Chapter 3

Microlocalization

3.1 The Global FBI Transform

For $u \in \mathcal{S}'(\mathbf{R}^n)$ we would like to treat simultaneously the local behavior of u and that of its h-Fourier transform $\mathcal{F}_h u$ (in this case, we say that we study the *microlocal* behavior of u).

For this purpose, we set

$$\begin{aligned} Tu(x,\xi;h) &= \underbrace{2^{-\frac{n}{2}}(\pi h)^{-\frac{3n}{4}}}_{=\alpha_{n,h}} \int e^{i(x-y)\xi/h-(x-y)^2/2h} u(y)\,dy \\ &\stackrel{\text{def}}{=} \alpha_{n,h} \left\langle u_y, e^{i(x-y)\xi/h-(x-y)^2/2h} \right\rangle_{\mathcal{S}',\mathcal{S}}, \end{aligned} \tag{3.1.1}$$

which belongs to $C^\infty(\mathbf{R}^{2n})$. Tu is called the *Fourier–Bros–Iagolnitzer* (for short, FBI) transform of u, and it has been used by many authors and for many purposes, in particular by J. Sjöstrand, who has also developed a systematic study for it in [Sj1, Sj2] (see also [Del]).

A possible explanation for this definition relies on the uncertainty principle (1.1.5): Here one tries to have $\Delta x \sim \Delta\xi \sim \sqrt{h}$, so we start by localizing u near x up to $\mathcal{O}\left(\sqrt{h}\right)$ by multiplying it with the Gaussian function $e^{-(x-y)^2/2h}$. Then one also tries to localize $\mathcal{F}_h u$ near ξ by just taking the Fourier transform with respect to y (the multiplication by $e^{ix\xi/h}$ is done only for the convenience of having a convolution operator). The fact that we have in this way localized $\mathcal{F}_h u$ near ξ up to $\mathcal{O}\left(\sqrt{h}\right)$ can be seen in the relation

$$Tu(x,\xi;h) = e^{ix\xi/h} T\mathcal{F}_h u(\xi,-x),$$

the proof of which is postponed (see Remark 3.4.20). As we shall see hereinafter in Proposition 3.1.1, the coefficient $\alpha_{n,h}$ is just a normalization factor.

Notice that all this can also be done by just doing a convolution with a Gaussian function, because one has

$$Tu(x,\xi;h) = \alpha_{n,h} e^{-\xi^2/2h} \int e^{-(x-i\xi-y)^2/2h} u(y)dy,$$

that is, setting $z = x - i\xi \in \mathbf{C}^n$, we have

$$Tu(x,\xi;h) = \alpha_{n,h} e^{-\xi^2/2h} \widetilde{T}u(z;h), \tag{3.1.2}$$

where $\widetilde{T}u(z;h) = \displaystyle\int e^{-(z-y)^2/2h} u(y)dy$ is called the *Bargman* transform of u. Also, writing $\phi_\xi(x) = (\pi h)^{-n/4} e^{ix\xi/h - x^2/2h}$ (the so-called *coherent state* centered at $(0,\xi)$, see also Exercise 1 of this chapter), one has (denoting by $*$ the usual convolution of functions)

$$Tu(x,\xi;h) = (2\pi h)^{-n/2} u * \phi_\xi(x),$$

and since $\mathcal{F}_h \phi_\xi(\eta) = (\pi h)^{-n/4} e^{-(\eta-\xi)^2/2h}$, this operation is also equivalent to the multiplication of $\mathcal{F}_h u$ by a Gaussian function.

The main elementary properties of this transform are as follows:

Proposition 3.1.1

(i) *For all $u \in \mathcal{S}'(\mathbf{R}^n)$, $e^{\xi^2/h} Tu(x,\xi;h)$ is a holomorphic function of $z = x - i\xi$ on $\mathbf{C}^n$.*

(ii) *If $u \in L^2(\mathbf{R}^n)$, then $Tu \in L^2(\mathbf{R}^{2n})$ and*

$$\|Tu\|_{L^2(\mathbf{R}^{2n})} = \|u\|_{L^2(\mathbf{R}^n)}$$

(that is, T maps $L^2(\mathbf{R}^n)$ isometrically into $L^2(\mathbf{R}^{2n})$.

(iii) *For all $u \in \mathcal{S}'(\mathbf{R}^n)$,*

$$hD_x Tu = (\xi + ihD_\xi) Tu.$$

Proof

Assertion (i) is an immediate consequence of (3.1.2).

To prove (ii), it is enough to prove the equality for $u \in C_0^\infty(\mathbf{R}^n)$. Then for all $M, N > 0$ one has

$$\|Tu\|^2_{L^2\{|x|\leq M\ ;\ |\xi|\leq N\}}$$
$$= \frac{1}{2^n(\pi h)^{3n/2}} \int_{\substack{|x|\leq M\\ |\xi|\leq N}} e^{i(y'-y)\xi/h-(x-y)^2/2h-(x-y')^2/2h} u(y)\overline{u(y')}dy\,dy'\,d\xi\,dx.$$

But, using, e.g., (2.4.5), we see that

$$\frac{1}{(2\pi h)^n}\int_{|\xi|\leq N} e^{i(y'-y)\xi/h}d\xi \underset{N\to+\infty}{\longrightarrow} \delta(y'-y) \text{ in } \mathcal{D}'(\mathbf{R}^{2n}_{y,y'}),$$

and thus

$$\|Tu\|^2_{L^2\left(\substack{|x|\leq M\\ |\xi|\leq N}\right)} \underset{N\to+\infty}{\longrightarrow} (\pi h)^{-n/2}\int_{|x|\leq M} e^{-(x-y)^2/h}|u(y)|^2 dy\,dx.$$

Since we also have

$$\int_{\mathbf{R}^n} e^{-(x-y)^2/h}dx = \int_{\mathbf{R}^n} e^{-x^2/h}dx = h^{n/2}\int_{\mathbf{R}^n} e^{-x^2}dx = (\pi h)^{n/2},$$

we finally obtain, by the dominated convergence theorem,

$$\|Tu\|^2_{L^2(\mathbf{R}^{2n})} = \|u\|^2_{L^2(\mathbf{R}^n)},$$

from which the result follows by the density of $C_0^\infty(\mathbf{R}^n)$ in $L^2(\mathbf{R}^n)$.

Assertion (iii) is obtained immediately by differentiation under the summation sign. $\diamond$

Remark 3.1.2 Still writing $z = x - i\xi$, one has

$$\partial_x - i\partial_\xi = 2\frac{\partial}{\partial \bar{z}},$$

so that actually (iii) can be seen as a consequence of (3.1.2) and (i).

Remark 3.1.3 As one can see in Exercise 2 at the end of this chapter, the image of $L^2(\mathbf{R}^n)$ by T is given by

$$T(L^2(\mathbf{R}^n)) = L^2(\mathbf{R}^{2n}) \cap e^{-\xi^2/2h}\mathcal{H}(\mathbf{C}^n_{x-i\xi}), \tag{3.1.3}$$

where $\mathcal{H}(\mathbf{C}^n_{x-i\xi})$ denotes the space of holomorphic functions with respect to $x - i\xi \in \mathbf{C}^n$.

An immediate consequence of (ii) is the following:

Corollary 3.1.4 *For all $u \in L^2(\mathbf{R}^n)$, one has $u = T^*Tu$.*

Remark 3.1.5 Of course, this does not mean that $T : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^{2n})$ is invertible. Indeed, Remark 3.1.3 proves that it is not, and actually, one can show (see Exercise 2 of this chapter) that TT^* is the orthogonal projector onto $L^2(\mathbf{R}^{2n}) \cap e^{-\xi^2/2h}\mathcal{H}(\mathbf{C}^n_{x-i\xi})$.

As another more general consequence of Proposition 3.1.1, we have the following:

Proposition 3.1.6 *T maps $\mathcal{S}'(\mathbf{R}^n)$ into $\mathcal{S}'(\mathbf{R}^{2n})$ continuously, and has its image included in $\mathcal{S}'(\mathbf{R}^{2n}) \cap C^\infty(\mathbf{R}^{2n})$. Moreover, for all $u \in \mathcal{S}'(\mathbf{R}^n)$ one has*

$$u = T^*Tu,$$

where for $v \in \mathcal{S}'(\mathbf{R}^{2n}) \cap C^\infty(\mathbf{R}^{2n})$ we have set

$$T^*v(y) = \alpha_{n,h} \int e^{-i(x-y)\xi/h-(x-y)^2/2h} v(x,\xi)\,dx\,d\xi,$$

which has to be interpreted as an oscillatory integral with respect to the ξ-variables.

Remark 3.1.7 In fact, one can prove (see Exercise 2 of this chapter) that

$$T(\mathcal{S}'(\mathbf{R}^n)) = \mathcal{S}'(\mathbf{R}^{2n}) \cap e^{-\xi^2/2h}\mathcal{H}(\mathbf{C}^n_{x-i\xi}).$$

Proof Since we already know that $T^*T = 1$ on $L^2(\mathbf{R}^n) \supset \mathcal{S}(\mathbf{R}^n)$, it is enough by duality to see that $T : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^{2n})$ and $T^* : \mathcal{S}(\mathbf{R}^{2n}) \to \mathcal{S}(\mathbf{R}^n)$ are continuous. Setting

$$L = \frac{1}{1+\xi^2}(1 - i\xi h D_y)$$

we have for all $N \in \mathbf{N}$,

$$Tu(x,\xi) = \alpha_{n,h} \int e^{i(x-y)\xi/h} ({}^tL)^N (e^{-(x-y)^2/2h}u(y))dy,$$

and therefore, since $({}^tL)^N$ is of order N in D_y and has coefficients that are $\mathcal{O}(\langle\xi\rangle^{-N})$, we get for all $\alpha, \beta \in \mathbf{N}^n$,

$$\partial_x^\alpha \partial_\xi^\beta Tu = \alpha_{n,h} \int e^{i(x-y)\xi/h-(x-y)^2/h} \mathcal{O}\left(\langle\xi\rangle^{|\alpha|-N} \langle x-y\rangle^{|\alpha|+|\beta|+N} \sum_{|\gamma|\leq N} |\partial^\gamma u(y)| \right) dy.$$

Now, using that

$$\begin{cases} \langle x\rangle^k = \mathcal{O}(\langle y\rangle^k + \langle x-y\rangle^k), \\ \langle x-y\rangle^k e^{-(x-y)^2/2h} = \mathcal{O}(1), \end{cases}$$

for all $k \geq 0$, we get for any $k, k' \in \mathbf{N}$,

$$\begin{aligned} &\langle x\rangle^k \langle \xi\rangle^{k'} \partial_x^\alpha \partial_\xi^\beta Tu \\ &\quad = \int \mathcal{O}\left(\underbrace{\langle \xi\rangle^{|\alpha|+k'-N}}_{=\mathcal{O}(1) \text{ if } N\gg 1} \underbrace{\langle x-y\rangle^{k+|\alpha|+|\beta|+N} e^{-(x-y)^2/2h}}_{\in L^1(\mathbf{R}^n_y)} \langle y\rangle^k \sum_{|\gamma|\leq N} |\partial^\gamma u(y)| \right) dy \\ &\quad = \mathcal{O}\left(\sum_{|\gamma|\leq N} \sup_{y\in\mathbf{R}^n} \langle y\rangle^k |\partial^\gamma u(y)| \right) \end{aligned}$$

uniformly with respect to x and ξ, which proves that T maps $\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}(\mathbf{R}^{2n})$ continuously.

The same type of arguments (but actually in a simpler way and without integration by parts) also show that $T^* : \mathcal{S}(\mathbf{R}^{2n}) \to \mathcal{S}(\mathbf{R}^n)$ is continuous, and this finishes the proof of Proposition 3.1.6. $\diamond$

From the previous result, we see that if we know Tu on $\mathbf{R}^{2n}$, then we also know u. Moreover, since we have an explicit formula restoring u from Tu, we can also hope to derive properties of u just by knowing some properties of Tu. By definition, the local properties of Tu will be called *microlocal* properties of u.

For instance, the fact that $Tu = \mathcal{O}(h^\infty)$ near some point (x_0, ξ_0) will also be expressed by saying that u is *microlocally* $\mathcal{O}(h^\infty)$ near (x_0, ξ_0).

3.2 Microsupport

From now on, we are mainly interested in the exponential decay properties of Tu as h tends to 0. As we shall see, in some contexts (such as the semiclassical quantum mechanics) this will correspond to the exponential decay of u itself, but in another context (see [Sj1] and Remark 3.2.10 below), these properties of Tu are related to the (microlocal) analytic singularities of u.

Definition 3.2.8 *For $u \in \mathcal{S}'(\mathbf{R}^n)$ (h-dependent) and $(x_0, \xi_0) \in \mathbf{R}^{2n}$, we say that u is* **microlocally exponentially small** *near (x_0, ξ_0) if there exists some*

$\delta > 0$ such that

$$Tu(x,\xi;h) = \mathcal{O}(e^{-\delta/h})$$

uniformly for (x,ξ) in a neighborhood of (x_0,ξ_0), and $h > 0$ small enough. The complementary set of such points (x_0,ξ_0) is called the **microsupport** *of u, and is denoted* $\mathrm{MS}(u) \subset \mathbf{R}^{2n}$.

In other words, $\mathrm{MS}(u)$ is the subset of $\mathbf{R}^{2n}$ consisting of the points near which u is not microlocally exponentially small as $h \to 0$.

Remark 3.2.9 By definition, $\mathrm{MS}(u)$ is a *closed* subset of $\mathbf{R}^{2n}$.

