \dots, z_n^\mu) \,:\, \mu \in \mathbb{N}\}$ be an infinite discrete subset of G. Then there is an i such that (z_i^μ) has no cluster point in G_i, and there is a holomorphic function f in G_i with $\lim_{\mu\to\infty}\big|\, f(z_i^\mu)\,\big| = \infty$. The function $\widehat{f}$ in G, defined by $\widehat{f}(z_1, \dots, z_n) := f(z_i)$, is holomorphic in G and unbounded on D. ■

Remark. The same proof shows that every Cartesian product of domains of holomorphy is again a domain of holomorphy.

Complete Reinhardt Domains

Let $G \subset \mathbb{C}^n$ be a complete Reinhardt domain (see Section I.1). We will give criteria for G to be a domain of holomorphy. For that purpose we define a map log from the absolute value space $\mathscr{V}$ to $\mathbb{R}^n$ by

$$\log(r_1, \dots, r_n) := (\log r_1, \dots, \log r_n).$$

Definition. A Reinhardt domain G is called *logarithmically convex* if $\log\tau(G \cap (\mathbb{C}^*)^n)$ is an affine convex domain in $\mathbb{R}^n$.

Remark. For $\mathbf{z} = (z_1, \dots, z_n) \in G$ we have $\log\tau(\mathbf{z}) = (\log|z_1|, \dots, \log|z_n|)$. If $\mathbf{z} \in (\mathbb{C}^*)^n$, then $|z_i| > 0$ for each i, and $\log\tau(\mathbf{z})$ is in fact an element of $\mathbb{R}^n$.

7.4 Proposition. *The domain of convergence of a power series* $S(\mathbf{z}) = \sum_{\nu\ge 0} a_\nu \mathbf{z}^\nu$ *is logarithmically convex.*

Proof: Let G be the domain of convergence of $S(\mathbf{z})$, and $M := \log\tau(G \cap (\mathbb{C}^*)^n) \subset \mathbb{R}^n$. We consider two points $\mathbf{x}, \mathbf{y} \in M$ and points $\mathbf{z}, \mathbf{w} \in G \cap (\mathbb{C}^*)^n$ with $\log\tau(\mathbf{z}) = \mathbf{x}$ and $\log\tau(\mathbf{w}) = \mathbf{y}$. If $\lambda > 1$ is small enough, $\lambda\mathbf{z}$ and $\lambda\mathbf{w}$ still belong to $G \cap (\mathbb{C}^*)^n$. Since $S(\mathbf{z})$ is convergent in $\lambda\mathbf{z}, \lambda\mathbf{w}$, there is a constant $C > 0$ such that

$$|a_\nu| \cdot \lambda^{|\nu|} \cdot |\mathbf{z}^\nu| \le C \quad \text{and} \quad |a_\nu| \cdot \lambda^{|\nu|} \cdot |\mathbf{w}^\nu| \le C, \text{ for every } \nu \in \mathbb{N}_0^n.$$

Thus

$$|a_\nu| \cdot \lambda^{|\nu|} \cdot |\mathbf{z}^\nu|^t \cdot |\mathbf{w}^\nu|^{1-t} \le C, \text{ for every } \nu \text{ and } 0 \le t \le 1.$$

It follows from Abel's lemma that $S(\mathbf{z})$ is convergent in a neighborhood of

$$\mathbf{z}_t := \big(|z_1|^t |w_1|^{1-t}, \dots, |z_n|^t |w_n|^{1-t}\big).$$

This means that $\mathbf{z}_t \in G$ and $t\mathbf{x} + (1-t)\mathbf{y} = \log\tau(\mathbf{z}_t) \in M$, for $0 \le t \le 1$. Therefore, M is convex. ■

7.5 Proposition. *Let G be a complete Reinhardt domain. If G is logarithmically convex, then it is holomorphically convex.*

PROOF: Let K be a relatively compact subset of G. Since G is a complete Reinhardt domain and $\overline{K}$ a compact subset of G, there are points $\mathbf{z}_1, \dots, \mathbf{z}_k \in G \cap (\mathbb{C}^*)^n$ such that

$$K \subset G' := \bigcup_{i=1}^{k} \mathsf{P}^n(\mathbf{0}, \mathbf{q}_i) \subset G, \quad \text{where } \mathbf{q}_i := \tau(\mathbf{z}_i).$$

We consider the set $\mathscr{M} = \{m(\mathbf{z}) = \mathbf{z}^\nu \,:\, \nu \in \mathbb{N}_0^n\}$ of monomials, which is a subset of $\mathcal{O}(G)$. For $\mathbf{z} \in \mathsf{P}^n(\mathbf{0}, \mathbf{q}_i)$ and $m \in \mathscr{M}$ we have

$$|m(\mathbf{z})| = |\mathbf{z}^\nu| < \mathbf{q}_i^\nu = |m(\mathbf{q}_i)|.$$

Let $Z := \{\mathbf{z}_1, \dots, \mathbf{z}_k\}$. Then for $\mathbf{z} \in \widehat{K}$ it follows that

$$|m(\mathbf{z})| \le \sup_K |m| \le \sup_{G'} |m| \le \sup_Z |m|, \text{ for every } m \in \mathscr{M}.$$

Suppose that $\widehat{K}$ is not relatively compact in G. Then $\widehat{K}$ has a cluster point $\mathbf{z}_0$ in ∂G, and it follows that $|m(\mathbf{z}_0)| \le \sup_Z |m|$, for every $m \in \mathscr{M}$.

Let $h(\mathbf{z}) := \log \tau(\mathbf{z})$, for $\mathbf{z} \in (\mathbb{C}^*)^n$. Since G is logarithmically convex, the domain $G_0 := h(G \cap (\mathbb{C}^*)^n) \subset \mathbb{R}^n$ is affine convex. For the time being we assume that $\mathbf{z}_0 \in (\mathbb{C}^*)^n$. Then $\mathbf{x}_0 := h(\mathbf{z}_0) \in \partial G_0$, and there is a real linear function $\lambda(\mathbf{x}) = a_1 x_1 + \cdots + a_n x_n$ such that $\lambda(\mathbf{x}) < \lambda(\mathbf{x}_0)$ for $\mathbf{x} \in G_0$.

Let $\mathbf{x} = \log \tau(\mathbf{z})$ be a point of G_0, and $\mathbf{u} \in \mathbb{R}^n$ with $u_j \le x_j$ for $j = 1, \dots, n$. Then $e^{u_j} \le e^{x_j} = |z_j|$, and therefore (since G is a complete Reinhardt domain) $\mathbf{w} = (e^{u_1}, \dots, e^{u_n}) \in G \cap (\mathbb{C}^*)^n$ and $\mathbf{u} \in G_0$. In particular,

$$\lambda(\mathbf{x}) - n a_j = \lambda(\mathbf{x} - n\mathbf{e}_j) < \lambda(\mathbf{x}_0), \text{ for every } n \in \mathbb{N}.$$

Therefore, $a_j \ge 0$ for $j = 1, \dots, n$.

Now we choose rational numbers $r_j > a_j$ and define $\widetilde{\lambda}(\mathbf{x}) := r_1 x_1 + \cdots + r_n x_n$. If we choose the r_j sufficiently close to a_j, the inequality $\widetilde{\lambda}(\mathbf{q}_i) < \widetilde{\lambda}(\mathbf{x}_0)$ holds for $i = 1, \dots, k$, and it still holds after multiplying by the common denominator of the r_j. Therefore, we may assume that the r_j are natural numbers, and we can define a special monomial m_0 by $m_0(\mathbf{z}) := z_1^{r_1} \cdots z_n^{r_n}$. Then

$$|m_0(\mathbf{z}_i)| = e^{\widetilde{\lambda}(\mathbf{q}_i)} < e^{\widetilde{\lambda}(\mathbf{x}_0)} = |m_0(\mathbf{z}_0)|, \text{ for } i = 1, \dots, k.$$

So $|m_0(\mathbf{z}_0)| > \sup_Z |m_0|$, and this is a contradiction.

If $\mathbf{z}_0 \notin (\mathbb{C}^*)^n$, then after a permutation of the coordinates we may assume that $z_1^{(0)} \cdots z_l^{(0)} \ne 0$ and $z_{l+1}^{(0)} = \cdots = z_n^{(0)} = 0$. We can project on the space

$\mathbb{C}^l$ and work with monomials in the variables $z_1, \ldots, z_l$. Then the proof goes through as above. ∎

Now we get the following result:

7.6 Theorem. *Let $G \subset \mathbb{C}^n$ be a complete Reinhardt domain. Then the following statements are equivalent:*

1. *G is the domain of convergence of a power series.*
2. *G is logarithmically convex.*
3. *G is holomorphically convex.*
4. *G is a domain of holomorphy.*

PROOF: We have only to show that if G is a complete Reinhardt domain and a domain of holomorphy, then it is the domain of convergence of a power series. By hypothesis, there is a function f that is holomorphic in G and completely singular at every boundary point. In Section I.5 we proved that for every holomorphic function in a proper Reinhardt domain there is a power series $S(\mathbf{z})$ that converges in G to f. By the identity theorem it does not converge on any domain strictly larger than G. ∎

Analytic Polyhedra

Let $G \subset \mathbb{C}^n$ be a domain.

