Preface

This book is based on lectures on geometry at the University of Bergen, Norway. Over the years these lectures have covered many different aspects and facets of this wonderful field. Consequently it has of course never been possible to give a full and final account of geometry as such, at an undergraduate level: A carefully considered selection has always been necessary. The present book constitutes the main central themes of these selections.

One of the groups I am aiming at, is future teachers of mathematics. All too often the geometry which goes into the syllabus for teacher-students present the material as pedantic and formalistic, suppressing the very powerful and dynamic character of this old – and yet so young! – field. A field of mathematical insight, research, history and source of artistic inspiration. And not least important, a foundation for our common cultural heritage.

Another motivation is to provide an invitation to *mathematics in general*. It is an unfortunate fact that today, at a time when mathematics and knowledge of mathematics is more important than ever, phrases like *math avoidance* and *math anxiety* are very much in the public vocabulary. An important task is seriously attempting to heal these ills. Ills perhaps inflicted on students at an early age, through deficient or even harmful teaching practices. Thus the book also aims at an informed public, interested in making a new beginning in math. And in doing so, learning more about this part of our cultural heritage.

The book is divided into two parts. Part 1 is called *A Cultural Heritage.* The section contains material which is normally not included into a mathematical text. For example, we relate some of the stories told by the Greek historian, *Herodotus*, in [17]. We also include some excursions into the history of geometry. These excursions do not represent an attempt at writing the history of geometry. To write an introduction to the history of geometry would be a quite different and very challenging undertaking. Even for the period up to the beginning of the Middle Ages it would vastly surpass what is presently undertaken in [21].

To write the History of Geometry is therefore definitely not my aim with Part I of the present book. Instead, I wish to seek out the roots of the themes to be treated in Part 2, *Introduction to Geometry.* These roots include not only the geometric ideas and their development, but also the historical con-text. Also relevant are the legends and tales – really *fairy-tales* – told about, for example, *Pythagoras.* Even if some of the more or less fantastic events of *Iamblichus'* writings are unsubstantiated, these stories very much became our *perception* of the geometry of Pythagoras, and thus became part of the *heritage of geometry*, if not of its *history.*

In Chapters 1 and 2 we go back to the beginnings of science. As geometry represents one of the two oldest fields of mathematics, we find it in evidence from the early beginnings. The other field being *Number Theory*, they go back as far as written records exist. Moreover, in the first written accounts from ancient civilizations they present themselves as already well developed and sophisticated disciplines.

Thus we find that problems which ancient mathematicians thought about several thousand years ago, in many cases are the same problems which are stumbling stones for the students of today. As we move on in Chapters 3 and 4, we find that great minds like Archimedes, Pythagoras, Euclid and many, many others should be allowed to speak to the people of today, young and old. They are unsurpassable tutors.

The mathematical insight which Archimedes regarded as his most profound theorem, was a theorem on geometry which was inscribed on Archimedes' tombstone. All of us, from college student to established mathematician, must feel humbled by it. What does it say? Simply that if a sphere is inscribed in a cylinder, then the proportion of the volume of the cylinder to that of the sphere, is equal to the proportion of the corresponding surface areas, counting of course top and bottom of the cylinder. The common proportion is $\frac{3}{2}$. This is a truly remarkable achievement for someone who did not know about integration, not know about limits, not know about... Its beauty and simplicity beckons us. How did Archimedes arrive at this result? Archimedes deserves to be remembered for this, rather for the silly affair of him having run out into the street as God had created him, shouting – *Eureka, Eureka!* But the story may well be true, his absentmindedness under pressure cost him dearly in the end.