Remark 3.2.10 In the case where u does not depend on h, $\mathrm{MS}(u)$ is closely related to the so-called *analytic wave front set* of u (see [Sj1]), which describes the microlocal analytic singularities of u. In fact, denoting by $\mathrm{WF}_a(u)$ this set, it is easy to prove that one then has (see Exercise 4 of Chapter 4)

$$\mathrm{MS}(u) = \mathrm{WF}_a(u) \,\cup\, [\mathrm{Supp}\, u \times \{0\}]\,.$$

Remark 3.2.11 In the definition, we have used the L^∞-norm of Tu in a real neighborhood of (x_0,ξ_0). In fact, using assertion (i) of Proposition 3.1.1, we see that $Tu(x,\xi)$ can be extended to a holomorphic function on $\mathbf{C}^{2n}$, and for all $x,\xi,t,\tau \in \mathbf{R}^n$, we have

$$Tu(x+it,\xi+i\tau) = e^{t^2/2h+\tau^2/2h-\xi(t+i\tau)/h}Tu(x+\tau,\xi-t). \tag{3.2.1}$$

As a consequence, $|Tu|$ will be exponentially small in a real neighborhood of (x_0,ξ_0) if and only if it is so in a *complex* neighborhood of (x_0,ξ_0), and by the Cauchy formulae, this is again equivalent to the fact that $\|Tu\|_{L^p(\mathcal{V})}$ is exponentially small for some $p \geq 1$ and some complex neighborhood $\mathcal{V}$ of (x_0,ξ_0). But still using (3.2.1), this is also equivalent to the fact that $\|Tu\|_{L^p(\mathcal{W})}$ is exponentially small for some $p \geq 1$ and some *real* neighborhood $\mathcal{W}$ of (x_0,ξ_0). Therefore, in Definition 3.2.8, one can equivalently replace the local L^∞-norm of Tu by any local L^p-norm, $p \geq 1$.

As we shall see, in practice it is often more convenient to try to obtain L^2-type estimates on Tu. Then the previous discussion shows that they will automatically give uniform estimates on Tu (and as well on all the derivatives of Tu).

Moreover, in the applications it is also sometimes useful to try to localize more in the x-variables than in the ξ-ones, or vice versa. A possible way to do that is to slightly modify the definition of T as follows: Fix $\mu > 0$, and for $u \in \mathcal{S}'(\mathbf{R}^n)$ set

$$\begin{aligned} T_\mu u(x,\xi;h) &= \mu^{\frac{n}{4}} 2^{-\frac{n}{2}} (\pi h)^{-\frac{3n}{4}} \int e^{i(x-y)\xi/h - \mu(x-y)^2/2h} u(y)dy \\ &= \mu^{-\frac{n}{2}} T u\left(x, \frac{\xi}{\mu}; \frac{h}{\mu}\right). \end{aligned} \tag{3.2.2}$$

Then $T_1 = T$, $\|T_\mu u\|_{L^2} = \|u\|_{L^2}$, and when $\mu \to 0_+$, $\mu^{-n/4} T_\mu u(x,\xi;h)$ tends (e.g., in $\mathcal{S}'(\mathbf{R}^{2n})$) to $(\pi h)^{-n/4} e^{ix\xi/h} \mathcal{F}_h u(\xi)$, while when $\mu \to +\infty$, $\mu^{-3n/4}\, T_\mu u(x,\xi;h)$ tends to $\alpha_{n,h} u(x)$. As a consequence, for μ small $T_\mu u$ localizes more in ξ than in x, and the contrary holds for μ large.

Now the question is to know whether the previous definition of $\mathrm{MS}(u)$ is related to the special choice $\mu = 1$ that we have made. The answer is no, as stated in the following result:

Proposition 3.2.12 *Let $u \in \mathcal{S}'(\mathbf{R}^n)$. Then for all $\mu, \mu' > 0$ and $(x_0, \xi_0) \in \mathbf{R}^{2n}$ one has that $T_\mu u$ is exponentially small near (x_0,ξ_0) (as $h \to 0_+$) if and only if the same is true for $T_{\mu'} u$.*

Proof We are going to show that $T_\mu u$ is exponentially small if and only if the same is true for $T_1 u$. Assume first that for some $\delta > 0$, $T_1 u = \mathcal{O}\left(e^{-\delta/h}\right)$ in a neighborhood $\mathcal{V}_0$ of (x_0,ξ_0). We know that $u = T_1^* T_1 u$, and thus

$$\begin{aligned} T_\mu u(x,\xi) &= (T_\mu T_1^*) T_1 u(x,\xi) \\ &= \mu^{\frac{n}{4}} \alpha_{n,h}^2 \int e^{(2i(x-y)\xi - \mu(x-y)^2 - 2i(z-y)\zeta - (z-y)^2)/2h} T_1 u(z,\zeta) dz\, d\zeta\, dy \\ &= \mu^{\frac{n}{4}} \alpha_{n,h}^2 \int e^{(2iy(\zeta-\xi) + 2i(x\xi - z\zeta) - \mu(x-y)^2 - (z-y)^2)/2h} T_1 u(z,\zeta) dz\, d\zeta\, dy. \end{aligned}$$

Now,

$$\begin{aligned} \mu(x-y)^2 + (z-y)^2 &= \mu x^2 + z^2 + (1+\mu) y^2 - 2y(\mu x + z) \\ &= (1+\mu)\left(y - \frac{\mu x + z}{1+\mu}\right)^2 + \frac{\mu}{1+\mu}(x-z)^2 \end{aligned}$$

and

$$\begin{aligned} \int e^{iy(\zeta-\xi)/h - (1+\mu)(y + \frac{\mu x + z}{1+\mu})^2/2h} dy &= e^{-i\frac{\mu x + z}{1+\mu}(\zeta-\xi)/h} \int e^{iy(\zeta-\xi)/h - (1+\mu)y^2/2h} dy \\ &= \left(\frac{2\pi h}{1+\mu}\right)^{\frac{n}{2}} e^{-i\frac{\mu x+z}{1+\mu}(\zeta-\xi)/h} e^{-(\xi-\zeta)^2/2(1+\mu)h}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
T_\mu u(x,\xi) &= \left(\frac{2\pi h\sqrt{\mu}}{1+\mu}\right)^{\frac{n}{2}} \alpha_{n,h}^2 \int e^{i(x\xi - z\zeta)/h - i\frac{\mu x + z}{1+\mu}(\zeta-\xi)/h} \\
&\qquad\qquad \times e^{-\frac{\mu}{1+\mu}(x-z)^2/2h - \frac{1}{(1+\mu)}(\xi-\zeta)^2/2h} T_1 u(z,\zeta)\, dz\, d\zeta,
\end{aligned}
$$

and taking $\chi \in C_0^\infty(\mathcal{V}_0)$ such that $\chi = 1$ near (x_0,ξ_0), we can write

$$
T_\mu u(x,\xi) = A(x,\xi) + B(x,\xi)
$$

with

$$
\begin{aligned}
A(x,\xi) &= \left(\frac{2\pi h\sqrt{\mu}}{1+\mu}\right)^{\frac{n}{2}} \alpha_{n,h}^2 \int \chi(z,\zeta) e^{i(x\xi - z\zeta)/h - i\frac{\mu x + z}{1+\mu}(\zeta-\xi)/h} \\
&\qquad\qquad \times e^{-\frac{\mu}{1+\mu}(x-z)^2/2h - \frac{1}{(1+\mu)}(\xi-\zeta)^2/2h} T_1 u(z,\zeta)\, dz d\zeta \\
&= \mathcal{O}(e^{-\delta/h}),
\end{aligned}
$$

where the last equality comes from the assumption we have made on $T_1 u$, and the fact that

$$
h^{\frac{n}{2}} \alpha_{n,h}^2 \int e^{-\frac{\mu}{1+\mu}(x-z)^2/2h - \frac{1}{(1+\mu)}(\xi-\zeta)^2/2h} dz\, d\zeta = \mathcal{O}(1)
$$

uniformly with respect to h. Moreover, if $(z,\zeta) \in \mathrm{Supp}\,(1-\chi)$, then $(x_0-z)^2 + (\xi_0-\zeta)^2 \geq \dfrac{1}{C}$ for some positive constant C, and therefore $(x-z)^2 + (\xi-\zeta)^2 \geq \dfrac{1}{2C}$ if (x,ξ) is sufficiently close to (x_0,ξ_0). As a consequence, we see that

$$
B(x,\xi) = \mathcal{O}(e^{-\delta'/h})
$$

with, e.g., $\delta' = \dfrac{\min(\mu,1)}{5(1+\mu)C} > 0$. Therefore, we get that $T_\mu u(x,\xi)$ is exponentially small near (x_0,ξ_0).

Conversely, the result follows in the same way by writing $T_1 u = (T_1 T_\mu^*) T_\mu u$. $\diamond$

Remark 3.2.13 Actually, the previous proof also shows that if $T_1 u$ satisfies an estimate of the type

$$
\| T_1 u(x,\xi;h) \|_{L^2(\mathcal{V})} \leq r(h),
$$

where $\mathcal{V}$ is a neighborhood of (x_0, ξ_0), then there exists $\delta > 0$ such that

$$\|T_\mu u(x, \xi; h)\|_{L^2(\mathcal{V}')} \leq \frac{2^{\frac{n}{2}}(1+\mu)^{\frac{n}{2}}}{\mu^{\frac{n}{4}}} r(h) + e^{-\delta/h}$$

uniformly for h small enough, where $\mathcal{V}'$ is a (possibly smaller) neighborhood of (x_0, ξ_0). As a consequence, we have a similar invariance for the analogues of $\mathrm{MS}(u)$ that are obtained by replacing the decay $\mathcal{O}(e^{-\delta/h})$ for the local L^2-norm of $T_1 u$ by $\mathcal{O}(h^\infty)$, $\mathcal{O}(h^s)$ or $\mathcal{O}(e^{-\delta/h^{1/\alpha}})$ ($s \in \mathbf{R}$ and $\alpha \geq 1$ fixed). These sets are respectively denoted by $\mathrm{FS}(u)$ (the *frequency set* of u: see also Section 2.9 and Exercise 3 at the end of this chapter), $\mathrm{FS}^{(s)}(u)$, and $\mathrm{MS}_\alpha(u)$. When u does not depend on h, they are associated with the microlocal C^∞ (respectively H^s and G_α) singularities of u, where H^s is the usual Sobolev space of order s, and G_α is the Gevrey space of order α.

Of course, other notions of microsupports or frequency sets can be considered, by modifying the choice of the local decay of $T_1 u$ as $h \to 0_+$. To obtain an invariant definition, however, it is necessary for this decay to be at most exponential.

Remark 3.2.14 The notion of $\mathrm{MS}(u)$ is *local*, in the sense that if u and v are two tempered distributions that coincide on some open set $\Omega \subset \mathbf{R}^n$, then $\mathrm{MS}(u) \cap (\Omega \times \mathbf{R}^n) = \mathrm{MS}(v) \cap (\Omega \times \mathbf{R}^n)$. This is an easy consequence of the presence of the Gaussian localization factor $e^{-(x-y)^2/2h}$ in the definition of T.

3.3 Action of the FBI Transform on ΨDOs

As we shall see in this section, a very pleasant property of the global FBI transform is that it transforms in a very explicit way the pseudodifferential operators (for short, ΨDOs) on $\mathbf{R}^n$ into pseudodifferential operators on $\mathbf{R}^{2n}$. This will be very useful for getting information on the solutions of partial differential equations (in particular, on their microsupport), since the combination of the following result with that of Section 3.5 will permit us to relate their properties to the geometric ones of the symbol of the equation.

For any symbol $p \in S_{2n}(1)$ as defined in Chapter 2, we have the following result:

Proposition 3.3.15 *For any $t \in [0,1]$, one has*

$$T \circ \mathrm{Op}_h^t(p) = \mathrm{Op}_h^t(\widetilde{p}) \circ T,$$

where $\widetilde{p} \in S_{4n}(1)$ is defined by

$$\widetilde{p}(x, \xi, x^*, \xi^*) = p(x - \xi^*, x^*).$$

Here x^* and ξ^* denote the dual variables of x and ξ, respectively, so that $\mathrm{Op}(x^*) = hD_x$ and $\mathrm{Op}(\xi^*) = hD_\xi$.

Remark 3.3.16 As one can notice, this formula is exact (that is, without any smaller remainder term), and the symbol $\widetilde{p}$ that is obtained does not depend on the choice of the quantization (i.e., on t).

Proof For $u \in C_0^\infty(\mathbf{R}^n)$, we have

$$\begin{aligned}&\mathrm{Op}_t(p(x-\xi^*, x^*))Tu(x,\xi) \\ &\quad = \frac{\alpha_{n,h}}{(2\pi h)^{2n}} \int_{\mathbf{R}^{5n}} e^{i\Phi/h} p((1-t)x + tx' - \xi^*, x^*) u(y) dy\, dx'\, d\xi'\, dx^*\, d\xi^*\end{aligned} \tag{3.3.1}$$

with

$$\Phi = (x - x')x^* + (\xi - \xi')\xi^* + (x' - y)\xi' + i(x' - y)^2/2.$$

Then integrating first with respect to ξ' and using the fact that

$$\int e^{i(x'-y-\xi^*)\xi'/h} d\xi' = (2\pi h)^n \delta_{\xi^* = (x'-y)}$$

we get from (3.3.1),

$$\begin{aligned}&\mathrm{Op}_t(p(x-\xi^*, x^*))Tu(x,\xi) \\ &\quad = \frac{\alpha_{n,h}}{(2\pi h)^n} \int e^{i\Phi_1/h} p((1-t)(x-x') + y, x^*) u(y) dy\, dx'\, dx^*\end{aligned} \tag{3.3.2}$$

with

$$\Phi_1 = (x' - y)\xi + (x - x')x^* + i(x' - y)^2/2.$$

Finally, making the change of variables $x' \mapsto z = x - x' + y$ in (3.3.2), we obtain

$$\mathrm{Op}_t(p(x-\xi^*, x^*))Tu(x,\xi) = \frac{\alpha_{n,h}}{(2\pi h)^n} \int e^{i\Phi_2/h} p((1-t)z + ty, x^*) u(y) dy\, dz\, dx^*$$

with

$$\Phi_2 = (x - z)\xi + i(x - z)^2/2 + (z - y)x^*$$

and therefore

$$\mathrm{Op}_t(p(x-\xi^*, x^*))Tu(x,\xi) = T(\mathrm{Op}_h^t(p)u)(x,\xi).$$

$\diamond$

3.4 Action of the FBI Transform on FIOs

In Remark 2.5.2 we already have had a taste of Fourier integral operators (for short, FIOs). In this paragraph we consider a special kind of FIOs, namely those associated with *linear canonical transformations* in a sense that will become clear only in Chapter 5 (see in particular Exercise 1 of Chapter 5): Here we must stress the fact that this section does not contain results used in the sequel, and it can therefore be skipped at a first reading. However, it is interesting to note that the considerations of this section always lead to completely explicit and exact formulae (such FIOs are also related to the so-called *exact Egorov theorem*, an example of which is given in Exercise 10 of Chapter 4). Moreover, these results can be useful in some problems requiring local canonical changes of variables. For similar considerations one may also consult [Fo].