Definition. Let $U \subset G$, $V_1, \ldots, V_k \subset \mathbb{C}$ open subsets, and $f_1, \ldots, f_k$ holomorphic functions in G. The set

$$P := \{\mathbf{z} \in U \,:\, f_i(\mathbf{z}) \in V_i, \text{ for } i = 1, \ldots, k\}$$

is called an *analytic polyhedron* in G if $P \subset\subset U$.

If, in addition, $V_1 = \cdots = V_k = \mathsf{D}$, then one speaks of a *special analytic polyhedron* in G.

Remark. An analytic polyhedron P need not be connected. The set U in the definition ensures that each union of connected components of P is also an analytic polyhedron if it has a positive distance from every other connected component of P.

7.7 Theorem. *Every connected analytic polyhedron P in G is a domain of holomorphy.*

PROOF: We have only to show that P is a weak domain of holomorphy. If $\mathbf{z}_0 \in \partial P$, then there is an i such that $f_i(\mathbf{z}_0) \in \partial V_i$. Therefore, $f(\mathbf{z}) := (f_i(\mathbf{z}) - f_i(\mathbf{z}_0))^{-1}$ is holomorphic in P and completely singular at $\mathbf{z}_0$. ∎

Example

Let $q < 1$ be a positive real number, and

$$P := \big\{\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1| < 1,\, |z_2| < 1 \text{ and } |z_1 \cdot z_2| < q\big\}.$$

Then P (see Figure II.7) is clearly an analytic polyhedron, but neither affine

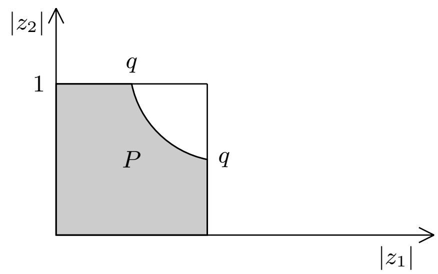


Figure II.7. An analytic polyhedron

convex nor a Cartesian product of domains. So the analytic polyhedra enrich our stock of examples of domains of holomorphy.

We will show that every domain of holomorphy is "almost" an analytic polyhedron.

7.8 Theorem. *If $G \subset \mathbb{C}^n$ is a domain of holomorphy, then there exists a sequence (P_ν) of special analytic polyhedra in G with $P_\nu \subset\subset P_{\nu+1}$ and $\bigcup_{\nu=1}^{\infty} P_\nu = G$.*

PROOF: Let (K_ν) be a normal exhaustion of G with $\widehat{K}_\nu = K_\nu$. If $\mathbf{z} \in \partial K_{\nu+1}$ is an arbitrary point, then $\mathbf{z}$ does not lie in $K_\nu \subset (K_{\nu+1})^\circ$, and therefore not in $\widehat{K}_\nu$. Hence there exists a function f holomorphic in G for which $q := \sup_{K_\nu} |f| < |f(\mathbf{z})|$. By multiplication by a suitable constant we obtain $q < 1 < |f(\mathbf{z})|$, and then there is an entire neighborhood $U = U(\mathbf{z})$ such that $|f| > 1$ on U.

Since the boundary $\partial K_{\nu+1}$ is compact, we can find finitely many open neighborhoods $U_{\nu,j}$ of $\mathbf{z}_{\nu,j} \in \partial K_{\nu+1}$, $j = 1, \ldots, k_\nu$, and corresponding functions $f_{\nu,j}$ holomorphic in G such that $|f_{\nu,j}| > 1$ on $U_{\nu,j}$, and $\partial K_{\nu+1} \subset \bigcup_{j=1}^{k_\nu} U_{\nu,j}$. We define

$$P_\nu := \big\{\mathbf{z} \in (K_{\nu+1})^\circ \,:\, |f_{\nu,j}(\mathbf{z})| < 1 \text{ for } j = 1, \ldots, k_\nu\big\}.$$

Clearly, $K_\nu \subset P_\nu \subset (K_{\nu+1})^\circ$. Furthermore, $M := K_{\nu+1} - (U_{\nu,1} \cup \cdots \cup U_{\nu,k_\nu})$ is a compact set with $P_\nu \subset M \subset (K_{\nu+1})^\circ$. Consequently, $P_\nu \subset\subset K_{\nu+1}$. Thus P_ν is a special analytic polyhedron in G. It follows trivially that the sequence (P_ν) exhausts the domain G. ∎

In the theory of Stein manifolds one proves the converse of this theorem.

Exercises

1. If R is a domain in the real number space $\mathbb{R}^n$, then
$$T_R = R + \mathrm{i}\mathbb{R}^n := \{\mathbf{z} \in \mathbb{C}^n \,:\, (\mathrm{Re}(z_1), \dots, \mathrm{Re}(z_n)) \in R\}$$
is called the *tube domain* associated with R. Prove that the following properties are equivalent:
(a) R is convex.
(b) T_R is (affine) convex.
(c) T_R is holomorphically convex.
(d) T_R is pseudoconvex.
Hint: To show $(d) \implies (a)$ choose $\mathbf{x}_0, \mathbf{y}_0 \in R$. Then the function $\varphi(\zeta) := -\ln \delta_{T_R}(\mathbf{x}_0 + \zeta(\mathbf{y}_0 - \mathbf{x}_0))$ is subharmonic in D. Since $\delta_{T_R}(\mathbf{x} + \mathrm{i}\mathbf{y}) = \delta_R(\mathbf{x})$, one concludes that $t \mapsto -\ln \delta_R(\mathbf{x}_0 + t(\mathbf{y}_0 - \mathbf{x}_0))$ assumes its maximum at $t = 0$ or $t = 1$.
2. Let $G \subset \mathbb{C}^n$ be a domain. A domain $\widehat{G} \subset \mathbb{C}^n$ is called the *envelope of holomorphy* of G if every holomorphic function f in G has a holomorphic extension to $\widehat{G}$. Prove:
(a) If $R \subset \mathbb{R}^n$ is a domain and $H(R)$ its affine convex hull, then $\widehat{G} := H(R) + \mathrm{i}\mathbb{R}^n$ is the envelope of holomorphy of the tube domain $G = R + \mathrm{i}\mathbb{R}^n$.
(b) If $G \subset \mathbb{C}^n$ is a Reinhardt domain and $\widehat{G}$ the smallest logarithmically convex complete Reinhardt domain containing G, then $\widehat{G}$ is the envelope of holomorphy of G. Hint: Use the convex hull of $\log \tau(G)$.
3. Construct the envelope of holomorphy of the domain
$$G_q := \mathsf{P}^2(\mathbf{0}, (1, q)) \cup \mathsf{P}^2(\mathbf{0}, (q, 1)) \,.$$
4. A domain $G \subset \mathbb{C}^n$ is called a *Runge domain* if for every holomorphic function f in G there is a sequence (p_ν) of polynomials converging compactly in G to f.

   Prove that the Cartesian product of n simply connected subdomains of $\mathbb{C}$ is a Runge domain in $\mathbb{C}^n$.
5. A domain $G \subset \mathbb{C}^n$ is called *polynomially convex* if it is convex with respect to the family of all polynomials (cf. Exercise 5.5). Prove that every polynomially convex domain is a holomorphically convex Runge domain.

8. Riemann Domains over $\mathbb{C}^n$

Riemann Domains. It turns out that for general domains in $\mathbb{C}^n$ the envelope of holomorphy (cf. Exercise 7.2) cannot exist in $\mathbb{C}^n$. Therefore, we have to consider domains covering $\mathbb{C}^n$.

Definition. A *(Riemann) domain over* $\mathbb{C}^n$ is a pair (G, π) with the following properties:

1. G is a connected Hausdorff space.[4]
2. $\pi : G \to \mathbb{C}^n$ is a local homeomorphism (that is, for each point $x \in G$ and its "base point" $\mathbf{z} := \pi(x) \in \mathbb{C}^n$ there exist open neighborhoods $U = U(x) \subset X$ and $V = V(\mathbf{z}) \subset \mathbb{C}^n$ such that $\pi : U \to V$ is a homeomorphism).

Remarks

1. Let (G, π) be a Riemann domain. Then G is pathwise connected, and the map $\pi : G \to \mathbb{C}^n$ is continuous and open. The latter means that the images of open sets are again open.
2. If (G_ν, π_ν) are domains over $\mathbb{C}^n$ for $\nu = 1, \ldots, l$, and $x_\nu \in G_\nu$ are points over the same base point $\mathbf{z}_0$, then there are open neighborhoods $U_\nu = U_\nu(x_\nu) \subset G_\nu$ and a connected open neighborhood $V = V(\mathbf{z}_0) \subset \mathbb{C}^n$ such that $\pi_\nu|_{U_\nu} : U_\nu \to V$ is a homeomorphism for $\nu = 1, \ldots, l$.

Examples

1. If G is a domain in $\mathbb{C}^n$, then (G, id_G) is a Riemann domain.
2. The Riemann surface of $\sqrt{z}$ (without the branch point) is the set

$$G := \{(z, w) \in \mathbb{C}^* \times \mathbb{C} \,:\, w^2 = z\}.$$

If G is provided with the topology induced from $\mathbb{C}^* \times \mathbb{C}$, then it is a Hausdorff space. The mapping $\varphi : \mathbb{C}^* \to G$ defined by $\zeta \mapsto (\zeta^2, \zeta)$ is continuous and bijective. Therefore, G is connected. The mapping φ is called a *uniformization* of G.