Pythagoras and his followers certainly did not discover the so-called *Pythagorean Theorem.* The Babylonians, and before them the Sumerians, not only knew this fact very well, they also had the sophisticated number-theoretical tools for constructing all *Pythagorean triples*, that is to say, all natural numbers a, b and d such that $a^2 + b^2 = d^2$. And the astronomers and engineers – or, if we prefer, the astrologers and priests – of ancient Babylonia, or Mesopotamia, used these insights to construct trigonometric tables. Tables which were simple, accurate and powerful thanks to the *sexagesimal* system they used for representing numbers. What a challenging project for interested college students to understand the math of the Plimpton 322 tablet at Columbia University! And to correct and explain the four mistakes in it. Or finally to construct the successor or the predecessor of this tablet in the series it must have belonged to.

So what did Pythagoras discover himself? We know nothing with certainty of Pythagoras' life before he appeared on the Greek scene in midlife. Some say that he traveled to Egypt, where he was taken prisoner by the legendary, in part infamous, Persian King *Cambyses II*, who also ruled Babylon, which had been captured by his father *Cyrus II.* Pythagoras was subsequently brought to Babylon as a prisoner, but soon befriended the priests, the *Magi*, and was initiated into the priesthood in the temple of Marduk. We tell this story as related by Herodotus and Iamblichus in [17] and [22]. However, the accounts given in these classical books are not always historically correct, the reader should consult the footnotes in [17] to get a flavor of the present state of Herodotus, *The Father of History*, by some of his critics labeled *The Father of Lies!* But Herodotus is a fascinating story-teller, and the place occupied by Pythagoras today has considerably more to do with the legends told about him than with what actually happened. So with this warning, do enjoy the story.

Euclid's Elements represents a truly towering masterpiece in the development of mathematics. Its influence runs strong and clear throughout, leading to non-Euclidian geometry, Hilbert's axioms and a deeper understanding of the foundations of mathematics. The era which Euclid was such an eminent representative of, ended with the murder of another geometer: Hypatia of Alexandria.

In Chapter 5 we describe how the foundation of present day geometry was created. Elementary Geometry is tied to straight lines and *circles.* The theorems are closely tied to constructions with straightedge and compass, reflecting the postulates of Euclid. In *higher geometry* one moves on to the more general class of conic sections, as well as curves of higher degrees.

Descartes introduced – or reintroduced, depending on your point of view – algebra into the geometry. At any rate, he is credited with the invention of the *Cartesian coordinate system*, which is named after him.

In Chapter 6, the last chapter of Part 1, we discuss the relations between geometry and the real world. The qualitative study of *catastrophes* is of a geometric nature. We explain the simplest one among *Thom's Elementary Catastrophes*, the so-called *Cusp catastrophe.* It yields an amazing insight into occurrences of *abrupt events* in the real world.

Also tied to the real world are the fractal structures in nature. Fractals are geometric objects whose *dimensions are not integers,* but which instead have a *real number* as dimension. Strange as this sounds, it is a natural outgrowth of *Felix Hausdorff's* theory of dimensions. Hausdorff was one of the pioneers of the modern transformation of geometry, referred to in his time as the *High Priest of point-set topology.* In the end, this all did not help him. He knew, being a Jew, what to expect when he was ordered to report the next morning for deportation. This was in 1942 in his home town of Bonn, Germany. Instead of doing so, Hausdorff and his wife committed suicide.

The *Geometry of fractals* shows totally new and unexpected geometric phenomena. Amazingly, what was thought of as *pathology*, as useless curiosities, may turn out to give the most precise description of the world we live in.

In Part 2, *Introduction to Geometry*, we take as our starting point the axiomatic treatment of geometry flowing from Euclid.

Euclid's original system of axioms and postulates passed remarkably well the test of modern demands to rigor. But an explanation was nevertheless very much called for, as his original system was set on a somewhat shaky foundation by our current mathematical standards. This clarifying explanation of the foundations was provided by Hilbert. The search for a proof of Euclid's Fifth Postulate at an earlier age, had met with no success. One version of this postulate asserts that there is one and only one line parallel to a given line through a point outside it.