In order to describe the action of the FBI transform on FIOs, we need to generalize again the class of FBI transforms we work with. If $A = A_1 + iA_2$ is a symmetric $n \times n$ matrix such that $A_1 = (A + \overline{A})/2$ is positive definite, we set, for $x \in \mathbf{R}^n$,

$$q_A(x) = \langle Ax, x\rangle \,,$$

and we set, for $u \in \mathcal{S}'(\mathbf{R}^n)$,

$$T_A u(x,\xi;h) = (\det A_1)^{\frac{1}{4}}\alpha_{n,h}\int e^{i(x-y)\xi/h - q_A(x-y)/2h}u(y)dy. \tag{3.4.1}$$

Then we see that

$$T_A u(x,\xi;h) = e^{i\langle A_2 x,x\rangle/2h}T_{A_1}(e^{-i\langle A_2 y,y\rangle/2h}u)(x,\xi - A_2 x)$$

and

$$T_{A_1}u(x,\xi;h) = (\det A_1)^{-\frac{1}{4}}T(u\circ A_1^{-\frac{1}{2}})(A_1^{\frac{1}{2}}x, A_1^{-\frac{1}{2}}\xi;h)$$

(where $T = T_I$ is the usual FBI transform defined previously). As a consequence T_A is an isometry from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{2n})$, too.

At first, we consider the three following types of (unitary) Fourier integral operators (next, we shall anyway limit our study to FIOs with real quadratic phase):

Type 1:

$$J_B \;:\; u \mapsto |\det B|^{\frac{1}{2}}(u\circ B),$$

where B is an invertible $n \times n$ matrix.

Type 2:

$$K_C \ : \ u \mapsto K_C u(x) = e^{-i\langle Cx,x\rangle/2h}u(x),$$

where C is a real symmetric $n \times n$ matrix.

Type 3:

$$L_j \ : \ u \mapsto L_j u(x_1,\dots,x_{j-1},\xi_j,x_{j+1},\dots,x_n) = \frac{1}{\sqrt{2\pi h}}\int e^{-ix_j\xi_j/h}u(x)dx_j,$$

where $j \in \{1,\dots,n\}$.

With these operators we associate respectively the three following transformations on $\mathbf{R}^{2n}$:

Type 1:

$$j_B \ : \ (x,\xi) \mapsto (Bx, {}^tB^{-1}\xi);$$

Type 2:

$$k_C \ : \ (x,\xi) \mapsto (x, \xi + Cx);$$

Type 3:

$$\ell_j \ : \ (x,\xi) \mapsto ((x_1,\dots,x_{j-1},-\xi_j,x_{j+1},\dots,x_n),(\xi_1,\dots,\xi_{j-1},x_j,\xi_{j+1},\dots,\xi_n)).$$

As is easy to verify, these transformations have the property of leaving unchanged the so-called *canonical 2-form* (or *canonical symplectic form*) σ on $\mathbf{R}^{2n}$, defined by

$$\sigma((x,\xi),(y,\eta)) = \xi y - x\eta,$$

in the sense that if κ denotes any one of them, it satisfies

$$\sigma(\kappa(X),\kappa(Y)) = \sigma(X,Y) \tag{3.4.2}$$

for all $X, Y \in \mathbf{R}^{2n}$. For this reason, such transformations are called *canonical* or *symplectic* transformations (see also Chapter 5 for more general considerations about this type of transformations). Moreover, it can be shown that the group of all the linear symplectic transformations on $\mathbf{R}^{2n}$ (i.e. satisfying (3.4.2)) is generated by those belonging to the three previous types; that is, any linear symplectic transformation on $\mathbf{R}^{2n}$ can be written as the composition of a finite number of j_B's, k_C's, and ℓ_j's (see, e.g., [Fo], Proposition (4.10)).

If κ is any linear canonical transformation on $\mathbf{R}^{2n}$, we define for $(x,\xi) \in \mathbf{R}^{2n}$,

$$\boxed{\theta_\kappa(x,\xi) := \frac{1}{2}\,(x\xi - y\eta)\,\Big|_{(y,\eta)=\kappa(x,\xi)},} \tag{3.4.3}$$

and we notice that if κ_1 and κ_2 are two such transformations, then

$$\theta_{\kappa_1\circ\kappa_2}(x,\xi) = \frac{1}{2}\,(x\xi - y\eta)\,\Big|_{(y,\eta)=\kappa_2(x,\xi)} + \frac{1}{2}\,(y\eta - z\zeta)\,\Big|_{\substack{(y,\eta)=\kappa_2(x,\xi)\\(z,\zeta)=\kappa_1(y,\eta)}}$$

and therefore

$$\theta_{\kappa_1\circ\kappa_2} = \theta_{\kappa_2} + \theta_{\kappa_1}\circ\kappa_2. \tag{3.4.4}$$

In the three particular cases above, we get

$$\theta_{j_B} = 0;\ \ \theta_{k_C}(x,\xi) = -\frac{1}{2}\,\langle Cx, x\rangle\,;\ \ \theta_{\ell_j}(x,\xi) = x_j\xi_j. \tag{3.4.5}$$

As we shall see, these functions will appear as phase shifts when we make T_A act on J_B, K_C, or L_j, respectively. Writing $A = (a_{j,k})_{1\le j,k\le n}$ we also define

$$\begin{aligned}
\mathcal{M}_{j_B}(A) &= {}^tB^{-1}AB^{-1},\\
\mathcal{M}_{k_C}(A) &= A + iC,\\
\mathcal{M}_{\ell_j}(A) &= \widetilde{A}_j - \frac{1}{a_{j,j}}\,(a_{j,k}a_{j,l}(1-\delta_{j,k})(1-\delta_{j,l}))_{1\le k,l\le n}\,,
\end{aligned}$$

where $\delta_{j,k}$ is the usual Kronecker symbol and $\widetilde{A}_j$ is obtained from A by substituting

$$R_j = \frac{1}{a_{j,j}}(-ia_{j,1},\ldots,-ia_{j,j-1},1,-ia_{j,j+1},\ldots,-ia_{j,n})$$

into its jth row, and ${}^tR_j$ into its jth column.

In fact, there is a more systematic way to define $\mathcal{M}_\kappa(A)$ (which can be extended for any linear canonical transformation κ), as can be seen from the following result:

Lemma 3.4.17 *For $\kappa \in \{j_B, k_C, \ell_j\}$, one has*

$$\boxed{\kappa\Big(\{(x, i\overline{A}x)\ ;\ x\in\mathbf{C}^n\}\Big) = \{(y, i\overline{\mathcal{M}_\kappa(A)}y)\ ;\ y\in\mathbf{C}^n\}.} \tag{3.4.6}$$

More generally, for any linear canonical transformation κ on $\mathbf{R}^{2n}$, there exists a unique symmetric matrix $\mathcal{M}_\kappa(A)$ such that $\mathrm{Re}\mathcal{M}_\kappa(A)$ is positive definite and (3.4.6) is valid.

Proof If $\kappa \in \{j_B, k_C, \ell_j\}$, the identity (3.4.6) can be verified by a straightforward computation, and is left as an exercise to the reader. Now let us consider the case where κ is a general linear canonical transformation on $\mathbf{R}^{2n}$. For $X = (x, i\overline{A}x)$ with $x \in \mathbf{C}^n$, we have (extending σ on $\mathbf{C}^{2n} \times \mathbf{C}^{2n}$ by $\mathbf{C}$-linearity)

$$\sigma(X, \overline{X}) = iAx \cdot \overline{x} + ix \cdot \overline{A}\overline{x},$$

and therefore, since A is symmetric and $\mathrm{Re}A$ is positive definite,

$$\frac{1}{2i}\sigma\left(X, \overline{X}\right) = (\mathrm{Re}A)x \cdot \overline{x} \geq \frac{1}{C_0}|x|^2 \tag{3.4.7}$$

for some positive constant C_0. Now, extending κ to $\mathbf{C}^{2n}$ by $\mathbf{C}$-linearity, we also have

$$\sigma\left(\kappa(X), \overline{\kappa(X)}\right) = \sigma\left(\kappa(X), \kappa\left(\overline{X}\right)\right) = \sigma\left(X, \overline{X}\right), \tag{3.4.8}$$

where the last equality comes from the fact that κ is canonical. On the other hand, writing

$$(y, \eta) = \kappa(X),$$

one has

$$\sigma\left(\kappa(X), \overline{\kappa(X)}\right) = \eta \cdot \overline{y} - y \cdot \overline{\eta} = 2i\ \mathrm{Im}(\eta \cdot \overline{y}). \tag{3.4.9}$$

In particular, we deduce from (3.4.7)-(3.4.9) that for any $x \in \mathbf{C}^n$, if $(y, \eta) = \kappa(x, i\overline{A}x)$, then

$$|x|^2 \leq C_0 \mathrm{Im}(\eta \cdot \overline{y}). \tag{3.4.10}$$

As a consequence, setting $y = 0$ in (3.4.10) we deduce from it that

$$\kappa\left(\{(x, i\overline{A}x)\ ;\ x \in \mathbf{C}^n\}\right)\ \cap\ \{0\} \times \mathbf{C}^n = \{(0, 0)\}.$$

This means that the subspace $\kappa\left(\{(x, i\overline{A}x)\ ;\ x \in \mathbf{C}^n\}\right)$ of $\mathbf{C}^{2n}$ is transversal to $\{y = 0\}$, and thus there exists a (unique) matrix $\mathcal{M}_\kappa(A)$ such that

$$\kappa\left(\{(x, i\overline{A}x)\ ;\ x \in \mathbf{C}^n\}\right) = \{(y, i\overline{\mathcal{M}_\kappa(A)}y)\ ;\ y \in \mathbf{C}^n\}.$$

It remains to show that $\mathcal{M}_\kappa(A)$ is symmetric and has a positive definite real part. The fact that it is symmetric is just a consequence of the identity

$$\sigma(\kappa(x, i\overline{A}x), \kappa(x', i\overline{A}x')) = \sigma((x, i\overline{A}x), (x', i\overline{A}x')) = i\overline{A}x.x' - x.i\overline{A}x' = 0$$

for any $x, x' \in \mathbf{C}^n$, which gives

$$0 = \sigma((y, i\overline{\mathcal{M}_\kappa(A)}y), (y', i\overline{\mathcal{M}_\kappa(A)}y')) = i\overline{\mathcal{M}_\kappa(A)}y.y' - y.i\overline{\mathcal{M}_\kappa(A)}y'$$

for any $y, y' \in \mathbf{C}^n$. (More generally, a subspace $\Lambda \subset \mathbf{C}^{2n}$ is said to be *isotropic* if the application $\Lambda^2 \ni (X, Y) \mapsto \sigma(X, Y)$ vanishes identically, and it is clear that such a property is conserved by canonical transformations; if, moreover, $\dim\Lambda = n$, then Λ is said to be *Lagrangian*, and this is again conserved by canonical transformations.)

Finally, using (3.4.10), we get that for any $y \in \mathbf{R}^n \backslash \{0\}$, one has

$$\mathrm{Im}\,(i\overline{\mathcal{M}_\kappa(A)}y \cdot y) > 0,$$

that is,

$$\mathrm{Re}\mathcal{M}_\kappa(A)y \cdot y > 0,$$

which means that $\mathrm{Re}\mathcal{M}_\kappa(A)$ is positive definite. $\diamond$

Remark 3.4.18 If κ_1 and κ_2 are two linear canonical transformations on $\mathbf{R}^{2n}$, Lemma 3.4.17 (applied to $\mathcal{M}_{\kappa_2}(A)$ instead of A) permits us to define the matrix $\mathcal{M}_{\kappa_1}(\mathcal{M}_{\kappa_2}(A))$. Then by construction one also has

$$\mathcal{M}_{\kappa_1}(\mathcal{M}_{\kappa_2}(A)) = \mathcal{M}_{\kappa_1 \circ \kappa_2}(A). \tag{3.4.11}$$

Now we prove the following result:

Proposition 3.4.19 *If κ denotes any one of the previous transformations j_B, k_C, or ℓ_j, denote by $\mathcal{J}_\kappa$ the corresponding operator J_B, K_C, or L_j, respectively. Then for all $u \in C_0^\infty(\mathbf{R}^n)$ one has*

$$T_A \mathcal{J}_\kappa u = \beta e^{i\theta_\kappa/h} \left(T_{\mathcal{M}_\kappa(A)} u\right) \circ \kappa,$$

where $\beta = 1$ if $\kappa \in \{j_B, k_C\}$, and $\beta = \sqrt{\dfrac{|a_{j,j}|}{a_{j,j}}}$ if $\kappa = \ell_j$ (here the determination of the square root on $\mathbf{R}_+^* + i\mathbf{R}$ is the one that assigns positive numbers to positive numbers).