Now let $\pi : G \to \mathbb{C}$ be defined by $\pi(z, w) := z$. Clearly, π is continuous. If $(z_0, w_0) \in G$ is an arbitrary point, then $z_0 \neq 0$, and we can find a simply connected neighborhood $V(\mathbf{z}_0) \subset \mathbb{C}^*$. Then there exists a holomorphic function f in V with $f^2(z) \equiv z$ and $f(z_0) = w_0$. We denote $f(z)$ by $\sqrt{z}$. The image $W := f(V)$ is open, and the set $\pi^{-1}(V)$ can be written as the union of two disjoint open sets

$$U_\pm := \{(z, \pm f(z)) \,:\, z \in V\} = (V \times (\pm W)) \cap G.$$

Let $\widehat{f}(z) := (z, f(z))$. Then $\widehat{f} : V \to G$ is continuous, and $\pi \circ \widehat{f}(z) \equiv z$. The open set $U := U_+$ is a neighborhood of (z_0, w_0), with $\widehat{f}(V) = U$ and $\widehat{f} \circ \pi(z, w) = (z, w)$ on U; that is, $\pi|_U : U \to V$ is topological. Hence (G, π) is a Riemann domain over $\mathbb{C}$.

[4] A general topological space X is said to be *connected* if it is not the union of two disjoint nonempty open sets. A space X is called *pathwise connected* if each two points of X can be joined by a continuous path in X. For open sets in $\mathbb{C}^n$ these two notions are equivalent.

The space G can be visualized in the following manner: We cover $\mathbb{C}$ with two additional copies of $\mathbb{C}$, cut both these "sheets" along the positive real axis, and paste them crosswise to one another (this is not possible in $\mathbb{R}^3$ without self intersection, but in higher dimensions, it is). This is illustrated in Figure II.8.

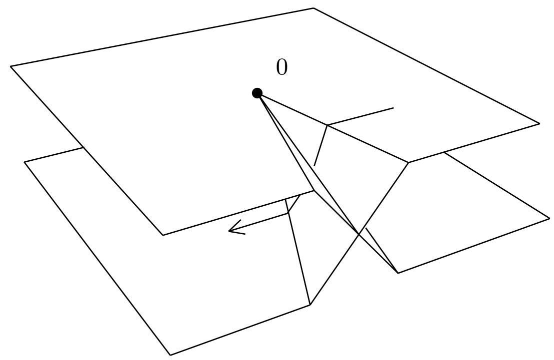


Figure II.8. The Riemann surface of $\sqrt{z}$

8.1 Proposition (on the uniqueness of lifting). *Let (G, π) be a domain over $\mathbb{C}^n$ and Y a connected topological space. Let $y_0 \in Y$ be a point and $\psi_1, \psi_2 : Y \to G$ continuous mappings with $\psi_1(y_0) = \psi_2(y_0)$ and $\pi \circ \psi_1 = \pi \circ \psi_2$. Then $\psi_1 = \psi_2$.*

PROOF: Let $M := \{y \in Y : \psi_1(y) = \psi_2(y)\}$. By assumption, $y_0 \in M$, so $M \neq \varnothing$. Since G is a Hausdorff space, it follows immediately that M is closed. Now let $y \in Y$ be chosen arbitrarily, and set $x := \psi_1(y) = \psi_2(y)$ and $\mathbf{z} := \pi(x)$. There are open neighborhoods $U = U(x) \subset G$ and $V = V(\mathbf{z}) \subset \mathbb{C}^n$ such that $\pi : U \to V$ is topological, and there is an open neigborhood $W = W(y)$ with $\psi_\lambda(W) \subset U$ for $\lambda = 1, 2$. Then

$$\psi_1|_W = (\pi|_U)^{-1} \circ \pi \circ \psi_1|_W = (\pi|_U)^{-1} \circ \pi \circ \psi_2|_W = \psi_2|_W,$$

and therefore $W \subset M$. Hence M is open, and since Y is connected, it follows that $M = Y$. ∎

Definition. Let $\mathbf{z}_0 \in \mathbb{C}^n$ be fixed. A *(Riemann) domain over $\mathbb{C}^n$ with distinguished point* is a triple $\mathcal{G} = (G, \pi, x_0)$ for which:

1. (G, π) is a domain over $\mathbb{C}^n$.
2. x_0 is a point of G with $\pi(x_0) := \mathbf{z}_0$.

Definition. Let $\mathcal{G}_j = (G_j, \pi_j, x_j)$ be domains over $\mathbb{C}^n$ with distinguished point. We say that $\mathcal{G}_1$ *is contained in* $\mathcal{G}_2$ (denoted by $\mathcal{G}_1 \prec \mathcal{G}_2$) if there is a continuous map $\varphi : G_1 \to G_2$ with the following properties:

1. $\pi_2 \circ \varphi = \pi_1$ (called "φ preserves fibers").
2. $\varphi(x_1) = x_2$.

8.2 Proposition. *If $\mathcal{G}_1 \prec \mathcal{G}_2$, then the fiber preserving map $\varphi : G_1 \to G_2$ with $\varphi(x_1) = x_2$ is uniquely determined.*

This follows immediately from the uniqueness of lifting.

8.3 Proposition. *The relation "$\prec$" is a weak ordering; that is:*

1. $\mathcal{G} \prec \mathcal{G}$.
2. $\mathcal{G}_1 \prec \mathcal{G}_2$ *and* $\mathcal{G}_2 \prec \mathcal{G}_3 \implies \mathcal{G}_1 \prec \mathcal{G}_3$.

The proof is trivial.

Definition. Two domains $\mathcal{G}_1, \mathcal{G}_2$ over $\mathbb{C}^n$ with fundamental point are called *isomorphic* or *equivalent* (symbolically $\mathcal{G}_1 \cong \mathcal{G}_2$) if $\mathcal{G}_1 \prec \mathcal{G}_2$ and $\mathcal{G}_2 \prec \mathcal{G}_1$.

8.4 Proposition. *Two domains $\mathcal{G}_j = (G_j, \pi_j, x_j)$, $j = 1, 2$, are isomorphic if and only if there exists a* ***topological***[5] *fiber preserving map $\varphi : G_1 \to G_2$ with $\varphi(x_1) = x_2$.*

PROOF: If we have fiber preserving mappings $\varphi_1 : G_1 \to G_2$ and $\varphi_2 : G_2 \to G_1$, with $\varphi_1(x_1) = x_2$ and $\varphi_2(x_2) = x_1$, it follows easily from the uniqueness of fiber preserving maps that $\varphi_2 \circ \varphi_1 = \mathrm{id}_{G_1}$ and $\varphi_1 \circ \varphi_2 = \mathrm{id}_{G_2}$. The other direction of the proof is trivial. ∎

Definition. A domain $\mathcal{G} = (G, \pi, x_0)$ with $\pi(x_0) = \mathbf{z}_0$ is called *schlicht* if it is isomorphic to a domain $\mathcal{G}_0 = (G_0, \mathrm{id}_{G_0}, \mathbf{z}_0)$ with $G_0 \subset \mathbb{C}^n$.

8.5 Proposition. *Let $\mathcal{G}_j = (G_j, \mathrm{id}_{G_j}, x_j)$, $j = 1, 2$, be two schlicht domains with $G_1, G_2 \subset \mathbb{C}^n$. Then $\mathcal{G}_1 \prec \mathcal{G}_2$ if and only if $G_1 \subset G_2$.*

Example

[5] Recall that a "topological map" is a homeomorphism!

Let $G_1 := \{(z,w) \in \mathbb{C}^2 \,:\, w^2 = z \text{ and } z \neq 0\}$ and $\pi_1(z,w) := z$. Then $\mathcal{G}_1 = (G_1, \pi_1, (1,1))$ is the Riemann surface of $\sqrt{z}$, with distinguished point $(1,1)$. The domain $\mathcal{G}_1$ is contained in the schlicht domain $\mathcal{G}_2 = (\mathbb{C}, \mathrm{id}_{\mathbb{C}}, 1)$, by $\varphi(z,w) := z$. But the two domains are not isomorphic.

Union of Riemann Domains. We begin with the definition of the union of two Riemann domains. Let $\mathcal{G}_j = (G_j, \pi_j, x_j)$, $j = 1,2$, be two Riemann domains over $\mathbb{C}^n$ with distinguished point, and $\mathbf{z}_0 := \pi_1(x_1) = \pi_2(x_2)$. We want to glue G_1, G_2 in such a way that x_1 and x_2 will also be glued.

To get a rough idea of the construction, assume that we already have a Riemann domain $\mathcal{G} = (G, \pi, x_0)$ that is in some sense the union of $\mathcal{G}_1$ and $\mathcal{G}_2$. Then there should exist continuous fiber preserving maps $\varphi_1 : G_1 \to G$ with $\varphi_1(x_1) = x_0$, and $\varphi_2 : G_2 \to G$ with $\varphi_2(x_2) := x_0$. If $\alpha : [0,1] \to G_1$ and $\beta : [0,1] \to G_2$ are two continuous paths with $\alpha(0) = x_1$, $\beta(0) = x_2$ and $\pi_1 \circ \alpha = \pi_2 \circ \beta$, then $\gamma_1 := \varphi_1 \circ \alpha$ and $\gamma_2 := \varphi_2 \circ \beta$ are continuous paths in G with $\pi \circ \gamma_1 = \pi \circ \gamma_2$ and $\gamma_1(0) = \gamma_2(0) = x_0$. Due to the uniqueness of lifting, it follows that $\gamma_1 = \gamma_2$. This means that $\alpha(t)$ and $\beta(t)$ have to be glued for every $t \in [0,1]$. Unfortunately, this is an ambiguous rule. For example, we could say that $x \in G_1$ and $y \in G_2$ have to be glued if $\pi_1(x) = \pi_2(y)$. Then the desired property is fulfilled, but it may be that there are no paths α from x_1 to x and β from x_2 to y with $\pi_1 \circ \alpha = \pi_2 \circ \beta$.