Assuming the converse, one wanted to derive a contradiction. But instead the relentless toil produced alternatives to Euclidian Geometry. This was a highly troubling development for an age in which non-Euclidian Geometry would appear as heretic as "Darwinism" appears in some circles today. We explain non-Euclidian Geometry in Chapter 9. But first we need to do some serious work on foundations. We start with *Logic and Set Theory.* In fact, the Intuitive Set Theory, even as put on a firmer foundation by *Cantor*, turned out to imply grave contradictions. The best known is the so-called *Russell's Paradox*, which we explain in Section 7.3.

Thus arouse the need for *Axiomatic set theory*, to which we give an introduction. The aim is to give a flavor of the field without going into the technical details at all.

We then explain the interplay between *axiomatic theories* and their *models* in Sections 7.3 and 7.4. The troubling result of *Gödel* is explained, in simplified terms, showing that a *mathematical Tower of Babel* as perhaps dreamt of by Hilbert, is not possible: Any axiomatic system without contradictions among its possible consequences, will have to live with some *undecidable* statements. That is to say, statements which are perfectly legal constructions within the system, being inherently undecidable: Their truth or falsehood cannot be ascertained from the system itself.

In Chapter 8 we apply these insights to axiomatic projective geometry. This is an extensive field in itself, and a complete treatment does of course, fall outside the scope of this book. But we give a basic set of axioms, to which other may be added, thus in the end culminating with a set which determines uniquely the *real, projective plane.* This is not on our agenda here. But we do give, in some detail, two important models for the basic system of axioms. The *Seven Point Plane* and the *real projective plane* $\mathbb{P}^2(\mathbb{R})$. That the simple axioms still hold intriguing open problems, is explained in Section 8.2. Use of powerful computers in conjunction with dexterous programming holds great promise for new insights, thus there exist ample possibilities for exiting research.

In Chapter 9 we are ready to explain models for non-Euclidian Geometry. In the *hyperbolic plane* there are infinitely many lines parallel to a given line through a point outside it. In the *elliptic plane* there are no parallel lines: Two lines always intersect. A model for this version is provided by $\mathbb{P}^2(\mathbb{R})$.

Plane non-Euclidian geometries have, of course, their spatial versions. This is best understood by turning to some of the basic facts from *Riemannian Geometry*, which we do in Section 9.5.

Chapter 10 contains some much needed mathematical tools, simple but essential. We need them for constructions to be carried out in following chapters, where we employ these standard techniques. The reader is advised to take the moments needed to ingest this material, which may well appear somewhat dry and barren at the first encounter.

In Chapter 11 we are now able to give coordinates in the projective plane, introduce projective n-space and discuss affine and projective coordinate systems. Again, the material may appear dry, but the reader will be rewarded in Chapter 12. There we use these techniques to give the remarkably simple proof of the theorem of Desargues. We introduce *duality for* $\mathbb{P}^2(\mathbb{R})$ and start the theory of conic sections in $\mathbb{R}^2$ and $\mathbb{P}^2(\mathbb{R})$ discussing tangency, degeneracy and the familiar classification of the conic sections. Pole and Polar belong to this picture, as well as a very simple proof of a famous theorem of Pascal. Using it, we then prove the theorem of Pappus by a classical technique known as *degeneration*, or as the *principle of continuity.* Here we give the first, naive, definition of an algebraic curve.

In Chapter 13 we find the transition to the study of curves of degrees greater than 2. This forms the fundament for Algebraic Geometry, and gives a glimpse into an important and very rich, active and expanding mathematical field. Here we encounter the *cubical parabola*, merely a fancy name for a familiar curve, but also the enigmatic *semi cubical parabola*, so important in modern *Catastrophe Theory.* However, as we shall see in the following chapter, in Chapter 14, from a projective point of view these two kinds of affine curves are the same! This is shown at the end of Section 14.5. We also learn about the *Folium of Descartes*, the *Trisectrix of Maclaurin*, of *Elliptical Curves* – which are by no means ellipses! – and much more. Chapter 14 concludes with Pascal's *Mysterium Hexagrammicum*, which may be obtained as a beautiful application of Pascal's Theorem: Dualizing it the Mystery of the Hexagram is revealed.