Proof Let us start with the first type. By the change of variables $y \mapsto z = By$, we have

$$
\begin{aligned}
&T_A J_B u(x,\xi)\\
&\quad= \alpha_{n,h}(\det A_1)^{\frac14}|\det B|^{\frac12}\int e^{i(x-y)\xi/h-q_A(x-y)/2h}u(By)dy\\
&\quad= \alpha_{n,h}(\det A_1)^{\frac14}|\det B|^{-\frac12}\int e^{i(x-B^{-1}z)\xi/h-q_A(x-B^{-1}z)/2h}u(z)dz\\
&\quad= \alpha_{n,h}(\det{}^tB^{-1}A_1B^{-1})^{\frac14}\int e^{i(Bx-z)^tB^{-1}\xi/h-q_{{}^tB^{-1}AB^{-1}}(Bx-z)/2h}u(z)dz\\
&\quad= T_{{}^tB^{-1}AB^{-1}}u(Bx,{}^tB^{-1}\xi).
\end{aligned}
$$

For the second type, we use the fact that

$$
q_A(x-y)+i\langle Cy,y\rangle = q_{A+iC}(x-y)-i\langle Cx,x\rangle+2i\langle Cx,y\rangle,
$$

which gives

$$
\begin{aligned}
&T_A K_C u(x,\xi)\\
&\quad= \alpha_{n,h}(\det A_1)^{\frac14}\int e^{i(x-y)\xi/h+i\langle Cx,x\rangle/2h-i\langle Cx,y\rangle/h-q_{A+iC}(x-y)/2h}u(y)dy\\
&\quad= \alpha_{n,h}(\det A_1)^{\frac14}e^{-i\langle Cx,x\rangle/2h}\int e^{i(x-y)(\xi+Cx)/h-q_{A+iC}(x-y)/2h}u(y)dy\\
&\quad= e^{-i\langle Cx,x\rangle/2h}T_{A+iC}u(x,\xi+Cx).
\end{aligned}
$$

For the third type, by a permutation of the variables (and an application of the result for the first type), we can assume that $j=1$. Then denoting $y'=(y_2,\ldots,y_n)\in\mathbf{R}^{n-1}$ for $y=(y_1,y_2,\ldots,y_n)\in\mathbf{R}^n$, we have

$$
\begin{aligned}
&T_A L_1 u(x,\xi)\\
&\quad= \frac{\alpha_{n,h}(\det A_1)^{\frac14}}{\sqrt{2\pi h}}\int e^{i(x-y)\xi/h-q_A(x-y)/2h-iy_1z_1/h}u(z_1,y')dz_1\,dy\\
&\quad= \frac{\alpha_{n,h}(\det A_1)^{\frac14}}{\sqrt{2\pi h}}\int e^{[i(x_1-y_1)(\xi_1+z_1)-q_A(x-y)/2-ix_1z_1+i(x'-y')\xi']/h}u(z_1,y')dz_1\,dy\\
&\quad= \frac{\alpha_{n,h}(\det A_1)^{\frac14}}{\sqrt{2\pi h}}\int e^{iy_1(\xi_1+z_1)/h-q_A(y)/2h-ix_1z_1/h+iy'\xi'/h}u(z_1,x'-y')dz_1\,dy,
\end{aligned}
$$

where the last equality comes from the change of variables: $y\mapsto x-y$. Now, $q_A(y)=a_{1,1}y_1^2+2\sum\limits_{j\ge2}a_{1,j}y_1y_j+\sum\limits_{j,k\ge2}a_{j,k}y_jy_k$ and since $\mathrm{Re}\,a_{1,1}>0$, we have

$$
\int e^{iy_1(\xi_1+z_1)/h-a_{1,1}y_1^2/2h-\sum_{j\ge2}a_{1,j}y_1y_j/h}dy_1=\sqrt{\frac{2\pi h}{a_{1,1}}}e^{-(\xi_1+z_1+i\sum_{j\ge2}a_{1,j}y_j)^2/2a_{1,1}h}.
$$

Therefore,

$$
\begin{aligned}
&T_A L_1 u(x,\xi)\\
&\quad = \frac{\alpha_{n,h}(\mathrm{det}A_1)^{\frac14}}{\sqrt{a_{1,1}}}\int e^{[-(\xi_1+z_1+i\sum_{j\geq 2}a_{1,j}y_j)^2/a_{1,1}-\sum_{j,k\geq 2}a_{j,k}y_jy_k-2ix_1z_1+2iy'\xi']/2h}\\
&\qquad\qquad \times u(z_1,x'-y')dz_1\,dy'. \qquad (3.4.12)
\end{aligned}
$$

On the other hand, by the same change of variables we have

$$
\begin{aligned}
&e^{ix_1\xi_1/h}T_{\mathcal{M}_{\ell_1}(A)}u(-\xi_1,x';x_1,\xi')\\
&\quad = \alpha_{n,h}\left(\det \mathrm{Re}\mathcal{M}_{\ell_1}(A)\right)^{\frac14}\int e^{-ix_1z_1/h+iy'\xi'/h-q_{\mathcal{M}_{\ell_1}(A)}(-\xi_1-z_1,y')/2h}\\
&\qquad\qquad \times u(z_1,x'-y')dz_1\,dy', \qquad (3.4.13)
\end{aligned}
$$

and using the definition of $\mathcal{M}_{\ell_1}(A)$, we see that

$$
\begin{aligned}
&q_{\mathcal{M}_{\ell_1}(A)}(-\xi_1-z_1,y')\\
&\quad = \frac{1}{a_{1,1}}(\xi_1+z_1)^2+2i\sum_{j\geq 2}\frac{a_{1,j}}{a_{1,1}}(\xi_1+z_1)y_j+\sum_{j,k\geq 2}\left(a_{j,k}-\frac{a_{1,j}a_{1,k}}{a_{1,1}}\right)y_jy_k\\
&\quad = \left(\xi_1+z_1+i\sum_{j\geq 2}a_{1,j}y_j\right)^2/a_{1,1}+\sum_{j,k\geq 2}a_{j,k}y_jy_k. \qquad (3.4.14)
\end{aligned}
$$

We deduce from (3.4.12)-(3.4.14) that there exists a complex constant γ such that

$$T_A L_1 u(x,\xi)=\gamma e^{ix_1\xi_1/h}T_{\mathcal{M}_{\ell_1}(A)}u(-\xi_1,x';x_1,\xi'). \qquad (3.4.15)$$

Moreover, since both $T_A\circ L_1$ and $T_{\mathcal{M}_{\ell_1}(A)}$ are isometries from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{2n})$, we have necessarily

$$|\gamma|=1, \qquad (3.4.16)$$

and applying (3.4.15) to $u=\delta$ (the Dirac measure at 0) and $(x,\xi)=(0,0)$, we get in particular (using also (3.4.12))

$$\frac{(\mathrm{det}A_1)^{\frac14}}{\sqrt{a_{1,1}}}=\gamma\left(\det \mathrm{Re}\mathcal{M}_{\ell_1}(A)\right)^{\frac14}. \qquad (3.4.17)$$

As a consequence, $(\gamma\sqrt{a_{1,1}}) \in \mathbf{R}_+^*$, and thus by (3.4.16),

$$\gamma = \sqrt{\frac{|a_{1,1}|}{a_{1,1}}}. \tag{3.4.18}$$

In view of (3.4.15), this finishes the proof of Proposition 3.4.19. $\diamond$

Remark 3.4.20 As a particular case, we get

$$\boxed{T\mathcal{F}_h u(x,\xi) = e^{ix\xi/h} Tu(-\xi, x).}$$

Remark 3.4.21 Incidentally, we have also proved

$$\det \mathrm{Re}\mathcal{M}_{\ell_j}(A) = \frac{1}{|a_{j,j}|^2} \det \mathrm{Re} A.$$

Remark 3.4.22 If κ_1 and κ_2 are two transformations of the type j_B, k_C, or ℓ_j, then applying Proposition 3.4.19 twice we get

$$\begin{aligned} T_A \mathcal{J}_{\kappa_2} \mathcal{J}_{\kappa_1} u &= \beta_2 e^{i\theta_{\kappa_2}/h} \left(T_{\mathcal{M}_{\kappa_2}(A)} \mathcal{J}_{\kappa_1} u \right) \circ \kappa_2 \\ &= \beta_2 e^{i\theta_{\kappa_2}/h} \left(\beta_1 e^{i\theta_{\kappa_1}/h} T_{\mathcal{M}_{\kappa_1}(\mathcal{M}_{\kappa_2}(A))} u \circ \kappa_1 \right) \circ \kappa_2 \end{aligned}$$

(with $|\beta_1| = |\beta_2| = 1$) and therefore, using (3.4.4) and (3.4.11),

$$T_A \mathcal{J}_{\kappa_2} \mathcal{J}_{\kappa_1} u = \beta_1 \beta_2 e^{i\theta_{\kappa_1 \circ \kappa_2}/h} T_{\mathcal{M}_{\kappa_1 \circ \kappa_2}(A)} u \circ (\kappa_1 \circ \kappa_2). \tag{3.4.19}$$

Now, as we have already noticed, if κ is a general linear canonical transformation on $\mathbf{R}^{2n}$, one can show that it can always be written in the form (see, e.g., [Fo], Chapter 4, for a proof of this fact)

$$\kappa = \kappa_1 \circ \kappa_2 \circ \ldots \circ \kappa_N \tag{3.4.20}$$

for some $N \in \mathbf{N}$, where for any $\nu \in \{1, \ldots, N\}$, κ_ν belongs to one of the previous forms j_B, k_C, or ℓ_j. Of course, there is no unicity in the way of writing κ as in (3.4.20), but given such an expression, if we set

$$\mathcal{J}_\kappa = \mathcal{J}_{\kappa_N} \mathcal{J}_{\kappa_{N-1}} \ldots \mathcal{J}_{\kappa_1}, \tag{3.4.21}$$

then an iteration of (3.4.19) shows that for any $u \in C_0^\infty(\mathbf{R}^n)$, one has

$$\boxed{T_A \mathcal{J}_\kappa u = \beta_\kappa e^{i\theta_\kappa/h} \left(T_{\mathcal{M}_\kappa(A)} u \right) \circ \kappa,} \tag{3.4.22}$$

where β_κ is an h-independent complex constant of modulus 1.

3.5 Microlocal Exponential Estimates

In view of studying $\mathrm{MS}(u)$ for u a solution of a partial differential equation of the type

$$P(x, hD_x)u = 0$$

with $P(x,\xi)$ analytic, we first establish some a priori estimates involving Tu.

As has been proved in Proposition 3.3.15, any pseudodifferential operator on $\mathbf{R}^n$ is transformed by T into a pseudodifferential operator on $\mathbf{R}^{2n}$. Moreover, multiplying P by a convenient elliptic pseudodifferential operator, one can reduce to the case of a *bounded* pseudodifferential operator. For these reasons, we start by considering the case of a bounded pseudodifferential operator on $\mathbf{R}^{2n}$:

$$Q = \mathrm{Op}_h^t(q(x,\xi,x^*,\xi^*)),$$

where $q \in S_{4n}(1)$ and $t \in [0,1]$. Let also $\psi = \psi(x,\xi) \in S_{2n}(1)$ be a *real-valued* smooth function on $\mathbf{R}^{2n}$. Then we have the following theorem:

Theorem 3.5.23 *There exist $\widetilde{q}(x,\xi;h) \in S_{2n}(1)$ and $R(h) \in \mathcal{L}(L^2(\mathbf{R}^{2n}))$ such that for all $u, v \in L^2(\mathbf{R}^n)$, one has*

$$\left\langle Qe^{\psi/h}Tu, e^{\psi/h}Tv\right\rangle_{L^2(\mathbf{R}^{2n})} = \left\langle (\widetilde{q}(x,\xi;h) + R(h))e^{\psi/h}Tu, e^{\psi/h}Tv\right\rangle_{L^2(\mathbf{R}^{2n})}$$

and

$$\begin{cases} \widetilde{q}(x,\xi;h) \sim \sum_{j\geq 0} h^j \widetilde{q}_j(x,\xi) \text{ in } S_{2n}(1), \\ \widetilde{q}_0(x,\xi) = q(x,\xi,\xi-\partial_\xi\psi(x,\xi),\partial_x\psi(x,\xi)), \\ \|R(h)\|_{\mathcal{L}(L^2(\mathbf{R}^{2n}))} = \mathcal{O}(h^\infty), \end{cases}$$

uniformly as $h \to 0_+$.

Remark 3.5.24 In fact, by an argument of density it will follow from the proof that this formula can be extended to those $\psi \in C^\infty(\mathbf{R}^{2n}\,;\mathbf{R})$ such that $\nabla\psi \in S_{2n}(1)$ (ψ not necessarily bounded), on the condition that u and v belong to the space $\mathcal{H}_\psi$ defined as the completion of $C_0^\infty(\mathbf{R}^n)$ in the norm $\|u\|_\psi := \|e^{\psi/h}Tu\|_{L^2}$.

Proof of Theorem 3.5.23 The proof we present here is essentially taken from [Na2]. Let

$$r_1(x,\xi,x^*,\xi^*) = q(x,\xi,x^*,\xi^*) - q(x,\xi,\xi-\partial_\xi\psi,\partial_x\psi).$$

Then since r_1 vanishes on $\{x^* - \xi + \partial_\xi \psi = \xi^* - \partial_x \psi = 0\}$, by Taylor's formula there exist two (vector-valued) smooth functions $q_1 = q_1(x, \xi, x^*, \xi^*)$ and $q_2 = q_2(x, \xi, x^*, \xi^*)$ such that

$$r_1 = (x^* - \xi + \partial_\xi \psi) q_1 + (\xi^* - \partial_x \psi) q_2. \tag{3.5.1}$$

In fact, we have

$$\begin{aligned} q_1 &= \int_0^1 (\partial_{x^*} q)(x, \xi, \xi - \partial_\xi \psi + t(x^* - \xi + \partial_\xi \psi), \xi^*)\, dt, \\ q_2 &= \int_0^1 (\partial_{\xi^*} q)(x, \xi, \xi - \partial_\xi \psi, \partial_x \psi + t(\xi^* - \partial_x \psi))\, dt, \end{aligned}$$

and thus $q_1, q_2 \in S_{4n}(1)$. Set

$$\begin{aligned} F &= hD_x - \xi + \partial_\xi \psi, \\ G &= hD_\xi - \partial_x \psi, \\ Q_j &= \mathrm{Op}_h^t(q_j) \quad (j = 1, 2). \end{aligned} \tag{3.5.2}$$

Then, using the symbolic calculus of pseudodifferential operators (see Section 2.7), we can deduce from (3.5.1) that there exists $r_2 \in S_{4n}(1)$ such that

$$\mathrm{Op}_h^t(r_1) = \frac{1}{2}(Q_1 F + F Q_1) + \frac{1}{2}(Q_2 G + G Q_2) + h\mathrm{Op}_h^t(r_2).$$

Moreover, we have seen in Proposition 3.1.1 that

$$(hD_x - \xi)T = ihD_\xi T,$$

and therefore, setting

$$T_\psi : u \mapsto e^{\psi/h} T u$$

we also have

$$(hD_x - \xi + i\partial_x \psi)T_\psi = (ihD_\xi - \partial_\xi \psi)T_\psi,$$

that is,

$$\boxed{F T_\psi = i G T_\psi.} \tag{3.5.3}$$

It is this last equality that will permit to us to conclude, and let us notice that its main originality relies on the fact that it identifies the action of the *symmetric* operator F with that of the *antisymmetric* operator iG in the range of T_ψ.