Therefore, we proceed in the following way: Start with the disjoint union $G_1 \dot\cup G_2$, and take the "finest" equivalence relation $\sim$ on this set with the following property:

1. $x_1 \sim x_2$.
2. If there are continuous paths $\alpha : [0,1] \to G_1$ and $\beta : [0,1] \to G_2$ with $\alpha(0) = x_1$, $\beta(0) = x_2$, and $\pi_1 \circ \alpha = \pi_2 \circ \beta$, then $\alpha(1) \sim \beta(1)$.

One can equip $G := (G_1 \dot\cup G_2)/\sim$ with the structure of a Riemann domain. This will now be carried out in a more general context.

Let X be an arbitrary set. An equivalence relation on X is given by a partition $\mathscr{X} = \{X_\nu \,:\, \nu \in N\}$ of X into subsets with:

1. $\bigcup_{\nu \in N} X_\nu = X$.
2. $X_\nu \cap X_\mu = \varnothing$ for $\nu \neq \mu$.

The sets X_ν are the equivalence classes.

Now let a family $(\mathscr{X}_\iota)_{\iota \in I}$ of equivalence relations on X be given with $\mathscr{X}_\iota = \{X^\iota_{\nu_\iota} \,:\, \nu_\iota \in N_\iota\}$ for $\iota \in I$. We set $N := \prod_{\iota \in I} N_\iota$, and

$$X_\nu := \bigcap_{\iota \in I} X^\iota_{\nu_\iota}, \qquad \text{for } \nu := (\nu_\iota)_{\iota \in I} \in N.$$

Then $\mathscr{X} = \{X_\nu : \nu \in N\}$ is again an equivalence relation (simple exercise), and it is finer than any $\mathscr{X}_\iota$. This means that for every $\iota \in I$ and every $\nu \in N$, there is a $\nu_\iota \in N_\iota$ with $X_\nu \subset X^\iota_{\nu_\iota}$.

We apply this to the disjoint union $X = \dot{\bigcup}_{\lambda \in L} G_\lambda$, for a given family $(\mathcal{G}_\lambda)_{\lambda \in L}$ of Riemann domains $\mathcal{G}_\lambda = (G_\lambda, \pi_\lambda, x_\lambda)$ over $\mathbb{C}^n$ with distinguished point. An equivalence relation on X is said to have property (P) if the following hold:

1. $x_\lambda \sim x_\varrho$, for $\lambda, \varrho \in L$.
2. If $\alpha : [0,1] \to G_\lambda$ and $\beta : [0,1] \to G_\varrho$ are continuous paths with $\alpha(0) \sim \beta(0)$ and $\pi_\lambda \circ \alpha = \pi_\varrho \circ \beta$, then $\alpha(1) \sim \beta(1)$.

We consider the family of all equivalence relations on X with property (P). It is not empty, as seen above in the case of two domains. Therefore we can construct an equivalence relation (as above) that is finer than any equivalence relation with property (P). We denote it by $\sim_P$. It is clear that $\pi_\lambda(x) = \pi_\varrho(y)$ if $x \in G_\lambda$, $y \in G_\varrho$, and $x \sim_P y$. The relation $\sim_P$ also has property (P), and the elements of an equivalence class X_ν all lie over the same point $\mathbf{z} = \mathbf{z}(X_\nu)$. We define $\widetilde{G} := X/\sim_P$ and $\widetilde{\pi}(X_\nu) := \mathbf{z}(X_\nu)$. The equivalence class of all x_λ will be denoted by $\widetilde{x}$.

8.6 Lemma. *Let $y \in G_\lambda$ and $x \in G_\varrho$ be given with $\pi_\varrho(x) = \pi_\lambda(y) =: \mathbf{z}$. If we choose open neighborhoods $U = U(y) \subset G_\lambda$, $V = V(x) \subset G_\varrho$, and an open connected neighborhood $W = W(\mathbf{z})$ such that $\pi_\lambda : U \to W$ and $\pi_\varrho : V \to W$ are topological mappings, then for $\varphi := (\pi_\varrho|_V)^{-1} \circ \pi_\lambda : U \to V$ the following hold:*

1. *$\varphi(y) = x$.*
2. *If $x \sim_P y$, then $\varphi(y') \sim_P y'$ for every $y' \in U$.*

PROOF: The first statement is trivial. Now let $\alpha : [0,1] \to W$ be a continuous path with $\alpha(0) = \mathbf{z}$ and $\alpha(1) = \pi_\lambda(y')$ for some $y' \in U$. Then $\beta := (\pi_\lambda|_U)^{-1} \circ \alpha$ and $\gamma := \varphi \circ \beta$ are continuous paths in U and V with $\beta(0) = y \sim_P x = \varphi(y) = \gamma(0)$ and $\pi_\lambda \circ \beta = \pi_\varrho \circ \varphi \circ \beta = \pi_\varrho \circ \gamma$. Therefore, $y' = \beta(1) \sim_P \gamma(1) = \varphi(y')$. ∎

8.7 Theorem. *There is a topology on $\widetilde{G}$ such that*

$$\widetilde{\mathcal{G}} := \big(\widetilde{G}, \widetilde{\pi}, \widetilde{x}\big)$$

is a Riemann domain over $\mathbb{C}^n$ with distinguished point $\widetilde{x}$, and all maps $\varphi_\lambda : G_\lambda \to \widetilde{G}$ with

$$\varphi_\lambda(x) := \textit{equivalence class of } x$$

are continuous and fiber preserving.

PROOF: (1) Sets of the form $\varphi_\lambda(M)$ for M open in G_λ together with $\widetilde{G}$ constitute a base of a topology for $\widetilde{G}$. To see this it remains to show that the intersection of two such sets is again of this form.

Let $M \subset G_\lambda$ and $N \subset G_\varrho$ be open subsets. Then

$$\varphi_\lambda(M) \cap \varphi_\varrho(N) = \varphi_\varrho(N \cap \varphi_\varrho^{-1}(\varphi_\lambda(M))).$$

But $\varphi_\varrho^{-1}(\varphi_\lambda(M))$ is open in G_ϱ. In fact, let $x \in \varphi_\varrho^{-1}(\varphi_\lambda(M))$ be given, and $y \in M$ be chosen such that $\varphi_\lambda(y) = \varphi_\varrho(x)$ (and therefore $y \sim_P x$). Let $\mathbf{z} := \pi_\lambda(y) = \pi_\varrho(x)$. Then there exist open neighborhoods $U = U(y)$ and $V = V(x)$ and an open connected neighborhood $W = W(\mathbf{z})$ such that $\pi_\lambda : U \to W$ and $\pi_\varrho : V \to W$ are topological mappings. Let $\varphi := (\pi_\varrho|_V)^{-1} \circ \pi_\lambda : U \to V$. By the lemma, $\varphi(y) = x$ and $\varphi(y') \sim_P y'$ for every $y' \in U$.

So $V' := \varphi(M \cap U)$ is a neighborhood of x in G_ϱ, and since $\varphi_\varrho(\varphi(y')) = \varphi_\lambda(y')$ for every $y' \in U$, it follows that $V' \subset \varphi_\varrho^{-1}(\varphi_\lambda(M))$.

Consequently, every φ_λ is a continuous map.

(2) Remark: Since every $y \in \widetilde{G}$ is an equivalence class $\varphi_\lambda(x)$, we have

$$M = \bigcup_{\lambda \in L} \varphi_\lambda(\varphi_\lambda^{-1}(M)) \text{ for any subset } M \subset \widetilde{G}.$$

(3) $\widetilde{\pi} : \widetilde{G} \to \mathbb{C}^n$ is continuous: Let $V \subset \mathbb{C}^n$ be an arbitrary open set, and $M := \widetilde{\pi}^{-1}(V)$. Then $\varphi_\lambda^{-1}(M) = \pi_\lambda^{-1}(V)$ is open in G_λ, and therefore $M = \bigcup_{\lambda \in L} \varphi_\lambda(\varphi_\lambda^{-1}(M))$ is open in $\widetilde{G}$.

(4) $\widetilde{G}$ is a Hausdorff space: Let $y_1, y_2 \in \widetilde{G}$ with $y_1 \neq y_2$, and $\mathbf{z}_1 := \widetilde{\pi}(y_1)$, $\mathbf{z}_2 := \widetilde{\pi}(y_2)$.

There are two cases. If $\mathbf{z}_1 \neq \mathbf{z}_2$, then there are open neighborhoods $V_1(\mathbf{z}_1)$ and $V_2(\mathbf{z}_2)$ with $V_1 \cap V_2 = \varnothing$. Then $\widetilde{\pi}^{-1}(V_1)$ and $\widetilde{\pi}^{-1}(V_2)$ are disjoint open neighborhoods of y_1 and y_2. If $\mathbf{z}_1 = \mathbf{z}_2$, then we choose elements $x_1 \in G_\lambda$, $x_2 \in G_\varrho$ with $\varphi_\lambda(x_1) = y_1$ and $\varphi_\varrho(x_2) = y_2$, and since x_1 and x_2 are not equivalent, the above lemma implies that there are disjoint neigborhoods of y_1 and y_2.