In Chapter 15, as the title says, we sharpen the Sword of Algebra. The aim is to show how one finally disposes of the three problems, which have haunted mathematicians and amateurs for two millenia. And unfortunately, still does haunt the latter. The algebra derives in large part from the heritage of Euclid, relying as it does on *Euclid's algorithm.* This mathematics also constitutes the foundation for the important field of *Galois theory* and the theory of equations and their solvability by radicals. That theme is, however, not treated in the present book.

In Chapter 16 we use this algebra for proving that the three classical problems are insoluble: Trisecting an angle with legal use of straightedge and compass, doubling the cube using straightedge and compass, and finally we see how the transcendency of the number π precludes the squaring of the circle using straightedge and compass. Gauss' towering achievement on constructibility of regular polygons conclude the chapter. The solution of this problem by Gauss transformed the final answer to the geometric problem into a number theoretic problem on the existence of certain primes, namely primes of the form $F_r = 2^{2^r} + 1$, the so-called *Fermat primes* . For $r = 0, 1, 2, 3, 4$ the numbers P_r are $3, 5, 17, 257$ and 65537. They are all primes, but then no case of an r yielding a prime is known. Gauss proved that if q is a product of such primes p_r, all of them distinct, then the regular $n = 2^m q$-gon may be constructed with straightedge and compass, and that this is precisely all the constructible cases. Thus for example the regular 3-gon, the regular 5-gon and the regular 15-gon are all constructible with straightedge and compass, as is the regular 30-gon and the regular 60-gon. The first impossible case is the regular 7-con. Now Archimedes constructed the regular 7-gon, but he used means beyond legal use of straightedge and compass. In Section 4.4 we have given Archimedes' construction of the regular 7-gon, the regular *heptagon*, by a so-called *verging construction*. It is not possible by the *legal use* of compass and straightedge, but may be carried out by conic sections or by a curve of degree 3. In fact, such constructions were very much part of the motivation for passing from *elementary geometry* to *higher geometry* in the first place.

In Chapter 17 we take a closer look at the theory of fractals. We explain the computation of fractal dimensions.

Chapter 18 contains a mathematical treatment of introductory Catastrophe Theory. We explain the Cusp Catastrophe as an application of geometry on a cubic surface. For this we also explain some rudiments of Control Theory.

Several variations of courses may be taught from the present text. Two possibilities are outlined below, each with two hours of lecture and preferably one hour of discussion per week, each of one term duration. I have labeled them as follows:

– *Geometry 1: Historical Topics in Geometry*
– *Geometry 2: Introduction to Modern Geometry.*

They may well be merged into one course, then possibly with 3 hours of lecture and two hours of discussion. Geometry 1 would be taught before Geometry 2.

Geometry 1 would comprise all of Part 1, and in addition a somewhat cursory treatment of selected sections from Part 2.

Geometry 2 would essentially consist of Part 2, with an in-depth treatment of the material from Part 2 having been more summarily taught in Geometry 1.

Geometry 1 requires modest background in mathematics, and could be offered to elementary school teachers, possibly in the settings of a Community College.

Sufficient background for Geometry 2 would be high school math with trigonometry and analytic geometry, or a pre calculus course. However, some prerequisites may be dealt with in an extra discussion hour.

With some faculty guidance a freshman seminar combining elements of the two alternatives is also a possibility.

Some of the material in this book has been published in the author's [19] and [21]. The material is included here with the permission of *Fagbokforlaget*, the publisher of [19] and [21]. A large number of the illustrations are created by the marvelous system Cinderella, [28], some of them were made by Ulrich H. Kortenkamp, one of the authors of the system. Others were made by the author, who would like to take this opportunity to thank Professor Kortenkamp for his efforts in making these illustrations, as well as for his valuable advice and assistance during this work. A few illustrations are made with the aid of the Computer Algebra system MAPLE, and finally some were made by Springer's illustrator, based on sketches by the author.