Now, for all $u, v \in C_0^\infty(\mathbf{R}^n)$, we have

$$\begin{aligned}\left\langle \mathrm{Op}_h^t(r_1) T_\psi u, T_\psi v \right\rangle & \\ = & \frac{1}{2}\left(\langle (Q_1 F + F Q_1) T_\psi u, T_\psi v\rangle + \langle (Q_2 G + G Q_2) T_\psi u, T_\psi v\rangle\right) \\ & + h \left\langle \mathrm{Op}_h^t(r_2) T_\psi u, T_\psi v \right\rangle,\end{aligned} \tag{3.5.4}$$

and by (3.5.3),

$$\begin{aligned}\langle F Q_1 T_\psi u, T_\psi v\rangle &= \langle Q_1 T_\psi u, F T_\psi v\rangle \\ &= \langle Q_1 T_\psi u, i G T_\psi v\rangle \\ &= -i \langle G Q_1 T_\psi u, T_\psi v\rangle \\ &= -i \langle Q_1 G T_\psi u, T_\psi v\rangle + i \langle [Q_1, G] T_\psi u, T_\psi v\rangle \\ &= -\langle Q_1 F T_\psi u, T_\psi v\rangle + i \langle [Q_1, G] T_\psi u, T_\psi v\rangle,\end{aligned}$$

which gives

$$\frac{1}{2}\langle (F Q_1 + Q_1 F) T_\psi u, T_\psi v\rangle = \frac{i}{2}\langle [Q_1, G] T_\psi u, T_\psi v\rangle. \tag{3.5.5}$$

In a similar way, we also have

$$\frac{1}{2}\langle (Q_2 G + G Q_2) T_\psi u, T_\psi v\rangle = \frac{i}{2}\langle [F, Q_2] T_\psi u, T_\psi v\rangle \tag{3.5.6}$$

and therefore, substituting in (3.5.4), we get

$$\left\langle \mathrm{Op}_h^t(r_1) T_\psi u, T_\psi v \right\rangle = \left\langle \left(\frac{i}{2}[Q_1, G] + \frac{i}{2}[F, Q_2] + h \mathrm{Op}_h^t(r_2)\right) T_\psi u, T_\psi v \right\rangle. \tag{3.5.7}$$

Now we observe (still using the symbolic calculus of Section 2.7) that

$$\frac{i}{2}[Q_1, G] + \frac{i}{2}[F, Q_2] + h \mathrm{Op}_h^t(r_2) = h \mathrm{Op}_h^t(q')$$

for some $q' \in S_{4n}(1)$, and thus

$$\left\langle \mathrm{Op}_h^t(r_1) T_\psi u, T_\psi v \right\rangle = h \left\langle \mathrm{Op}_h^t(q') T_\psi u, T_\psi v \right\rangle. \tag{3.5.8}$$

Summing up, until now we have proved that for any $q \in S_{4n}(1)$, there exists $q' \in S_{4n}(1)$ such that

$$\left\langle \mathrm{Op}_h^t(q) T_\psi u, T_\psi v \right\rangle = \langle q(x,\xi,\xi - \partial_\xi \psi, \partial_x \psi) T_\psi u, T_\psi v\rangle + h \left\langle \mathrm{Op}_h^t(q') T_\psi u, T_\psi v \right\rangle. \tag{3.5.9}$$

Performing the same argument for q', and iterating the procedure, we get that there exist a sequence $(\widetilde{q}_j)_{j\in\mathbf{N}}$ of elements of $S_{2n}(1)$, and a sequence $(q^{(j)})_{j\in\mathbf{N}}$ of elements of $S_{4n}(1)$, such that at any order $N \in \mathbf{N}$ one has

$$\left\langle \mathrm{Op}_h^t(q) T_\psi u, T_\psi v \right\rangle = \left\langle \sum_{j=0}^{N-1} h^j \widetilde{q}_j T_\psi u, T_\psi v \right\rangle + h^N \left\langle \mathrm{Op}_h^t(q^{(N)}) T_\psi u, T_\psi v \right\rangle \tag{3.5.10}$$

with $\widetilde{q}_0(x,\xi) = q(x,\xi,\xi - \partial_\xi\psi, \partial_x\psi)$.

Now let $\widetilde{q} \in S_{2n}(1)$ be a resummation of $\sum_{j\geq 0} h^j \widetilde{q}_j$ in the sense of Proposition 2.3.2, and write

$$R = \Pi_\psi \left(\mathrm{Op}_h^t(q) - \widetilde{q} \right) \Pi_\psi, \tag{3.5.11}$$

where Π_ψ denotes the orthogonal projection from $L^2(\mathbf{R}^{2n})$ onto the image of T_ψ (which is closed in $L^2(\mathbf{R}^{2n})$, since the image of T is isometric to $L^2(\mathbf{R}^n)$ by Proposition 3.1.1, and the map $w \mapsto e^{\psi/h} w$ is closed on $L^2(\mathbf{R}^{2n})$). By (3.5.10) and the Calderón–Vaillancourt theorem, we see that for any $N \in \mathbf{N}$, one has

$$\langle R T_\psi u, T_\psi v\rangle = \mathcal{O}(h^N) \|T_\psi u\| \cdot \|T_\psi v\|$$

and thus, since R leaves the image of T_ψ stable and vanishes on its orthogonal:

$$\|R\|_{\mathcal{L}(L^2(\mathbf{R}^{2n}))} = \mathcal{O}(h^\infty). \tag{3.5.12}$$

In view of the definition (3.5.11) of R, the estimate (3.5.12) gives exactly the required result, and this finishes the proof of Theorem 3.5.23. $\diamond$

Now let $a, b > 0$ and $p \in S_{2n}(1)$ such that p extends holomorphically to the complex strip

$$\Sigma(a,b) := \{(x,\xi) \in \mathbf{C}^{2n} \; ; \; |\mathrm{Im} x| < a \; , \; |\mathrm{Im}\xi| < b\}$$

and satisfies

$$\forall \, \alpha \in \mathbf{N}^{2n} \; , \; \partial^\alpha p = \mathcal{O}(1) \text{ uniformly in } \Sigma(a,b). \tag{3.5.13}$$

Assume also that the real-valued function $\psi \in S_{2n}(1)$ satisfies

$$\sup_{\mathbf{R}^{2n}} |\nabla_x \psi| < b \qquad \sup_{\mathbf{R}^{2n}} |\nabla_\xi \psi| < a, \tag{3.5.14}$$

and for $t \in [0,1]$ fixed set

$$P = \mathrm{Op}_h^t(p).$$

Then we have the following important corollary of Theorem 3.5.23:

Corollary 3.5.25 *Let $f \in S_{2n}(1)$. Under assumptions (3.5.13)-(3.5.14), there exist $\widetilde{p}(x,\xi;h) \in S_{2n}(1)$ and $R(h) \in \mathcal{L}(L^2(\mathbf{R}^{2n}))$ such that for all $u, v \in L^2(\mathbf{R}^n)$, one has*

$$\left\langle f e^{\psi/h} T P u, e^{\psi/h} T v \right\rangle_{L^2(\mathbf{R}^{2n})} = \left\langle (\widetilde{p}(x,\xi;h) + R(h)) e^{\psi/h} T u, e^{\psi/h} T v \right\rangle_{L^2(\mathbf{R}^{2n})}$$

and

$$\begin{cases} \widetilde{p}(x,\xi;h) \sim \sum_{j \geq 0} h^j \widetilde{p}_j(x,\xi) \text{ in } S_{2n}(1), \\ \widetilde{p}_0(x,\xi) = f(x,\xi) p(x - 2\partial_z \psi(x,\xi), \xi + 2i\partial_z \psi(x,\xi)), \\ \|R(h)\|_{\mathcal{L}(L^2(\mathbf{R}^{2n}))} = \mathcal{O}(h^\infty), \end{cases}$$

where

$$\partial_z := \frac{1}{2}(\nabla_x + i\nabla_\xi)$$

is holomorphic differentiation with respect to $z = x - i\xi$.

Proof By Proposition 3.3.15, we have

$$f e^{\psi/h} T P u = Q e^{\psi/h} T u$$

with

$$Q = f e^{\psi/h} \mathrm{Op}_h^t(p(x - \xi^*, x^*)) e^{-\psi/h}. \tag{3.5.15}$$

Therefore, in view of applying Theorem 3.5.23 we first have to prove the following lemma:

Lemma 3.5.26 *The operator $f e^{\psi/h} \mathrm{Op}_h^t(p(x - \xi^*, x^*)) e^{-\psi/h}$ is a semiclassical pseudodifferential operator on $\mathbf{R}^{2n}$, and its symbol $q \in S_{4n}(1)$ admits an asymptotic expansion of the form*

$$q \sim \sum_{j \geq 1} h^j q_j \text{ in } S_{4n}(1)$$

with

$$q_0(x,\xi,x^*,\xi^*) = f(x,\xi) p(x - \xi^* - i\partial_\xi \psi, x^* + i\partial_x \psi).$$

Proof Here again we follow [Na2]. Set $\widetilde{P} = \mathrm{Op}_h^t(p(x-\xi^*, x^*))$. For $w \in C_0^\infty(\mathbf{R}^{2n})$ and $\nu \in \mathbf{R}$, we write

$$
\begin{aligned}
&e^{i\nu\psi/h}\widetilde{P}e^{-i\nu\psi/h}w(x,\xi)\\
&\quad= \frac{1}{(2\pi h)^{2n}}\int e^{i(x-y)x^*/h+i(\xi-\eta)\xi^*/h+i\nu(\psi(x,\xi)-\psi(y,\eta))/h}\\
&\qquad\qquad\times p((1-t)x+ty-\xi^*, x^*))\,w(y,\eta)dy\,d\eta\,dx^*\,d\xi^*,
\end{aligned}
$$

and we also have by Taylor's formula

$$\psi(x,\xi)-\psi(y,\eta) = (x-y)\,\phi_1(x,y,\xi,\eta) + (\xi-\eta)\,\phi_2(x,y,\xi,\eta),$$

where $\phi_1, \phi_2 \in [S_{4n}(1)]^n$ are real-valued.

Then we make the change of variables $(x^*, \xi^*) \mapsto (\widetilde{x}^*, \widetilde{\xi}^*)$ given by

$$
\left\{
\begin{aligned}
&\widetilde{x}^* = x^* + \nu\phi_1(x,y,\xi,\eta),\\
&\widetilde{\xi}^* = \xi^* + \nu\phi_2(x,y,\xi,\eta).
\end{aligned}
\right.
\tag{3.5.16}
$$

We obtain

$$
\begin{aligned}
&e^{i\nu\psi/h}\widetilde{P}e^{-i\nu\psi/h}w(x,\xi)\\
&\quad= \frac{1}{(2\pi h)^{2n}}\int e^{i(x-y)\widetilde{x}^*/h+i(\xi-\eta)\widetilde{\xi}^*/h}\\
&\qquad\qquad\times p((1-t)x+ty-\widetilde{\xi}^*+\nu\phi_2, \widetilde{x}^*-\nu\phi_1)w(y,\eta)dy\,d\eta\,d\widetilde{x}^*\,d\widetilde{\xi}^*,
\end{aligned}
\tag{3.5.17}
$$

and since $|\phi_1| \leq \sup|\nabla_x\psi| < b$ and $|\phi_2| \leq \sup|\nabla_\xi\psi| < a$, we see that the function

$$\nu \mapsto p((1-t)x+ty-\widetilde{\xi}^*+\nu\phi_2, \widetilde{x}^*-\nu\phi_1)$$

can be extended holomorphically in a complex neighborhood of $\{\nu \in \mathbf{C}\,;\ |\nu| \leq 1\}$, with values in $S_{6n}(1)$. As a consequence, the right-hand side of (3.5.17) can be extended too, and since this is also obviously true for the left-hand side and both sides are equal for $\nu \in \mathbf{R}$, by analytic continuation they remain equal for $\nu \in \mathbf{C}$, $|\nu| \leq 1$. In particular, for $\nu = -i$ we get

$$e^{\psi/h}\widetilde{P}e^{-\psi/h} = \mathrm{Op}(p((1-t)x+ty-\xi^*-i\phi_2, x^*+i\phi_1)),$$

where the quantization is given by the general case of symbols in $S_{3d}(1)$ (with $d = 2n$ here) as in Definition 2.5.1. Then, using Theorem 2.7.1, we obtain

$$e^{\psi/h}\widetilde{P}e^{-\psi/h} = \mathrm{Op}_h^t(p_t(x,\xi,x^*,\xi^*;\ h)), \tag{3.5.18}$$

where p_t admits an asymptotic expansion in $S_{4n}(1)$, with first term given by

$$\begin{aligned} p_t^0(x,\xi,x^*,\xi^*) &= p((1-t)x+ty-\xi^*-i\phi_2, x^*+i\phi_1)\Big|_{\substack{y=x\\ \eta=\xi}} \\ &= p(x-\xi^*-i\partial_\xi\psi, x^*+i\partial_x\psi). \end{aligned} \tag{3.5.19}$$

Therefore, the lemma follows from (3.5.18) and (3.5.19) by multiplying (3.5.18) by $f(x,\xi)$. $\diamond$

End of the proof of Corollary 3.5.25 From Lemma 3.5.26, we see that we can apply Theorem 3.5.23 with Q given in (3.5.15), and this gives exactly the result of Corollary 3.5.25. $\diamond$

In the same situation as for Corollary 3.5.25 there is another consequence of Theorem 3.5.23 that will be rather useful in the applications:

Corollary 3.5.27 *Let $f \in S_{2n}(1)$. Under assumptions (3.5.13)-(3.5.14) one has*

$$\|fe^{\psi/h}TPu\|^2 = \|f(x,\xi)p(x-2\partial_z\psi, \xi+2i\partial_z\psi)e^{\psi/h}Tu\|^2 + \mathcal{O}(h)\|e^{\psi/h}Tu\|^2$$

uniformly with respect to $u \in L^2(\mathbf{R}^n)$ and $h > 0$ small enough.

Proof : With Q given in (3.5.15), one has

$$\|fe^{\psi/h}TPu\|^2 = \left\langle Q^*Qe^{\psi/h}Tu, e^{\psi/h}Tu\right\rangle$$

and from Lemma 3.5.26 and the symbolic calculus, Q^*Q is a semiclassical pseudodifferential operator whose symbol admits an asymptotic expansion with first term $|f(x,\xi)p(x-\xi^*-i\partial_\xi\psi, x^*+i\partial_x\psi)|^2$. Then the result follows directly from Theorem 3.5.23. $\diamond$

Remark 3.5.28 Analogous estimates are valid when p is not analytic, but, e.g., Gevrey. In this case, the weight $e^{\psi/h}$ has to be replaced by $e^{\psi/h^{1/s}}$, where $s > 1$ is the Gevrey index, see [Ju]. The proof relies essentially on the almost analytic extensions introduced by Melin and Sjöstrand in [MeSj], and can also be adapted in the general C^∞ case with weights of the type $h^{\psi(x,\xi)}$ (see Exercise 4 of this chapter).