(5) $\widetilde{G}$ is connected: Let $y = \varphi_\lambda(x)$ be an arbitrary point of $\widetilde{G}$. Then there is a continuous path $\alpha : [0,1] \to G_\lambda$ that connects the distinguished point x_λ to x. Then $\varphi_\lambda \circ \alpha$ connects $\widetilde{x}$ to y.

(6) $\widetilde{\pi}$ is locally topological: Let $y = \varphi_\lambda(x)$ be a point of $\widetilde{G}$, and $\mathbf{z} = \widetilde{\pi}(y) = \pi_\lambda(x)$. Then there exist open neighborhoods $U = U(x) \subset G_\lambda$ and $W = W(\mathbf{z}) \subset \mathbb{C}^n$ such that $\pi_\lambda : U \to W$ is a topological mapping. $\widetilde{U} := \varphi_\lambda(U)$ is an open neighborhood of y, with $\widetilde{\pi}(\widetilde{U}) = \pi_\lambda(U) = W$. In addition, $\widetilde{\pi}|_{\widetilde{U}}$ is injective, since $\widetilde{\pi} \circ \varphi_\lambda = \pi_\lambda$ and $\pi_\lambda|_U$ is injective.

(7) Clearly, the maps $\varphi_\lambda : G_\lambda \to \widetilde{G}$ are fiber preserving, and it was already shown that they are continuous. ■

Now $\widetilde{\mathcal{G}}$ has the following properties:

1. $\mathcal{G}_\lambda \prec \widetilde{\mathcal{G}}$, for every $\lambda \in L$.
2. If $\mathcal{G}^*$ is a domain over $\mathbb{C}^n$ with $\mathcal{G}_\lambda \prec \mathcal{G}^*$ for every λ, then $\widetilde{\mathcal{G}} \prec \mathcal{G}^*$.

PROOF: (of the second statement)

If $\mathcal{G}^*$ is given, then there exist fiber preserving mappings $\varphi^*_\lambda : G_\lambda \to G^*$. We introduce a new equivalence relation $\simeq$ on the disjoint union X of the G_λ by

$$x \simeq x' :\iff x \in G_\lambda,\ x' \in G_\varrho \text{ and } \varphi^*_\lambda(x) = \varphi^*_\varrho(x').$$

It follows from the uniqueness of lifting that $\simeq$ has the property (P). Now we define a map $\varphi : \widetilde{G} \to G^*$ by

$$\varphi(\varphi_\lambda(x)) := \varphi^*_\lambda(x).$$

Since $\sim_P$ is the finest equivalence relation with property (P), φ is well defined. Also it is clear that φ is continuous and fiber preserving. ■

Therefore $\widetilde{\mathcal{G}}$ is the smallest Riemann domain over $\mathbb{C}^n$ that contains all domains $\mathcal{G}_\lambda$.

Definition. The domain $\widetilde{\mathcal{G}}$ constructed as above is called the *union of the domains* $\mathcal{G}_\lambda$, and we write $\widetilde{\mathcal{G}} = \bigcup_{\lambda \in L} \mathcal{G}_\lambda$.

Special cases:

1. From $\mathcal{G}_1 \prec \mathcal{G}$ and $\mathcal{G}_2 \prec \mathcal{G}$ it follows that $\mathcal{G}_1 \cup \mathcal{G}_2 \prec \mathcal{G}$.
2. From $\mathcal{G}_1 \prec \mathcal{G}_2$ it follows that $\mathcal{G}_1 \cup \mathcal{G}_2 \cong \mathcal{G}_2$.
3. $\mathcal{G} \cup \mathcal{G} \cong \mathcal{G}$.
4. $\mathcal{G}_1 \cup \mathcal{G}_2 \cong \mathcal{G}_2 \cup \mathcal{G}_1$.
5. $\mathcal{G}_1 \cup (\mathcal{G}_2 \cup \mathcal{G}_3) \cong (\mathcal{G}_1 \cup \mathcal{G}_2) \cup \mathcal{G}_3$.

Example

Let $\mathcal{G}_1 = (G_1, \pi_1, x_1)$ be the Riemann surface of $\sqrt{z}$ with distinguished point $x_1 = (1, 1)$ and $\mathcal{G}_2 = (G_2, \mathrm{id}, x_2)$ the schlicht domain

$$G_2 = \left\{ z \in \mathbb{C} \,:\, \frac{1}{2} < |z| < 2 \right\}$$

with distinguished point $x_2 = 1$.

Then $\mathcal{G}_1 \cup \mathcal{G}_2 = (\widetilde{G}, \widetilde{\pi}, \widetilde{x}_0)$, where $\widetilde{G} = (G_1 \dot{\cup} G_2)/\sim_P$.

Let $y \in \pi_1^{-1}(G_2) \subset G_1$. Then we can connect y to the point x_1 by a path α in $\pi_1^{-1}(G_2)$, and $\pi_1(y)$ to x_2 by the path $\pi_1 \circ \alpha$ in G_2. But $x_1 \sim_P x_2$, so

$y \sim_P \pi_1(y)$ as well. This shows that over each point of G_2 there is exactly one equivalence class.

Now let $z \in \mathbb{C} - \{0\}$ be arbitrary. The line through z and 0 in $\mathbb{C}$ contains a segment $\alpha : [0,1] \to \mathbb{C}^*$ that connects z to a point $z^* \in G_2$. There are two paths α_1, α_2 in G_1 with $\pi_1 \circ \alpha_1 = \pi_1 \circ \alpha_2 = \alpha$. Since $\alpha_1(1) \sim_P \alpha_2(1)$, it follows that $\alpha_1(0) \sim_P \alpha_2(0)$.

Then it follows that $\mathcal{G}_1 \cup \mathcal{G}_2 = (\mathbb{C} - \{0\}, \mathrm{id}, 1)$.

Exercises

1. For $\mathbf{t} = (t_1, \ldots, t_n) \in \mathscr{V}$ define $\Phi_{\mathbf{t}} : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Phi_{\mathbf{t}}(z_1, \ldots, z_n) := \big(e^{\mathrm{i}t_1} z_1, \ldots, e^{\mathrm{i}t_n} z_n\big).$$

   A Riemann domain $\mathcal{G} = (G, \pi, x_0)$ is called a *Reinhardt domain over* $\mathbb{C}^n$ if $\pi(x_0) = \mathbf{0}$ and for every $\mathbf{t} \in \mathscr{V} - (\mathbb{C}^*)^n$ there is an isomorphism $\varphi_{\mathbf{t}} : \mathcal{G} \to \mathcal{G}$ with $\pi \circ \varphi_{\mathbf{t}} = \Phi_{\mathbf{t}} \circ \pi$. Prove:
   (a) If $G \subset \mathbb{C}^n$ is a proper Reinhardt domain, then $\mathcal{G} = (G, \mathrm{id}, \mathbf{0})$ is a Reinhardt domain over $\mathbb{C}^n$.
   (b) Let $G_1, G_2 \subset \mathbb{C}^2$ be defined by

$$\begin{aligned} G_1 &:= \mathsf{P}^2(\mathbf{0}, 1) - \Big\{(z, w) \,:\, |z| = \frac{1}{2} \text{ and } |w| \le \frac{1}{2}\Big\}, \\ G_2 &:= \Big\{(z, w) \in \mathsf{P}^2(\mathbf{0}, 1) \,:\, |w| < \frac{1}{2}\Big\}. \end{aligned}$$

   Gluing G_1 and G_2 along $\big\{(z, w) \,:\, \frac{1}{2} < |z| < 1 \text{ and } |w| < \frac{1}{2}\big\}$ one obtains a Riemann domain over $\mathbb{C}^2$ that is a Reinhardt domain over $\mathbb{C}^2$, but not schlicht. Show that this domain can be obtained as the union of $\mathcal{G}_1 = \Big(G_1, \mathrm{id}, \big(\frac{3}{4}, \frac{1}{4}\big)\Big)$ and $\mathcal{G}_2 = \Big(G_2, \mathrm{id}, \big(\frac{3}{4}, \frac{1}{4}\big)\Big)$.

2. Let $J = \{0, 1, 2, 3, \ldots\} \subset \mathbb{N}_0$ be a finite or infinite sequence of natural numbers and $P_i = \mathsf{P}^n(\mathbf{z}_i, r_i)$, $i \in J$, a sequence of polydisks in $\mathbb{C}^n$. Assume that for every pair $(i, j) \in J \times J$ an "incidence number" $\varepsilon_{ij} \in \{0, 1\}$ is given such that the following hold:
   (a) $\varepsilon_{ij} = \varepsilon_{ji}$ and $\varepsilon_{ii} = 1$.
   (b) $\varepsilon_{ij} = 0$ if $P_i \cap P_j = \varnothing$.
   (c) For every $i > 0$ in J there is a $j < i$ with $\varepsilon_{ij} = 1$.
   (d) If $P_i \cap P_j \cap P_k \ne \varnothing$ and $\varepsilon_{ij} = 1$, then $\varepsilon_{ik} = \varepsilon_{jk}$.