Thanks are also due to Dr. Even Førland of the Norwegian Statoil Corporation, who has red parts of the manuscript and provided valuable comments.

It is a great pleasure to thank Springer Verlag, in particular the Mathematics Editor Dr. Martin Peters as well as Mrs. Ruth Allewelt, for their enthusiasm and support. Mrs. Daniela Brandt of Springer's Production Department has provided great support and assistance in getting the manuscript in shape for the printer.

Bergen, *Audun Holme*
November 2001

3 Greek and Hellenic Geometry

3.1 Early Greek Geometry. Thales of Miletus

The word *geometry* is derived from two Greek words, namely $\gamma\eta$, gē, which means *earth* and $\mu\varepsilon\tau\rho o\nu$, metron, which means *measure*. Our sources on early Greek geometry – and mathematics in general, for that matter – are sparse. Indeed, as far as mathematical contents is concerned we have to rely on the work of the first serious *historian of mathematics*, namely *Eudemus of Rhodes*, 350 – 290 B.C. He was, probably, a student of Aristotle, at any rate a close associate and collaborator of him. But Eudemus of Rhodes should not be confused with *Eudemus of Cyprus*, another philosopher associated with Aristotle. In any case, our Eudemus is known to have written three works on the history of mathematics, namely *The History of Arithmetic, of Geometry* and *of Astronomy.* All three are lost now, but were available to Hellenistic mathematicians and used to the extent that at least some of their contents is known to us today. In particular Eudemus reports, in his History of Astronomy, that *Thales of Miletus* predicted a solar eclipse, which is presumed to be the one which occured on May 28, 585 B.C. But most historians of mathematics tend to be skeptical to this claim. The reason for this is that Thales is generally agreed to have been the first Greek astronomer, and that such abilities would have been unlikely at this early stage of Greek astronomy. However, it appears that the most plausible explanation is offered by *van der Waerden* in [35], where he writes:

> *The conclusion is inescapable that he must have drawn upon the experience of Oriental astronomers.*

By the way, the Greek historian *Herodotus* also makes this assertion concerning the prediction by Thales. The solar eclipse occured during a battle fought between the Lydians, under their King *Alyattes*, and the Medians under their King *Astyages*. The war had been going on for five years, and when the eclipse occured during an ongoing battle, the belligerent parties found it prudent to end the fighting and make peace. The Gods, evidently, did not approve of what they were doing. Thales had ties to the Lydian kingdom, and when Alyattes' son Croesus later went to war against the Persian King Cyrus II, who had meanwhile conquered the Median kingdom, Thales went along as an advisor to the Lydian King. Thales is credited with a clever scheme for splitting the river Halys, so that the Lydian troops could pass over.

Eudemus' historical works are lost. But their contents are, to some extent, known through later summaries. The last Greek philosopher and mathematician was *Proclus Diadochus*. He was head of the Neoplatonic Academy in Athens late in the fifth Century A.D., one of the last holdouts of classic civilization. At that time Eudemus' books were still extant. As all research towards the end of the classic civilization, Proclus' research is not very original. But as part of his work at the Academy he wrote a summary of Eudemus' History of Geometry, as an introduction to his own *Comments on Euclid's Elements*, Book I. This is essentially the only surviving source on early Greek geometry, frequently referred to as *The Eudemian Summary*. There can be no doubt that Proclus amply deserves a honorable place in the history of geometry and mathematics for preserving this knowledge for posterity. Another important contribution by Proclus was the formulation of Euclid's Fifth Postulate as we state it today, usually referred to as *Playfair's Axiom*. See Section 4.1.

Thales is the first Greek mathematician whose name we know. He lived and worked in Miletus, a Greek city in Asia Minor, now in Turkey. He was born about 625 B.C. and died around 545 B.C., in Miletus. We may regard Thales as the *Founding Father of Greek Geometry*. His mother was *Cleobulina*, the first woman philosopher in Greece. Thales referred to her as *The Wise One*.