Of course, in the particular case where ψ vanishes identically, one can see from the proof of Corollary 3.5.25 that the assumption of analyticity made on p is no longer necessary. As a consequence, using also Corollary 3.1.4, we get the following result:

Corollary 3.5.29 *Let $p \in S_{2n}(1)$, $t \in [0,1]$, and let $P = \mathrm{Op}_h^t(p)$. Then there exist $\widetilde{p}(x,\xi;h) \in S_{2n}(1)$ and $R(h) \in \mathcal{L}(L^2(\mathbf{R}^{2n}))$ such that for all $u, v \in L^2(\mathbf{R}^n)$, one has*

$$\langle Pu, v\rangle_{L^2(\mathbf{R}^n)} = \langle(\widetilde{p}(x,\xi;h) + R(h))Tu, Tv\rangle_{L^2(\mathbf{R}^{2n})}$$

and

$$\begin{cases} \widetilde{p}(x,\xi;h) \sim \sum_{j\geq 0} h^j \widetilde{p}_j(x,\xi) \text{ in } S_{2n}(1), \\ \widetilde{p}_0(x,\xi) = p(x,\xi), \\ \|R(h)\|_{\mathcal{L}(L^2(\mathbf{R}^{2n}))} = \mathcal{O}(h^\infty). \end{cases}$$

As an application of this last result we have the following semiclassical version of a celebrated theorem (see also Exercise 22 of Chapter 2):

Theorem 3.5.30 (Sharp Gårding Inequality) *Let $p = p(x,\xi) \in S_{2n}(1)$ such that $p \geq 0$ on $\mathbf{R}^{2n}$. Then there exists a constant $C > 0$ such that for all $u \in L^2(\mathbf{R}^n)$ and $h > 0$ small enough, one has*

$$\left\langle \mathrm{Op}_h^W(p)u, u\right\rangle \geq -Ch\|u\|^2.$$

Remark 3.5.31 Since p is real-valued, the operator $\mathrm{Op}_h^W(p)$ is symmetric on $L^2(\mathbf{R}^n)$, and therefore the quantity $\left\langle \mathrm{Op}_h^W(p)u, u\right\rangle$ is necessarily real.

Remark 3.5.32 In other words, the fact that p is nonnegative everywhere implies that (in the sense of operators) $\mathrm{Op}_h^W(p)$ is nonnegative modulo $\mathcal{O}(h)$.

Remark 3.5.33 Of course, the result remains valid for any perturbation of p of order $\mathcal{O}(h)$ in $S_{2n}(1)$. In particular, the assumption $p \geq 0$ can be replaced by $p \geq -C'h$ for some constant C'. Indeed, a stronger result exists for $\mathrm{Op}_h^W(p)$ when $p \geq 0$: This is the so-called *Fefferman–Phong Inequality*, which asserts that in this situation one has

$$\left\langle \mathrm{Op}_h^W(p)u, u\right\rangle \geq -Ch^2\|u\|^2.$$

But the proof is very delicate, and we refer the interested reader to [FePh1] (see also [Bo, Tat]).

Proof of Theorem 3.5.30 It is an immediate consequence of Corollary 3.5.29 and the Calderón–Vaillancourt theorem. $\diamond$

There also exists a generalization of Theorem 3.5.30 to possibly unbounded pseudodifferential operators:

Corollary 3.5.34 *Let $m \in \mathbf{R}$, and $p \in S_{2n}(\langle\xi\rangle^m)$ such that $p \geq 0$ on $\mathbf{R}^{2n}$. Then there exists a constant $C > 0$ such that for all $u \in H^{m/2}(\mathbf{R}^n)$ one has*

$$\left\langle \mathrm{Op}_h^W(p)u, u \right\rangle \geq -Ch\|u\|^2_{H^{m/2}}.$$

Proof By the symbolic calculus, there exists $r \in S_{2n}(1)$ such that

$$\mathrm{Op}_h^W(\langle\xi\rangle^{-m/2})\,\mathrm{Op}_h^W(p)\,\mathrm{Op}_h^W(\langle\xi\rangle^{-m/2}) = \mathrm{Op}_h^W(\langle\xi\rangle^{-m}\,p) + h\mathrm{Op}_h^W(r). \quad (3.5.20)$$

Then the result for $u \in H^{m/2}(\mathbf{R}^n)$ follows from the Calderón–Vaillancourt theorem by applying Theorem 3.5.30 with the symbol $\langle\xi\rangle^{-m}\,p \in S_{2n}(1)$, and the function $v = \mathrm{Op}_h^W(\langle\xi\rangle^{m/2})u \in L^2(\mathbf{R}^n)$. $\diamond$

Remark 3.5.35 If, moreover, p satisfies $\partial^\alpha p = \mathcal{O}(1+p)$ for all $\alpha \in \mathbf{R}^{2n}$, then the previous inequality can be improved to

$$\left\langle \mathrm{Op}_h^W(p)u, u \right\rangle \geq -Ch\|u\|^2_{L^2}.$$

Actually, in this case a slight generalization of Corollary 3.5.29 gives

$$\langle Pu, u\rangle_{L^2(\mathbf{R}^n)} = \langle (p(x,\xi) + R(h))Tu, Tu\rangle_{L^2(\mathbf{R}^{2n})}$$

with $|\langle R(h)Tu, Tu\rangle| = \mathcal{O}(h\|u\|^2 + h\langle pTu, Tu\rangle)$. Thus the result follows by taking h small enough.

Many other generalizations of Corollary 3.5.25 can be made, including a version whose framework is the so-called *Weyl–Hörmander calculus*, and which should contain all the previous cases. With the notation of [Ho2], and in the case where $f = 1$ and $\psi = 0$, it reads as follows:

For any $a \in S(m, g)$,

$$\left\langle \mathrm{Op}_1^W(a)u, v \right\rangle = \langle (\tilde{a} + R)Tu, Tv\rangle\,,$$

where T is now defined without the semiclassical parameter, $\tilde{a} \in S(m, g)$ is a symbol with an expansion that can be explicitly computed, and R satisfies, for any $N > 0$,

$$\|Rw\| = \mathcal{O}\left(\left\|h^N w\right\|\right)$$

uniformly with respect to $w \in L^2(\mathbf{R}^{2n})$. Here h is related to the metric g by the formula $h^2 = g/g^\sigma$ (see [Ho2] for the definition of g^σ).

As a final remark of this section, let us note that when ψ depends only on a group of variables, say $(x_{i_1}, \ldots, x_{i_k}, \xi_{j_1}, \ldots, \xi_{j_\ell})$, then the analyticity assumption in Corollary 3.5.25 can be relaxed with respect to the variables other than $(x_{j_1}, \ldots, x_{j_\ell}, \xi_{i_1}, \ldots, \xi_{i_k})$. In particular, if $\psi = \psi(x)$ depends only on x (respectively $\psi = \psi(\xi)$ depends only on ξ), then one needs only the analyticity of $p(x, \xi)$ with respect to the variable ξ (respectively x).

3.6 Exercises and Problems

1. **Coherent States -** For $(x, \xi) \in \mathbf{R}^{2n}$ denote by $\phi_{x,\xi}$ the function on $\mathbf{R}^n$ defined by

$$\phi_{x,\xi}(y) = (\pi h)^{-n/4} e^{i(y-x)\xi/h-(y-x)^2/2h}$$

(the so-called *coherent state* centered at (x, ξ)).

(i) Prove that for all $(x, \xi), (x', \xi') \in \mathbf{R}^{2n}$ one has

$$\langle \phi_{x,\xi}, \phi_{x',\xi'} \rangle_{L^2(\mathbf{R}^n)} = e^{i(x'-x)(\xi'+\xi)/2h-(x-x')^2/4h-(\xi-\xi')^2/4h}$$

(in particular, $\|\phi_{x,\xi}\|_{L^2(\mathbf{R}^n)} = 1$).

(ii) Deduce from (i) that for any $(x, \xi) \in \mathbf{R}^{2n}$ one has

$$\mathrm{MS}(\phi_{x,\xi}) = \{(x, \xi)\}.$$

(iii) Use Proposition 3.1.6 to prove that for all $u \in \mathcal{S}'(\mathbf{R}^n)$ one has

$$u = (2\pi h)^{-n/2} \int_{\mathbf{R}^{2n}} Tu(x, \xi)\, \phi_{x,\xi}\, dx\, d\xi$$

(which means that u can be written as a superposition of coherent states).

(iv) If A is a bounded operator on $L^2(\mathbf{R}^n)$ with distribution kernel K_A, then prove the formula

$$K_A(x,y) = (2\pi h)^{-n} \int_{\mathbf{R}^{2n}} (A\phi_{z,\zeta})(x)\overline{\phi_{z,\zeta}(y)}dz\,d\zeta.$$

Hint: Just write $A = AT^*T$ and observe that

$$Tu(z,\zeta) = (2\pi h)^{-n/2} \int \overline{\phi_{z,\zeta}(y)}u(y)dy.$$

Note: For other applications of coherent states one may consult, e.g., [CoRo], [PaUr], and references therein.

2. **Range of T -** On $L^2(\mathbf{R}^{2n})$ we consider the operator

$$\Pi := TT^*,$$

where T is the FBI transform defined in (3.1.1).

(i) Prove that $\Pi^2 = \Pi = \Pi^*$ and $T^*\Pi = T^*$, and deduce that Π is the orthogonal projector onto $T(L^2(\mathbf{R}^n))$.

(ii) Let $v \in L^2(\mathbf{R}^{2n})$ be of the form

$$v(x,\xi) = e^{-\xi^2/2h}a(x - i\xi),$$

where a is an entire function on $\mathbf{C}^n$. Prove that

$$\begin{aligned}&\Pi v(x,\xi)\\ &= \frac{1}{(2\pi h)^n}\int e^{-[(y-x)^2+(\eta-\xi)^2]/4h-[\eta^2+i(y-x)(\eta+\xi)]/2h}a(y-i\eta)\,dy\,d\eta.\end{aligned}$$

(iii) Making (and justifying) the change of contour of integration

$$\mathbf{R}^{2n} \ni (y,\eta) \mapsto (y + i\eta - i\xi, \eta),$$

show that $\Pi v = v$. (Hint: Interpret the previous integral as an oscillatory one, e.g., passing through the limit when $\varepsilon \to 0_+$ of the integral obtained by multiplying the integrated function with $e^{-\varepsilon\eta^2}$.)

(iv) Deduce from the previous questions that

$$T(L^2(\mathbf{R}^n)) = L^2(\mathbf{R}^{2n}) \cap e^{-\xi^2/2h}\mathcal{H}(\mathbf{C}^n_{x-i\xi}),$$

where $\mathcal{H}(\mathbf{C}^n_{x-i\xi})$ denotes the space of entire functions of $x - i\xi$ on $\mathbf{C}^n$.

(v) Following the same procedure, prove that

$$T(\mathcal{S}'(\mathbf{R}^n)) = \mathcal{S}'(\mathbf{R}^{2n}) \cap e^{-\xi^2/2h}\mathcal{H}(\mathbf{C}^n_{x-i\xi}).$$

3. **Frequency Set -** Using Definition 2.9.1, prove that a point $(x_0,\xi_0) \in \mathbf{R}^{2n}$ is not in the frequency set of $u \in L^2(\mathbf{R}^n)$ if and only if $Tu(x,\xi) = \mathcal{O}(h^\infty)$ uniformly in a neighborhood of (x_0,ξ_0).

   Hint: First prove that if $\chi = \chi(y,\eta) \in S_{2n}(1)$ vanishes near some point (y_0,η_0), then $\chi(y,hD_y)(e^{i(y-x)\xi/h-(x-y)^2/2h}) = \mathcal{O}(h^\infty)$ for (x,ξ) close enough to (y_0,η_0) (here $\chi(y,hD_y)$ denotes any quantization of χ). Then for the necessary condition write $u(y) = \chi(y,hD_y)u + (1 - \chi(y,hD_y))u$ in the expression of Tu, and for the sufficient condition just use that $\mathrm{Op}_h^W(\chi)u = \mathrm{Op}_h^W(\chi)T^*(Tu)$.

4. **Weighted Estimates in the General C^∞ Case -** Let $p \in S_{2n}(1)$. By using an almost-analytic extension of p (see Exercise 23 of Chapter 2), prove an estimate similar to the one of Corollary 3.5.25 but with $e^{\psi/h}$ replaced by h^ψ.
   Hint: Just mimic the proof, working with $(h\ln h)\psi$ instead of ψ, and use the Stokes formula to justify modulo $\mathcal{O}(h^\infty)$ the change of contour given in (3.5.16) directly with $\nu = -i$. The final estimate is

$$\left\langle fh^\psi TPu, h^\psi Tv\right\rangle = \left\langle (\widetilde{p}(x,\xi;h) + R(h))h^\psi Tu, h^\psi Tv\right\rangle$$

   with $\|R(h)\| = \mathcal{O}(h^\infty)$ and

$$\widetilde{p}(x,\xi;h) \sim f(x,\xi)p_a(x - 2h\ln h\partial_z\psi, \xi + 2ih\ln h\partial_z\psi) + \sum_{k\geq 1} h^h \widetilde{p}_k(x,\xi,h)$$

   in $S_{2n}(1)$, where p_a denotes an almost-analytic extension of p.

5. **Weighted Estimates in the Gevrey Case -** Assume now that $p \in S_{2n}(1)$ is s-*Gevrey* for some $s > 1$, that is,

$$\sup_{\mathbf{R}^{2n}} |\partial^\alpha p| \leq C^{1+|\alpha|}(\alpha!)^s$$

   for all $\alpha \in \mathbf{N}^{2n}$ and for some constant $C > 0$. Prove again an estimate similar to the one of Corollary 3.5.25 but with $e^{\psi/h}$ replaced by $e^{\psi/h^{\frac{1}{s}}}$.

6. **Levi–Mizohata Uniqueness Theorem -** In 1 dimension, one considers for $h > 0$ the differential operator $A = hD_t + it$.

   (i) Determinate all the solutions $u \in \mathcal{D}'(\mathbf{R})$ of the equation $Au = 0$, and show that the only one that belongs to $L^2(\mathbf{R})$ is the trivial one. Is it still true when A is replaced by $B = hD_t - it$?