   Points $\mathbf{z} \in P_i$ and $\mathbf{w} \in P_j$ are called equivalent ($\mathbf{z} \sim \mathbf{w}$) if $\mathbf{z} = \mathbf{w}$ and $\varepsilon_{ij} = 1$. Prove that $G := \dot{\bigcup} P_i / \sim$ carries in a natural way the structure of a Riemann domain over $\mathbb{C}^n$.

   Let $\pi : G \to \mathbb{C}^n$ be the canonical projection and suppose that there is a point $\mathbf{z}_0 \in \bigcap_{i \in J} P_i$. Is there a point $x_0 \in G$ such that (G, π, x_0) can be written as the union of the Riemann domains $(P_j, \mathrm{id}, \mathbf{z}_0)$?

9. The Envelope of Holomorphy

Holomorphy on Riemann Domains

Definition. Let (G, π) be a domain over $\mathbb{C}^n$. A function $f : G \to \mathbb{C}$ is called *holomorphic at a point* $x \in G$ if there are open neighborhoods $U = U(x) \subset G$ and $V = V(\pi(x)) \subset \mathbb{C}^n$ such that $\pi|_U : U \to V$ is topological and $f \circ (\pi|_U)^{-1} : V \to \mathbb{C}$ is holomorphic. The function f is called *holomorphic on* G if f is holomorphic at every point $x \in G$.

Remark. A holomorphic function is always continuous. For schlicht domains in $\mathbb{C}^n$ the new notion of holomorphy agrees with the old one.

Definition. Let $\mathcal{G}_j = (G_j, \pi_j, x_j)$, $j = 1, 2$, be domains over $\mathbb{C}^n$ with distinguished point, and $\mathcal{G}_1 \prec \mathcal{G}_2$ by virtue of a continuous mapping $\varphi : G_1 \to G_2$. For every function f on G_2 we define $f|_{G_1} := f \circ \varphi$.

9.1 Proposition. *If $f : G_2 \to \mathbb{C}$ is holomorphic and $\mathcal{G}_1 \prec \mathcal{G}_2$, then $f|_{G_1}$ is holomorphic on G_1.*

PROOF: Trivial, since φ is a local homeomorphism with $\pi_2 \circ \varphi = \pi_1$. ∎

Definition.

1. Let (G, π) be a domain over $\mathbb{C}^n$ and $x \in G$ a point. If f is a holomorphic function defined near x, then the pair (f, x) is called a *local holomorphic function* at x.
2. Let (G_1, π_1), (G_2, π_2) be domains over $\mathbb{C}^n$, and $x_1 \in G_1$, $x_2 \in G_2$ points with $\pi_1(x_1) = \pi_2(x_2) =: \mathbf{z}$. Two local holomorphic functions (f_1, x_1), (f_2, x_2) are called *equivalent* if there exist open neighborhoods $U_1(x_1) \subset G_1$, $U_2(x_2) \subset G_2$, $V(\mathbf{z})$, and topological mappings $\pi_1 : U_1 \to V$, $\pi_2 : U_2 \to V$ with $f_1 \circ (\pi_1|_{U_1})^{-1} = f_2 \circ (\pi_2|_{U_2})^{-1}$.
3. The equivalence class of a local holomorphic function (f, x) is denoted by f_x.

Remark. If $(f_1)_{x_1} = (f_2)_{x_2}$, then clearly, $f_1(x_1) = f_2(x_2)$. In particular, if $G_1 = G_2$, $\pi_1 = \pi_2$, and $x_1 = x_2$, then it follows that f_1 and f_2 coincide in an open neighborhood of $x_1 = x_2$.

9.2 Proposition. *Let* (G_1, π_1), (G_2, π_2) *be domains over* $\mathbb{C}^n$, $\alpha_\lambda : [0,1] \to G_\lambda$ *continuous paths with* $\pi_1 \circ \alpha_1 = \pi_2 \circ \alpha_2$. *Additionally, let* f_λ *be holomorphic on* G_λ, *for* $\lambda = 1, 2$. *If* $(f_1)_{\alpha_1(0)} = (f_2)_{\alpha_2(0)}$, *then also* $(f_1)_{\alpha_1(1)} = (f_2)_{\alpha_2(1)}$.

PROOF: Let $M := \{t \in [0,1] : (f_1)_{\alpha_1(t)} = (f_2)_{\alpha_2(t)}\}$. Then $M \neq \varnothing$, since $0 \in M$. It is easy to see that M is open and closed in $[0,1]$, because of the identity theorem for holomorphic functions. So $M = [0,1]$. ∎

9.3 Proposition. *Let* $\mathcal{G}_j = (G_j, \pi_j, x_j)$, $j = 1, 2$, *be domains over* $\mathbb{C}^n$ *with distinguished point, and* $\mathcal{G}_1 \prec \mathcal{G}_2$. *Then for every holomorphic function* f *on* G_1 *there is at most one holomorphic function* F *on* G_2 *with* $F|_{G_1} = f$, *i.e., a possible holomorphic extension of* f *is uniquely determined.*

PROOF: Let F_1, F_2 be holomorphic extensions of f to G_2. We choose neighborhoods $U_\lambda(x_\lambda) \subset G_\lambda$ such that the given fiber-preserving map $\varphi : G_1 \to G_2$ maps U_1 topologically onto U_2. We have $F_j \circ \varphi|_{U_1} = f|_{U_1}$, for $j = 1, 2$, and therefore $F_1|_{U_2} = F_2|_{U_2}$. It follows that $(F_1)_{x_2} = (F_2)_{x_2}$. Since each point of G_2 can be joined to x_2, the equality $F_1 = F_2$ follows. ∎

Envelopes of Holomorphy

Definition. Let $\mathcal{G} = (G, \pi, x_0)$ be a domain over $\mathbb{C}^n$ with distinguished point and $\mathscr{F}$ a nonempty set of holomorphic functions on G.

Let $(\mathcal{G}_\lambda)_{\lambda \in L}$ be the system of all domains over $\mathbb{C}^n$ with the following properties:

1. $\mathcal{G} \prec \mathcal{G}_\lambda$ for every $\lambda \in L$.
2. For every $f \in \mathscr{F}$ and every $\lambda \in L$ there is a holomorphic function F_λ on G_λ with $F_\lambda|_G = f$.

Then $H_{\mathscr{F}}(\mathcal{G}) := \bigcup_{\lambda \in L} \mathcal{G}_\lambda$ is called the *$\mathscr{F}$-hull* of $\mathcal{G}$.

If $\mathscr{F} = \mathcal{O}(G)$ is the set of all holomorphic functions on G, then $H(\mathcal{G}) := H_{\mathcal{O}(G)}(\mathcal{G})$ is called the *envelope of holomorphy* of $\mathcal{G}$. If $\mathscr{F} = \{f\}$ for some holomorphic function f on G, then $H_f(\mathcal{G}) := H_{\{f\}}(\mathcal{G})$ is called the *domain of existence* of the function f.

9.4 Theorem. *Let* $\mathcal{G} = (G, \pi, x_0)$ *be a domain over* $\mathbb{C}^n$, $\mathscr{F}$ *a nonempty set of holomorphic functions on* G, *and* $H_{\mathscr{F}}(\mathcal{G}) = (\widehat{G}, \widehat{\pi}, \widehat{x}_0)$ *the* $\mathscr{F}$*-hull. Then the following hold:*

1. $\mathcal{G} \prec H_{\mathscr{F}}(\mathcal{G})$.
2. *For each function* $f \in \mathscr{F}$ *there exists exactly one holomorphic function* F *on* $\widehat{G}$ *with* $F|_G = f$.

3. *If* $\mathcal{G}_1 = (G_1, \pi_1, x_1)$ *is a domain over* $\mathbb{C}^n$ *such that* $\mathcal{G} \prec \mathcal{G}_1$ *and every function* $f \in \mathscr{F}$ *can be holomorphically extended to* G_1*, then* $\mathcal{G}_1 \prec H_{\mathscr{F}}(\mathcal{G})$.

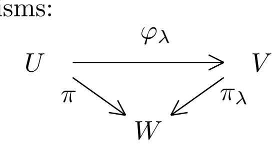
Proof: $H_{\mathscr{F}}(\mathcal{G})$ is the union of all Riemann domains $\mathcal{G}_\lambda = (G_\lambda, \pi_\lambda, x_\lambda)$ to which each function $f \in \mathscr{F}$ can be extended. We have fiber-preserving maps $\varphi_\lambda : G \to G_\lambda$ and $\widehat{\varphi}_\lambda : G_\lambda \to \widehat{G}$.

Let $\sim_P$ be the finest equivalence relation on $X := \dot{\bigcup}_{\lambda \in L} G_\lambda$ with property (P).[6] Then $\widehat{G}$ is the set of equivalence classes in X relative to $\sim_P$. We define a new equivalence relation $\simeq$ on X by

$$x \simeq x' \quad :\Longleftrightarrow \quad \begin{array}{l} x \in G_\lambda,\ x' \in G_\varrho,\ \pi_\lambda(x) = \pi_\varrho(x'),\ \text{and for each } f \in \mathcal{F} \\ \text{and its holomorphic extensions } F_1, F_2 \text{ on } G_\lambda,\ \text{respectively } G_\varrho, \\ \text{we have } (F_\lambda)_x = (F_\varrho)_{x'}. \end{array}$$

Then $\simeq$ has property (P):

(i) For any λ we can find open neighborhoods $U = U(x_0)$, $V = V(x_\lambda)$, and $W = W(\pi(x_0))$ such that all mappings in the following commutative diagram are homeomorphisms:



Then for $f \in \mathscr{F}$ and its holomorphic extension F_λ on G_λ we have that $F_\lambda \circ (\pi_\lambda|_V)^{-1} = F_\lambda \circ \varphi_\lambda \circ (\pi|_U)^{-1} = f \circ (\pi|_U)^{-1}$ is independent of λ. Therefore, all distinguished points x_λ are equivalent.