There are reports that Thales was *of Phoenician descent*, but others refute this by asserting that *"... the majority opinion considered him a true Milesian, and of a distinguished family."* Do we sense a trace of bigotry here? Perhaps the infusion of some Phoenician blood through Thales did the Greeks and their science some real good...

Thales is supposed to have estimated the height of a pyramid in Egypt by measuring its shadow at the time when the shadow cast by himself was equal in length to his own height. Eudemus ascribes to Thales a method for finding the distance between two ships at sea. We do not know exactly what this method was, but van der Waerden in [35] supposes that it might be something like the method described by the Roman surveyor *Marcus Junius Nipsius*, which goes as follows:

> In order to find the distance from A to the inaccessible point B, one erects in the plane a perpendicular AC to AB, of arbitrary length, and determines its mid point D. On C one constructs a line CE perpendicular to CA, in a direction opposite of AB, and one extends it to a point E, collinear with D and B. Then CE has the same length as AB.

The rule is illustrated in Figure 3.1.

Thales also is credited for discovering that the base angles of isosceles triangles are equal, and that vertical angles are equal. He is also said to have

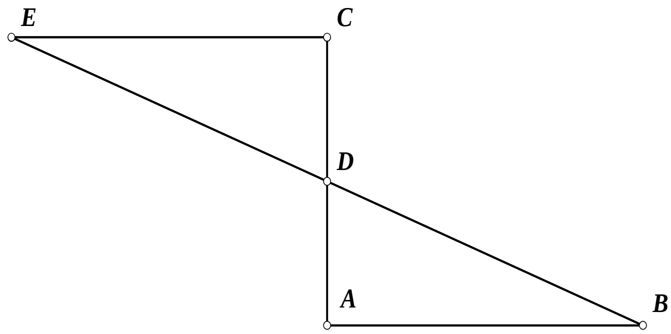

Fig. 3.1. To find the distance to an inaccessible point.

discovered that a diameter of a circle divides it in two equal parts. In what sense Thales "discovered" these geometrical facts is not clear, it does seem reasonable to assume that this knowledge would have predated Thales by perhaps more than a thousand years, in Egypt, Mesopotamia, and elsewhere in the East. He may, however, have studied this material, providing some sort of proofs for the above statements.

According to Aristotle, Thales was ridiculed by some Milesians for directing a lot of energy to activities which had no useful applications, and from which he made no profit. Thales then decided to show them that if he had thought it worthwhile, he could do better than most of them in this regard as well. Thus, noticing signs that a bumper crop of olives was in the comings, he bought up all the presses. When the bumper crop then subsequently *did* materialize, the growers had to buy or rent presses from him, at a substantial price.

3.2 The Story of Pythagoras and the Pythagoreans

Pythagoras of Samos is a rather enigmatic figure. It is frequently asserted in texts on the history of mathematics that we know practically nothing of his life and work prior to the time when he founded the *school of the Pythagoreans* in Croton, at which time Pythagoras may have been in his mid 50's. We do know however, that he appears at this precise point, and that he undoubtedly possessed extensive knowledge of mathematics in general and geometry in particular. Prior to that time this kind of knowledge is only very sparsely documented in Greece, and all of it comes to us from Thales. But in the East, in Egypt, Mesopotamia, in India and even, perhaps, in the early Indus valley civilization, as well as in China, we find evidence of extensive insights into these matters. Add to this the many stories which are told concerning his travels in Egypt and more widely. We have to realize, however, that for now there is no solid evidence on which the legends of Pythagoras' travels can be accepted as historical facts. So until some new papyrus is found in Egypt, or a tablet uncovered from ancient Babylon, relating the tale of the Greek *visiting priest* at the Temple, we might as well sit back and enjoy the stories. Some of them simply are too good not to be true!