   (ii) Now we try to find a generalization of this result. Let $p = p(x, \xi) \in S_{2n}(1)$, and denote $p_1 = \mathrm{Re}p$ and $p_2 = \mathrm{Im}p$. We assume that there exists some $\delta > 0$ such that for all $(x, \xi) \in \mathbf{R}^{2n}$ the following implication is true:

$$|p(x,\xi)| \leq \delta \Longrightarrow \{p_1, p_2\}(x,\xi) \geq \delta, \tag{3.5.1}$$

   where $\{.,.\}$ is the Poisson bracket. We set $P = Op_h^W(p)$ and $P_j = Op_h^W(p_j)$ $(j = 1, 2)$, and we denote by T the global FBI transform defined in (3.1.1).

   (ii.a) Prove the a priori estimate

$$\|Pu\|^2 = \|P_1 u\|^2 + \|P_2 u\|^2 + h\langle\{p_1, p_2\}Tu, Tu\rangle_{L^2(\mathbf{R}^{2n})} + \mathcal{O}(h^2\|u\|^2)$$

   uniformly with respect to $h > 0$ small enough and $u \in L^2(\mathbf{R}^n)$. (The norms without index are those in $L^2(\mathbf{R}^n)$.)

   (ii.b) We set $\Sigma_\delta = \{(x,\xi) \in \mathbf{R}^{2n}\ ;\ |p(x,\xi)| \leq \delta\}$, and Σ_δ^C its complement in $\mathbf{R}^{2n}$. Show that if p satisfies (3.5.1), then there exists $C_\delta > 0$ such that for all $h > 0$ and for all $u \in L^2(\mathbf{R}^n)$ one has

$$\|P_1 u\|^2 + \|P_2 u\|^2 + h\langle\{p_1, p_2\}Tu, Tu\rangle_{L^2(\mathbf{R}^{2n})}$$

$$\geq \max\left\{\frac{1}{C_\delta}\|Tu\|^2_{\Sigma_\delta^C} - C_\delta h\|Tu\|^2_{\Sigma_\delta}\ ;\ \frac{h}{C_\delta}\|Tu\|^2_{\Sigma_\delta} - C_\delta h\|Tu\|^2_{\Sigma_\delta^C}\right\}.$$

   (ii.c) Deduce from (ii.a) and (ii.b) that if p satisfies (3.5.1), then for $h > 0$ small enough one has the implication

$$\left.\begin{array}{r} u \in L^2(\mathbf{R}^n) \\ Pu = 0 \end{array}\right\} \Longrightarrow u = 0. \tag{3.5.2}$$

   (ii.d) Let $Q = hD_t + itOp^W(a)$ with $a = a(x,\xi) \in S_{2n}(1)$ elliptic and positive. Reducing to $S_{2n}(1)$ by multiplying Q with a convenient operator, show that the implication (3.5.2) with P replaced by Q (and n replaced by $n + 1$) is true.

7. Calculate the second term $\widetilde{p}_1$ in Corollary 3.5.29 when $t = \frac{1}{2}$. (Hint: Notice that when $t = \frac{1}{2}$, then $r_2 = 0$ in the proof of Theorem 3.5.23. The final result is: $\widetilde{p}_1 = -\frac{1}{4}\Delta p$.)

8. **Agmon Estimates -** Let $p \in S_{2n}(1)$ be such that $p(x,\xi)$ extends holomorphycally with respect to ξ near $\{\xi \in \mathbf{C}^n \ ; \ |\mathrm{Im}\xi| \leq c_0\}$ for some $c_0 > 0$, and remains bounded there together with all its derivatives. Let also $\varphi = \varphi(x)$ be a real-valued smooth function on $\mathbf{R}^n$, bounded together with all its derivatives, and satisfying $|\nabla\varphi| \leq c_0$. We set $P = \mathrm{Op}_h^W(p)$.

   (i) Making a change of contour of integration, prove that the operator $P_\varphi := e^{\varphi/h}Pe^{-\varphi/h}$ is an h-pseudodifferential operator with symbol $p_\varphi(x,\xi,h) = p(x,\xi + i\nabla\varphi(x)) + \mathcal{O}(h)$ in $S_{2n}(1)$.

   (ii) Using Corollary 3.5.29, deduce from (i) that for all $u \in L^2(\mathbf{R}^n)$ one has

   $$\left\langle e^{\varphi/h}Pu, e^{\varphi/h}u\right\rangle = \left\langle p(x,\xi+i\nabla\varphi(x))Te^{\varphi/h}u, Te^{\varphi/h}u\right\rangle + \mathcal{O}(h\|e^{\varphi/h}u\|^2)$$

   uniformly with respect to h small enough and $u \in L^2(\mathbf{R}^n)$. (Hint: Just rewrite $\left\langle e^{\varphi/h}Pu, e^{\varphi/h}u\right\rangle = \langle P_\varphi v, v\rangle$ with $v = e^{\varphi/h}u$.)

   (iii) Give a generalization of the previous estimate when $p \in S_{2n}(\langle\xi\rangle^m)$ with $m \geq 1$.

   (iv) In the particular case of the Schrödinger operator $P_V = -h^2\Delta + V(x)$ with $V \in S_n(1)$, deduce for h small enough the following inequality:

   $$\mathrm{Re}\left\langle e^{\varphi/h}P_Vu, e^{\varphi/h}u\right\rangle \geq \left\langle (V(x) - |\nabla\varphi(x)|^2)e^{\varphi/h}u, e^{\varphi/h}u\right\rangle - Ch\|e^{\varphi/h}u\|^2,$$

   where C is some positive constant.

   (v) Make $e^{\varphi/h}P_Ve^{-\varphi/h}$ explicit by a direct computation, and deduce that one actually has the so-called *Agmon inequality*:

   $$\mathrm{Re}\left\langle e^{\varphi/h}P_Vu, e^{\varphi/h}u\right\rangle \geq \left\langle \left(V(x) - |\nabla\varphi(x)|^2\right)e^{\varphi/h}u, e^{\varphi/h}u\right\rangle.$$

   (vi) If $u \in L^2(\mathbf{R}^n)$ satisfies $P_Vu = Eu$ for some $E \in \mathbf{R}$, and $\|u\| = 1$, then deduce from (v) (or even from (iv)) that for any $\varepsilon > 0$ and any φ such that $|\nabla\varphi(x)|^2 \leq V(x) - E - \varepsilon$ on $\mathrm{Supp}\varphi$, one has $\|e^{\varphi/h}u\| = \mathcal{O}(1)$ uniformly as $h \to 0_+$ (*Agmon estimates*).

Note: This type of estimate has been used by many authors to obtain precise exponential decay of eigenfunctions of Schrödinger operators, see, e.g., [Ag, BrCoDu, He1, HeSj1, HeSj2, HiSi, Si]. In particular, when V admits a nondegenerate minimum at some point x_0, such estimates make it possible to get the WKB asymptotics near x_0 (i.e., asymptotics of the form $(\sum a_j(x)h^j)e^{-\varphi/h}$ as in Exercise 6 of Chapter 2) of the first eigenfunctions of P_V ([He1, HeSj1]).

9. Rewrite Theorem 3.5.23 when T is replaced by T_A defined in (3.4.1). In particular, calculate $\widetilde{p}_0$ of Corollary 3.5.25 explicitly when $A = \mu I$ with $\mu > 0$.

   Answer: $\widetilde{q}_0(x,\xi)$ becomes

   $$q(x,\xi,\xi-(A_1+A_2A_1^{-1}A_2)\partial_\xi\psi - A_2A_1^{-1}\partial_x\psi, A_1^{-1}\partial_x\psi + A_1^{-1}A_2\partial_\xi\psi),$$

   and when $A = \mu I$, then $\widetilde{p}_0$ becomes

   $$p(x-\mu^{-1}\partial_x\psi - i\partial_\xi\psi, \xi + i\partial_x\psi - \mu\partial_\xi\psi).$$

   Note: In the case where $\psi = \psi(x)$ does not depend on ξ, and μ is taken very large, then the previous $\widetilde{p}_0$ becomes arbitrarily close to $p(x,\xi+i\partial_x\psi)$, which is the quantity appearing in the Agmon estimates (see Exercise 8 above).

10. **Semiclassical Measures -** Let $u = (u_h)_{h\in(0,h_0]}$ be a family of functions in $L^2(\mathbf{R}^n)$ such that $\|u_h\|_{L^2} = 1$ (so that $|T_hu_h(x,\xi)|^2dxd\xi$ is a probability measure on $L^2(\mathbf{R}^{2n})$, where T_h denotes the h-dependent FBI transform defined in (3.1.1)). An (h-independent) probability measure $d\mu$ on $\mathbf{R}^{2n}$ is called a *semiclassical measure of* u if there exists some sequence $(h_j)_{j\in\mathbf{N}}$ converging to 0_+ such that

    $$|T_{h_j}u_{h_j}(x,\xi)|^2dx\,d\xi \to d\mu \text{ weakly, as } j\to+\infty.$$

    (The function $|Tu(x,\xi)|^2$ is called the *Husimi function* attached to u.)

    (i) Let $d\mu$ be a semiclassical measure of u, corresponding to a sequence $(h_j)_{j\in\mathbf{N}}$ converging to 0_+, and let $A(x,hD_x)$ denote a semiclassical pseudodifferential operator admitting a principal symbol $a_0\in S_{2n}(1)$. Using Corollary 3.5.25 with $\psi=0$, prove that

    $$\left\langle A(x,h_jD_x)u_{h_j},u_{h_j}\right\rangle_{L^2} \to \int_{\mathbf{R}^{2n}} a_0(x,\xi)d\mu(x,\xi) \qquad (j\to+\infty). \tag{3.5.3}$$

Conversely, prove that if the property (3.5.3) is true for all $a \in S_{2n}(1)$, then $d\mu$ is a semiclassical measure of u.

(ii) Assume that u is a solution of the equation $Pu = 0$, where $P = P(x, hD_x)$ is a semiclassical pseudodifferential operator admitting a principal symbol $p_0 \in S_{2n}(1)$. Applying (i) with $A(x, hD_x) \circ P(x, hD_x)$ instead of $A(x, hD_x)$, prove that if $d\mu$ is a semiclassical measure of u, then it satisfies $p_0 d\mu = 0$.

(iii) Assume now that $u_h = u_h(t) \in C^1(\mathbf{R}_t\ ;\ L^2(\mathbf{R}^n))$ is solution of

$$\begin{cases} (hD_t + P_h)u_h = 0, \\ u_h\,|_{t=0} = v_h, \end{cases}$$

with $P_h = \mathrm{Op}_h^W(p)$, $p \in S_{2n}(1)$ real-valued, $p = p_0 + \mathcal{O}(h)$ in $S_{2n}(1)$ (p_0 independent of h), and $\|v_h\|_{L^2} = 1$. Assume, moreover, that v admits a semiclassical measure $d\nu$ corresponding to a sequence $(h_j)_{j\in\mathbf{N}}$. Then prove that for all $t \in \mathbf{R}$, $u(t)$ admits a semiclassical measure $d\mu_t$ corresponding to the same sequence $(h_j)_{j\in\mathbf{N}}$, and that it satisfies

$$\begin{cases} \partial_t(d\mu_t) + \{p_0, d\mu_t\} = 0, \\ d\mu_0 = d\nu. \end{cases} \tag{3.5.4}$$

(Hint: Use the equation to show that for all pseudodifferential operator $A = A(x, hD_x)$ one has $hD_t \langle Au, u\rangle = \langle [P_h, A]u, u\rangle$, and deduce that any weak limit $d\mu_t$ of a subsequence of $|T_{h_j} u_{h_j}(x,\xi)|^2 dx\, d\xi$ is a solution of (3.5.4); then conclude by observing that the system (3.5.4) admits a unique solution.)

(iv) In the particular case where u_h (independent of t) is a solution of $P_h u_h = 0$, deduce from (iii) a result of propagation for the semiclassical measures of u.

Note: For further results on the semiclassical measures, one may consult, e.g., [GeP].

11. **Lagrangian States -** An h-dependent function $u \in L^2(\mathbf{R}^n)$ is called *Lagrangian* if it can be written in the form $I_\varphi(a)$ defined in (2.4.8) with the conditions (2.4.6) and (2.4.7).

(i) In this case, prove that

$$\mathrm{FS}(u) \subset \{(x, \nabla_x \varphi(x,\theta))\ ;\ \nabla_\theta \varphi(x,\theta) = 0\}.$$

(ii) If, moreover, $a \in S^{hol}_{n+n'}(\langle\theta\rangle^m)$ and $\varphi(x,\theta)$ is analytic in a complex strip around $\mathbf{R}^{n+n'}$, prove that

$$\mathrm{MS}(u) \subset \{(x, \nabla_x\varphi(x,\theta))\ ;\ \nabla_\theta\varphi(x,\theta) = 0\}.$$

Hint: In the expression of $T_\mu u(x,\xi)$ with $\mu > 0$ large enough, write that $\varphi(y,\theta) = \varphi(x,\theta) + (y-x)\nabla_x\varphi(x,\theta) + \mathcal{O}(|x-y|^2)$; then use the operator $L = \left(1 + \frac{|\nabla_\theta\varphi|^2}{h^2}\right)^{-1}\left(1 + \frac{1}{h}\nabla_\theta\varphi.D_\theta\right)$ to make integrations by parts near a point x for which $\nabla_\theta\varphi(x,\theta) \neq 0$ for all θ; finally, make integrations by parts with respect to y in the C^∞ case, or make a change of contour of integration with respect to y in the analytic case, when $\xi \neq \nabla_x\varphi(x,\theta)$.

Note: $I_\varphi(a) \in L^2$ if, e.g., $|\nabla_\theta\varphi| \geq (\langle\theta\rangle^\rho + \langle x\rangle^\rho)/C$, or if $a = a(x) \in L^2$.

12. Give a result of microlocal decay at infinity, by using a sequence of bounded weight functions $(\psi_j)_{j\in\mathbf{N}}$ converging at infinity toward an unbounded function ψ (but with bounded gradient).

13. Do the same as in the previous exercise, but this time assuming only that $\mathrm{Hess}\,\psi$ is uniformly bounded (together with all its derivatives) and sufficiently small, while p is holomorphic (and admits symbol-type estimates) in a complex sector of the form

$$\mathcal{S}_\delta = \{(x,\xi) \in \mathbf{C}^{2n}\ ;\ |\mathrm{Im}(x,\xi)| < \delta\,\langle\mathrm{Re}(x,\xi)\rangle\}$$

with $\delta > 0$.

Bibliography

[Ag] Agmon, S., *Lectures on Exponential Decay of Solutions of Second Order Elliptic Equations, Princeton University Press*, Princeton (1982).