(ii) If $\alpha : [0,1] \to G_\lambda$ and $\beta : [0,1] \to G_\varrho$ are continuous paths with $\alpha(0) \simeq \beta(0)$ and $\pi_\lambda \circ \alpha = \pi_\varrho \circ \beta$, then $(F_\lambda)_{\alpha(0)} = (F_\varrho)_{\beta(0)}$. It follows that $(F_\lambda)_{\alpha(1)} = (F_\varrho)_{\beta(1)}$ as well, and therefore $\alpha(1) \simeq \beta(1)$.

Since $\mathcal{G} \prec \mathcal{G}_\lambda$ and $\mathcal{G}_\lambda \prec H_{\mathscr{F}}(\mathcal{G})$, it follows that $\mathcal{G} \prec H_{\mathcal{F}}(\mathcal{G})$. Furthermore, the fiber preserving map $\widehat{\varphi} := \widehat{\varphi}_\lambda \circ \varphi_\lambda$ does not depend on λ.

Now let a function $f \in \mathscr{F}$ be given. We construct a holomorphic extension F on $\widehat{G}$ as follows:

If $y \in \widehat{G}$ is an arbitrary point, then there is a $\lambda \in L$ and a point $y_\lambda \in G_\lambda$ such that $y = \widehat{\varphi}_\lambda(y_\lambda)$, and we define

$$F(y) := F_\lambda(y_\lambda).$$

If $y = \widehat{\varphi}_\varrho(y_\varrho)$ as well, then $y_\lambda \sim_P y_\varrho$, and therefore $y_\lambda \simeq y_\varrho$ as well. It follows that $F_\lambda(y_\lambda) = F_\varrho(y_\varrho)$, and F is well defined.

[6] For the definition of property (P) see page 92.

We have $F \circ \widehat{\varphi} = F \circ \widehat{\varphi}_\lambda \circ \varphi_\lambda = F_\lambda \circ \varphi_\lambda = f$ on G. This shows that F is an extension of f, and from the equation $F \circ \widehat{\varphi}_\lambda = F_\lambda$ it follows that F is holomorphic (since $\widehat{\varphi}_\lambda$ is locally topological).

The maximality of $H_{\mathscr{F}}(\mathcal{G})$ follows by construction. ∎

The $\mathscr{F}$-hull $H_{\mathscr{F}}(\mathcal{G})$ is therefore the largest domain into which all functions $f \in \mathscr{F}$ can be holomorphically extended.

9.5 Identity theorem. *Let $\mathcal{G}_j = (G_j, \pi_j, x_j)$, $j = 1, 2$, be domains over $\mathbb{C}^n$, and $\widetilde{\mathcal{G}} = \big(\widetilde{G}, \widetilde{\pi}, \widetilde{x}\big)$ the union of $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $f_j : G_j \to \mathbb{C}$ be holomorphic functions and $\mathcal{G} = (G, \pi, x)$ a domain with $\mathcal{G} \prec \mathcal{G}_j$ for $j = 1, 2$ such that $f_1|_G = f_2|_G$. Then there is a holomorphic function $\widetilde{f}$ on $\widetilde{G}$ with $\widetilde{f}|_{G_j} = f_j$, for $j = 1, 2$.*

Proof: Let $f := f_1|_G = f_2|_G$, and $\mathscr{F} := \{f\}$. Since $\mathcal{G}_1 \prec H_{\mathscr{F}}(\mathcal{G})$ and $\mathcal{G}_2 \prec H_{\mathscr{F}}(\mathcal{G})$, it follows that $\mathcal{G}_1 \cup \mathcal{G}_2 \prec H_{\mathscr{F}}(\mathcal{G})$.

Let $\widehat{f}$ be a holomorphic extension of f to $\widehat{G}$ (where $H_{\mathscr{F}}(\mathcal{G}) = \big(\widehat{G}, \widehat{\pi}, \widehat{x}\big)$), and $\widetilde{f} := \widehat{f}|_{\widetilde{G}}$. Then

$$\big(\widetilde{f}|_{G_j}\big)|_G = \widetilde{f}|_G = \big(\widehat{f}|_{\widetilde{G}}\big)|_G = \widehat{f}|_G = f.$$

Therefore, $\widetilde{f}|_{G_j}$ is a holomorphic extension of f to G_j. Due to the uniqueness of holomorphic extension, $\widetilde{f}|_{G_j} = f_j$ for $j = 1, 2$. ∎

Pseudoconvexity

Pseudoconvexity. Let $\mathsf{P}^n \subset \mathbb{C}^n$ be the unit polydisk, $(\mathsf{P}^n, \mathsf{H})$ a Euclidean Hartogs figure, and $\Phi : \mathsf{P}^n \to \mathbb{C}^n$ an injective holomorphic mapping. Then $(\Phi(\mathsf{P}^n), \Phi(\mathsf{H}))$ is a generalized Hartogs figure. $\mathcal{P} = (\mathsf{P}^n, \Phi, \mathbf{0})$ and $\mathcal{H} = (\mathsf{H}, \Phi, \mathbf{0})$ are Riemann domains with $\mathcal{H} \prec \mathcal{P}$. We regard the pair $(\mathcal{P}, \mathcal{H})$ as a generalized Hartogs figure.

9.6 Proposition. *Let (G, π) be a domain over $\mathbb{C}^n$, $(\mathcal{P}, \mathcal{H})$ a generalized Hartogs figure, and $x_0 \in G$ a point for which $\mathcal{H} \prec \mathcal{G} := (G, \pi, x_0)$.*

Then every holomorphic function f on G can be extended holomorphically to $\mathcal{G} \cup \mathcal{P}$.

The proposition follows immediately from the identity theorem.

> **Definition.** A domain (G, π) over $\mathbb{C}^n$ is called *Hartogs convex* if the fact that $(\mathcal{P}, \mathcal{H})$ is a generalized Hartogs figure and $x_0 \in G$ a point with $\mathcal{H} \prec \mathcal{G} := (G, \pi, x_0)$ implies $\mathcal{G} \cup \mathcal{P} \cong \mathcal{G}$.
>
> A domain $\mathcal{G} = (G, \pi, x_0)$ over $\mathbb{C}^n$ is called a *domain of holomorphy* if there exists a holomorphic function f on G such that its domain of existence is equal to $\mathcal{G}$.

Remark. If $G \subset \mathbb{C}^n$ is a schlicht domain, then the new definition agrees with the old one.

9.7 Theorem.

1. *If $\mathcal{G} = (G, \pi, x_0)$ is a domain over $\mathbb{C}^n$ and $\mathscr{F}$ a nonempty set of holomorphic functions on G, then $H_{\mathscr{F}}(\mathcal{G})$ is Hartogs convex.*
2. *Every domain of holomorphy is Hartogs convex.*

PROOF: Let $(\mathcal{P}, \mathcal{H})$ be a generalized Hartogs figure with $\mathcal{H} \prec H_{\mathscr{F}}(\mathcal{G})$. Then every function $f \in \mathscr{F}$ has a holomorphic continuation to $H_{\mathscr{F}}(\mathcal{G}) \cup \mathcal{P}$. Therefore, $H_{\mathscr{F}}(\mathcal{G}) \cup \mathcal{P} \prec H_{\mathscr{F}}(\mathcal{G})$. On the other hand, we also have $H_{\mathscr{F}}(\mathcal{G}) \prec H_{\mathscr{F}}(\mathcal{G}) \cup \mathcal{P}$. So $H_{\mathscr{F}}(\mathcal{G}) \cup \mathcal{P} \cong H_{\mathscr{F}}(\mathcal{G})$. ∎

A Riemann domain (G, π) is called *holomorphically convex* if for every infinite discrete subset $D \subset G$ there exists a holomorphic function f on G that is unbounded on D.

9.8 Theorem (Oka, 1953). *If a Riemann domain (G, π) is Hartogs pseudoconvex, it is holomorphically convex (and therefore a domain of holomorphy).*

This is the solution of Levi's problem for Riemann domains over $\mathbb{C}^n$. We cannot give the proof here.

It seems possible to construct the holomorphic hull by adjoining Hartogs figures (cf. H. Langmaak, [La60]). It is conceivable that such a construction may be realized with the help of a computer, but until now (spring 2002) no successful attempt is known. We assume that parallel computer methods are necessary.

Boundary Points

In the literature other notions of pseudoconvexity are used. We want to give a rough idea of these methods.

Definition. Let X be a topological space. A *filter (basis)* on X is a nonempty set $\mathcal{R}$ of subsets of X with the following properties:

1. $\varnothing \notin \mathcal{R}$.
2. The intersection of two elements of $\mathcal{R}$ contains again an element of the set $\mathcal{R}$.

Example

1. If x_0 is a point of X, then every fundamental system of neighborhoods of x_0 in X is a filter, called a *neighborhood filter* of x_0.

2. Let (x_n) be a sequence of points of X. If we define $S_N := \{x_n : n \geq N\}$, then $\mathcal{R} := \{S_N : N \in \mathbb{N}\}$ is the so-called *elementary filter* of the sequence (x_n). A filter is therefore the generalization of a sequence.

   Definition. A point $x_0 \in X$ is called a *cluster point* of the filter $\mathcal{R}$ if $x_0 \in \overline{A}$, for every $A \in \mathcal{R}$. The point x_0 is called a *limit* of the filter $\mathcal{R}$ if every element of a fundamental system of neighborhoods of x_0 contains an element of $\mathcal{R}$.

For sequences the new notions agree with the old ones.

If $f : X \to Y$ is a continuous map, then the image of any filter on X is a filter on Y, the so-called *direct image*.

Definition. Let (G, π) be a Riemann domain over $\mathbb{C}^n$. An *accessible boundary point* of (G, π) is a filter $\mathcal{R}$ on G with the following properties:

1. $\mathcal{R}$ has no cluster point in G.
2. The direct image $\pi(\mathcal{R})$ has a limit $\mathbf{z}_0 \in \mathbb{C}^n$.
3. For every connected open neighborhood $V = V(\mathbf{z}_0) \subset \mathbb{C}^n$ there is exactly one connected component of $\pi^{-1}(V)$ that belongs to $\mathcal{R}$.
4. For every element $U \in \mathcal{R}$ there is a neighborhood $V = V(\mathbf{z}_0)$ such that U is a connected component of $\pi^{-1}(V)$.

Remark. For a Hausdorff space X the following hold:

1. A filter in X has at most one limit.
2. If a filter in X has the limit x_0, then x_0 is the only cluster point of this filter.

(for a proof see Bourbaki, [Bou66], § 8.1)

Therefore, the limit $\mathbf{z}_0$ in the definition above is uniquely determined.

There is an equivalent description of accessible boundary points that avoids the filter concept. For this consider sequences (x_ν) of points of G with the following properties:

1. (x_ν) has no cluster point in G.
2. The sequence of the images $\pi(x_\nu)$ has a limit $\mathbf{z}_0 \in \mathbb{C}^n$.
3. For every connected open neighborhood $V = V(\mathbf{z}_0) \subset \mathbb{C}^n$ there is an $n_0 \in \mathbb{N}$ such that for $n, m \geq n_0$ the points x_n and x_m can be joined by a continuous path $\alpha : [0,1] \to G$ with $\pi \circ \alpha([0,1]) \subset V$.

Two such sequences $(x_\nu), (y_\nu)$ are called equivalent if:

1. $\lim_{\nu\to\infty} \pi(x_\nu) = \lim_{\nu\to\infty} \pi(y_\nu) = \mathbf{z}_0$.
2. For every connected open neighborhood $V = V(\mathbf{z}_0)$ there is an n_0 such that for $n, m \geq n_0$ the points x_n and y_m can be joined by a continuous path $\alpha : [0,1] \to G$ with $\pi \circ \alpha([0,1]) \subset V$.

An accessible boundary point is an equivalence class of such sequences.

Let $\breve{\partial} G$ be the set of all accessible boundary points of G. Even if G is schlicht, this set may be different from the topological boundary ∂G. There may be points in ∂G that are not accessible, and it may be happen that an accessible boundary point is the limit of two inequivalent sequences.

We define $\breve{G} := G \cup \breve{\partial} G$. If $r_0 = [x_n]$ is an accessible boundary point, we define a neighborhood of r_0 in $\breve{G}$ as follows: Take a connected open set $U \subset G$ such that almost all x_n lie in U and $\pi(U)$ is contained in a neighborhood of $\mathbf{z}_0 := \lim_{n\to\infty} \pi(x_n)$. Then add all boundary points $r = [y_n]$ such that almost all y_n lie in U and $\lim_{n\to\infty} \pi(y_n)$ is a cluster point of $\pi(U)$. With this neighborhood definition $\breve{G}$ becomes a Hausdorff space, and $\breve{\pi} : \breve{G} \to \mathbb{C}^n$ with

$$\breve{\pi}(x) := \begin{cases} \pi(x) & \text{if } x \in G, \\ \lim\limits_{n\to\infty} \pi(x_n) & \text{if } x = [x_n] \in \breve{\partial} G, \end{cases}$$

is a continuous mapping.

Definition. A boundary point $r \in \breve{\partial} G$ is called *removable* if there is a connected open neighborhood $U = U(r) \subset \breve{G}$ such that $(U, \breve{\pi})$ is a schlicht Riemann domain over $\mathbb{C}^n$ and $\breve{\partial} G \cap U$ is locally contained in a proper analytic subset of U.

A subset $M \subset \breve{\partial} G$ is called *thin* if for every $r_0 \in M$ there is an open neighborhood $U = U(r_0) \subset \breve{G}$ and a nowhere identically vanishing holomorphic function f on $U \cap G$ such that for every $r \in M \cap U$ there exists a sequence (x_n) in $U \cap G$ converging to r such that $\lim_{n\to\infty} f(x_n) = 0$.

Example

Let $G \subset \mathbb{C}^n$ be a (schlicht) domain and $A \subset G$ a nowhere dense analytic subset. Then every point of A is a removable boundary point of $G' := G - A$.

The points of the boundary of the hyperball $\mathsf{B}_r(\mathbf{0}) \subset \mathbb{C}^n$ are all not removable.

Let B be a ball in the affine hyperplane $H = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : z_0 = 1\}$, and $G \subset \mathbb{C}^{n+1} - \{\mathbf{0}\}$ the cone over B. Then every boundary point of G is not removable, since locally the boundary has real dimension $2n+1$. The set $M := \{\mathbf{0}\}$ is thin in the boundary, as is seen by choosing $f(z_0, \ldots, z_n) := z_0$.

Analytic Disks. Let (G, π) be a Riemann domain over $\mathbb{C}^n$. If $\varphi : \overline{\mathsf{D}} \to \breve{G}$ is a continuous mapping, $\breve{\pi} \circ \varphi : \mathsf{D} \to \mathbb{C}^n$ holomorphic, and $(\breve{\pi} \circ \varphi)'(\zeta) \neq 0$ for $\zeta \in \mathsf{D}$, then $S := \varphi(\mathsf{D})$ is called an *analytic disk* in $\breve{G}$. The set $bS := \varphi(\partial \mathsf{D})$ is called its *boundary*.

Let $I := [0,1]$ be the unit interval. A family $(S_t)_{t\in I}$ of analytic disks $\varphi_t(\mathsf{D})$ in $\breve{G}$ is called continuous if the mapping $(z,t) \mapsto \varphi_t(z)$ is continuous. It is called distinguished if $S_t \subset G$ for $0 \le t < 1$ and $bS_t \subset G$ for $0 \le t \le 1$.

> **Definition.** The domain G is called *pseudoconvex* if for every distinguished continuous family $(S_t)_{t\in I}$ of analytic disks in $\breve{G}$ it follows that $S_1 \subset G$.
>
> The domain G is called *pseudoconvex at* $r \in \breve{\partial} G$ if there is a neighborhood $U = U(r) \subset \breve{G}$ and an $\varepsilon > 0$ such that for every distinguished continuous family $(S_t)_{t\in I}$ of analytic disks in $\breve{G}$ with $\breve{\pi}(S_t) \subset \mathsf{B}_\varepsilon(\breve{\pi}(r))$ it follows that $S_t \cap U \subset G$ for $t \in I$.

As in $\mathbb{C}^n$ one can show that a Riemann domain is pseudoconvex if and only if it is Hartogs pseudoconvex.

9.9 Theorem (Oka). *A Riemann domain (G,π) is pseudoconvex if and only if it is pseudoconvex at every point $r \in \breve{\partial} G$.*

9.10 Corollary. *If (G,π) is a domain of holomorphy, then G is pseudoconvex at every accessible boundary point.*

The converse theorem is Oka's solution of Levi's problem.

Finally, we mention the following result:

9.11 Theorem. *Let (G,π) be a Riemann domain over $\mathbb{C}^n$, and $M \subset \breve{\partial} G$ a thin set of nonremovable boundary points. If G is pseudoconvex at every point of $\breve{\partial} G - M$, then G is pseudoconvex.*

PROOF: See [GrRe56], §3, Satz 4. ∎

Exercises

1. Prove that a Reinhardt domain $\mathcal{G}$ over $\mathbb{C}^n$ must be schlicht if it is a domain of holomorphy.
2. Prove that if (G,π) is a Reinhardt domain, then for every $f \in \mathcal{O}(G)$ there is a power series $S(\mathbf{z})$ at the origin such that $f(x) = S(\pi(x))$ for $x \in G$.
3. Prove that the envelope of holomorphy of a Reinhardt domain is again a Reinhardt domain.
4. Prove that the Riemann surface of the function $f(z) = \log(z)$ has just one boundary point over $0 \in \mathbb{C}$.
5. Find a schlicht Riemann domain in $\mathbb{C}^2$ whose envelope of holomorphy is not schlicht.
6. Construct a Riemann domain $\mathcal{G} = (G,\pi,x_0)$ over $\mathbb{C}^2$ such that for all $x,y \in \pi^{-1}(\pi(x_0))$ and every $f \in \mathcal{O}(G)$ the equality $f(x) = f(y)$ holds.