[AlGe] Alinhac, S., Gérard, P., *Opérateurs Pseudo-différentiels et Théorème de Nash–Moser*, Interéditions, Savoirs Actuels (Paris) (1991).

[Be1] Beals, R., A General Calculus of Pseudodifferential Operators, *Duke Math. J.* **42**, 1–42 (1975).

[Be2] Beals, R., Characterization of Pseudodifferential Operators and Applications, *Duke Math. J.* **44**, 45–57 (1977).

[Bi] Bismut, J.-M., The Witten complex and the degenerate Morse inequalities, *J. Differ. Geom.* **23**, 207–240 (1986).

[Bo] Bony, J.-M., Sur l'inégalité de Fefferman–Phong, *Séminaire EDP de l'Ecole Polytechnique*, Palaiseau (France), exp. n.III, 1998–99.

[BoHeJo] Born, M., Heisenberg, W., Jordan, P., *Zur Begruendung der Matrizenmechanik, Dokumente der Naturwissenschaft, Abt. Physik*, Bd. 2. Stuttgart: Ernst Battenberg Verlag, 135 S. (1962).

[BrCoDu] Briet, P., Combes, J.-M., Duclos, P., On the Location of Resonances for Schrödinger Operators in the Semiclassical Limit II, *Comm. Math. Phys.* **12** (2), 201–222 (1987).

[CaVa1] Calderón, A.P., Vaillancourt, R., On the Boundedness of Pseudodifferential Operators, *J. Math. Soc. Japan* **23**, 374–378 (1972).

[CaVa2] Calderón, A.P., Vaillancourt, R., A class of Bounded Pseudodifferential Operators, *Proc. Nat. Acad. Sci. USA* **69**, 1185–1187 (1972).

[ChPi] Chazarain, J., Piriou, A., *Introduction à la théorie des équations aux dérivées partielles linéaires*, Gauthier Villars, Paris (1981).

[CdV1] Colin de Verdière, Y., *Spectres de graphes*, Cours Spécialisés n.4, Paris: Société Mathématique de France (1998).

[CdV2] Colin de Verdière, Y., Une introduction a la mecanique semi-classique, *Enseign. Math.*, II. Ser. 44, No.1-2, 23–51 (1998).

[CoRo] Combescure, M., Robert, D., Semiclassical Spreading of Quantum Wave Packets and Applications near Unstable Fixed Points of the Classical Flow, *Asymptotic Analysis* **14**, 377–404 (1997).

[CFKS] Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B., *Schrödinger Operators, with Applications to Quantum Mechanics and Global Geometry*, Texts and Monographs in Physics, Springer Verlag (1987).

[Da] Davies, E.B., The Functional Calculus,*J. London Math. Soc.* **52**, 166–176 (1995).

[DeB] De Broglie, L., Ondes et quanta, *Comptes Rendus Hebd. Séances Acad. Sci. Paris*, t.177, 507 (1923) [see also: Ann. d. Phys., série X, p. 2, (1925)].

[Del] Delort, J.M., *F.B.I. Transformation, Second Microlocalization and Semilinear Caustics*, Springer LNM 1522 (1992).

[DiSj] Dimassi, M., Sjöstrand, J., *Spectral Asymptotics in the Semiclassical Limit*, London Math. Soc. Lecture Notes Series 268, Cambridge University Press (1999).

[Du1] Duistermaat, J.J., *Fourier integral operators*, Lecture Notes, Courant Institute of Math. Sci., New York (1973).

[Du2] Duistermaat, J.J., Oscillatory integrals, Lagrange immersions and unfoldings of singularities. (English) *Commun. Pure Appl. Math.* **27**, 207–281 (1974).

[DuHo] Duistermaat, J.J., Hörmander, L., Fourier integral operators II,*Acta Math.* **128**, 183–269 (1972).

[Dy] Dyn'kin, E.M., An Operator Calculus Based upon the Cauchy–Green Formula, *J. Soviet Math.*, **4** (4), 329–334 (1975).

[Eg] Egorov, Yu.V., On Canonical Transformations of Pseudo-differential Operators, *Uspekhi Mat. Nauk* **25**, 235–236 (1969).

[En] Enss, V., *Introduction to Asymptotic Observables for Multi-Particle Quantum Scattering*, Lecture Notes in Math. vol. 1218, 61–92, Springer (1983).

[FeMa] Fedoriuk, M.V., Maslov, V.P., *Semiclassical Approximation in Quantum Mechanics*, Math. Phys. App. Math., Reidel, Dordrecht (1981).

[FePh1] Fefferman, C., Phong, D.H., On Positivity of Pseudodifferential Operators, *Proc. Nat. Sci. USA* **75**, 4673–4674 (1978).

[FePh2] Fefferman, C., Phong, D.H., The Uncertainty Principle and Sharp Gårding Inequalities, *Comm. Pures and Appl. Math.*, Vol. 34, 285–331 (1981).

[Fo] Folland, G.B., *Harmonic Analysis in Phase Space*, Princeton University Press (1989).

[Ga] Gårding, L., Dirichlet's Problem for Linear Elliptic Partial Differential Equations, *Math. Scand.* **1**, 55–72 (1953).

[GeP] Gérard, P., Mesures semi-classiques et ondes de Bloch, *Seminaire EDP de l'Ecole Polytechnique*, Palaiseau (France), Exposé n. XVI, 1990–91.

[GMS] Gérard, C., Martinez, A., Sjöstrand, S., A Mathematical Approach to the Effective Hamiltonian in Perturbed Periodic Problems, *Commun. Math. Phys.* **142**, n2 (1991).

[GrSj] Grigis, A., Sjöstrand, J., *Microlocal Analysis for Differential Operators, An Introduction*, London Math. Soc. Lecture Notes Series 196, Cambridge University Press (1994).

[GuSt] Guillemin, V., Sternberg, S., *Geometric Asymptotics*, Amer. Math. Soc. Survey 14 (1977).

[GRT] Guillot, J.C., Ralston, J., Trubowitz, E., Semi-Classical Asymptotics in Solid State Physics, *Commun. Math. Phys.* **116**, 401–415 (1988).

[Ha] Hanges, N., Parametrices and Propagation of Singularities for Operators with Non-Involutive Characteristics, *Indiana Univ. Math. J.* **28**, 87–97 (1979).

[He1] Helffer, B.,*Semiclassical analysis for the Schrödinger operator and applications*, Lecture Notes in Math. 1336, Springer Verlag (1988).

[He2] Helffer, B.,*Opérateurs Globalement Elliptiques*, Astérisque 112 (1984).

[HeRo] Helffer, B., Robert, D., Calcul fonctionnel par la transformation de Mellin et opérateurs admissibles, *J. Funct. Anal.* **53**, n.3, 246–268 (1983).

[HeSj1] Helffer, B., Sjöstrand, J., Multiple Wells in the Semiclassical Limit I, *Comm. Part. Diff. Eq.*, vol. 9, (4), 337–408 (1984).

[HeSj2] Helffer, B., Sjöstrand, J., *Résonances en limite semi-classique*, Mémoire Bull. Soc. Math. France, n. 24/25, tome 114 (1986).

[HeSj3] Helffer, B., Sjöstrand, J., Effet tunnel pour l'équation de Schrödinger avec champ magnétique, *Ann. Sc. Norm. Sup. di Pisa*, Ser. IV, **14**(4), 625–657 (1987).

[HeSj4] Helffer, B., Sjöstrand, J., *Semiclassical Analysis for Harper Equation III*, Mémoire Bull. Soc. Math. France, n.39, t.117, fasc.4 (1989).

[HeSj5] Helffer, B., Sjöstrand, J., Equation de Schrödinger avec champ magnétique et équation de Harper, *Springer L.N. Physics*, n. 345, 118–197 (1989).

[HeSj6] Helffer, B., Sjoestrand, J., Puits multiples en mecanique semi-classique - IV: Etude du complexe de Witten, *Comm. Part. Diff. Eq.* **10**, 245–340 (1985).

[HiSi] Hislop, P., Sigal, I.M., *Introduction to Spectral Theory*, Appl. Math. Sciences 113, Springer Verlag (1996).

[Ho1] Hörmander, L., *An Introduction to Complex Analysis in Several Variables.* North Holland–Mathematical library.

[Ho2] Hörmander, L., *The Analysis of Linear Partial Differential Operators*, Vols. I to IV. Springer Verlag (1985).

[Iv] Ivrii, V., *Microlocal Analysis and Precise Spectral Asymptotics*, Springer Verlag (1998).

[JeNa] Jensen, A., Nakamura, S., Mapping Properties of Functions of Schrödinger Operators between L^p-Spaces and Besov Spaces, *Adv. Stud. Pure Math.* **23**, 187–209 (1994).

[Ju] Jung, K., *Adiabatik und Semiklassik bei Regularität vom Gevrey-Typ*, PhD Thesis, T.U Berlin (1997).

[KK] Kawai, T., Kashiwara, Microhyperbolic Pseudodifferential Operators I, *J. Math. Soc. Japan* **27**, 359–404 (1975).

[KMSW] Klein, M., Martinez, A., Seiler, R., Wang, X., On the Born–Oppenheimer Expansion for Polyatomic Molecules, *Comm. Math. Phys.* **143**, 607–639 (1992).

[KoNi] Kohn, J.J., Nirenberg, L., An Algebra of Pseudo-differential Operators, *Comm. Pure Appl. Math.* **18**, 269–305 (1965).

[LaLi] Landau, L.D., Lifshitz, E.M., *Quantum Mechanics: Non-Relativistic Theory*, Pergamon Press, London (1958).

[Mar1] Martinez, A., Estimations sur l'effet tunnel microlocal, *Séminaire E.D.P. de l'Ecole Polytechnique* (1991–92).

[Mar2] Martinez, A., Precise Exponential Estimates in Adiabatic Theory, *J. Math. Phys.* **35** (8), 3889–3915 (1994).

[Mar3] Martinez, A., Estimates on Complex Interactions in Phase Space, *Math. Nachr.* **167**, 203–254 (1994).

[Mar4] Martinez, A., Microlocal exponential estimates and applications to tunneling, in *Microlocal Analysis and Spectral Theory*, NATO ASI Series C, Vol.490, (L. Rodino, ed.), Kluwer Acad. Publ., 349–376 (1997).

[MaSo] Martinez, A., Sordoni, V., Microlocal WKB expansions, *J. Funct. Analysis* **168**, 380–402 (1999).

[Mas] Maslov, V.P., *Théorie des Perturbations et Méthodes Asymptotiques*, Dunod, Paris (1972).

[MeSj] Melin, A., Sjöstrand, J., Fourier integral operators with Complex Valued Phase Functions, *Springer Lecture Notes in Math.*, **459**, 120–223 (1974).

[Mes] Messiah, A., *Quantum Mechanics*, North Holland (1970).

[Na1] Nakamura, S., On an Example of Phase-Space Tunneling, *Ann. Inst. H. Poincaré*, vol. 63, n. 2 (1995).

[Na2] Nakamura, S., On Martinez' Method of Phase Space Tunneling, *Rev. Math. Phys.*, vol. 7, n. 3, 431–441 (1995).

[PaUr] Paul, T., Uribe, A., The Semi-classical Trace Formula and Propagation of Wave Packets, *J. Funct. Analysis* **132**, 192–249 (1995).

[ReSi] Reed, M., B. Simon, B., Methods of Modern Mathematical Physics, Vols. I to IV, Academic Press (1972).

[Ro] Robert, D., *Autour de l'approximation semi-classique*, Birkhäuser (1987).

[SKK] Sato. M., Kawai, T., Kashiwara, M., Hyperfunctions and Pseudodifferential Equations, *Springer Lecture Notes in Math.* **287**, 265–529 (1973).

[Scha] Schapira, P., Conditions de positivité dans une variété symplectique complexe. Application à l'étude des microfonctions, *Ann. Sci. Ec. Norm. Sup.* 4ème série **14**, 121–139 (1981).

[Schr1] Schrödinger, E., Quantisierung als Eigenwertproblem, *Annalen der Physik* **79** (1926).

[Schr2] Schrödinger, E., *Collected Papers on Wave Mechanics*, London and Glasgow, Blackie and Son Limited (1928).

[Schw1] Schwartz, L., *Théorie des distributions*, Hermann, Paris (1966).

[Schw2] Schwartz, L., Théorie des distributions à valeurs vectorielles, *Ann. Inst. Fourier (Grenoble)* **7**, 1–141 (1957).

[Sh] Shubin, M.A., *Pseudodifferential Operators and Spectral Theory*, Springer series in Soviet Math., Springer (1987).

[SiSo] Sigal, I.M., Soffer, A., The N-particle Scattering Problem: Asymptotic Completeness for Short Range Systems, *Ann. Math.* **126**, 35–108 (1987).

[Si] Simon, B., Semiclassical Limit of Low Lying eigenvalues I - Non Degenerate minima, *Ann. Inst. H. Poincaré*, vol. 38, n. 3, 295–307 (1983).

[Sj1] Sjöstrand, J., *Singularités analytiques microlocales*, Astérisque 95 (1982).

[Sj2] Sjöstrand, J., Function Spaces Associated to Global I-Lagrangian Manifolds, Morimoto, M. (ed.) et al., *Structure of solutions of differential equations*, Proceedings of the Taniguchi symposium, Katata, Japan, June 26–30, 1995 and the RIMS symposium, Kyoto, Japan, July 3–7, 1995. Singapore: World Scientific. 369–423 (1996).

[St] Sternberg, S., *Lectures on Differential Geometry*, Prentice Hall (1965).

[Tat] Tataru, D., On the Fefferman–Phong Inequality and Related Problems, Preprint Northwestern University (2000).

[Tay] Taylor, M., *Pseudodifferential Operators and Spectral Theory*, Princeton Univ. Press, Princeton (1981).

[Tr] Treves, F., *Introduction to Pseudodifferential and Fourier integral operators*, Vols. 1 and 2, Plenum Press, New York (1980).

[VdW] Van der Waerden, B.L., From Matrix Mechanics and Wave Mechanics to Unified Quantum Mechanics, *Notices Am. Math. Soc.* **44**, n. 3, 323–329 (1997).

[Vo] Voros, A., The Return of the Quartic Oscillator - The complex WKB-Method, *Ann. Inst. H. Poincaré* **29**, 211–338 (1983).

[Wi] Witten, E., Supersymmetry and Morse theory, *J. Differ. Geom.* **17**, 661–692 (1982).