Pythagoras was born about 570 B.C. in Samos, one of the most fertile Greek islands, just off the coast of Asia Minor. It seems to be general agreement that he died in the Greek city of Metapontium, in southern Italy, probably some time during the first decades of the fifth Century B.C., one estimate being approximately 480 B.C. At any rate there are reports that he died at the advanced age of 90.

Some historians of mathematics think that Pythagoras was a student of Thales. Others feel that the age-gap between them makes this unlikely. But with the – admittedly hypothetical – dates of birth and death we have put down, Pythagoras would have been 25 at the time of Thales' death. This does not preclude him having been a student of Thales, but it is probable that Pythagoras at least also had other teachers, working in the same mathematical environment as Thales. In fact, Samos and Miletus were geographically close.

Iamblichus relates in [22] that Pythagoras *"...went to Pherecydes and to Anaximander, the natural philosopher, and also he visited Thales at Miletus. All of these teachers admired his natural endowments and imparted to him their doctrines. Thales, after teaching him such disciplines as he possessed, exhorted his pupil to sail to Egypt and associate with the Memphian and Diospolitan priests of Jupiter by whom he himself had been instructed, giving the assurance that he would thus become the wisest and most divine of men."*

So according to this source, Pythagoras followed in Thales' footsteps. Not only did he take up his geometry, he also made extensive travels in the known civilized world. In Samos *Polycrates* assumed dictatorial powers, but he was in many ways an enlightened ruler, and at least in the beginning Pythagoras may have had good relations with him.

Polycrates had allied himself with Amasis, the King of Egypt. Polycrates was very successful in the beginning, and he established Samos as a naval power, he build temples, harbors and aqueducts and he encouraged art and science including mathematics. Herodotus relates how Polycrates became worried when he received a message from his Egyptian ally, warning him that his good fortune would eventually make the Gods envious, thus bringing some kind of disaster down on him. The advice he gave was for Polycrates to throw away his most valued possession. The grief this would cause him, should suffice to placate the envious Gods. After thinking about it, Polycrates decided that a precious ring he owned would be a suitable object to loose, and he went out to sea on a boat, where threw his ring into the water. Some days later, however, a local fisherman caught a big fish. The fish was so extraordinary that the fisherman brought it to Polycrates, expecting to be rewarded lavishly. Polycrates was very pleased, and showed it by inviting the fisherman to his supper, where the fish was to be served. The cook started the preparations and cut the fish open, and in its stomach he found the ring. He brought the ring to Polycrates, who was not exactly overjoyed. When Amasis learned about this, he realized that Polycrates could bring him nothing but bad luck, and canceled his alliance with him. And in fact, towards the end of his reign Polycrates engaged in some ill-conceived schemes, trying to ally himself with the Persians against the Egyptians. This failed because of mutiny among the men he sent, who with good reason suspected that Polycrates really wanted to get rid of them. He himself was later lured into an ambush by the Persians and suffered a shameful death.

Returning to Pythagoras, he went to Egypt, some say around 535 B.C. Polycrates had supplied him with letters of recommendation, so he could gain access to the Temples there.[1] He visited many temples where he had discussions with the priests. He tried to gain admittance to the Order of the Temples, and finally succeeded when he was admitted into the Temple and Priesthood at Diospolis, near Memphis. Here he stayed for some time, and absorbed their customs and their geometry, as well as their magic and astrology.

But this quiet life was interrupted when there appeared on the scene a Persian King and warlord by the name *Cambyses*. He invaded Egypt in 525 B.C., and capturing Memphis came across Pythagoras in the Temple, as Iamblichus relates in [22]. Pythagoras was then taken prisoner by Cambyses, and if this story is true, he must have had some very exiting and interesting years, under Cambyses' rather heavy hand. In the beginning it would not have been too bad, Cambyses himself respected the Egyptians and showed great interest in their traditions and customs. He even had himself designated a *Pharo* under the name *Ramesut*. He also had himself initiated into the priesthood, and if Pythagoras were around this, he might have had something to do with it. In fact, Cambyses' father was *King Cyrus* II or *Cyrus the Great*. He could possibly have met Thales, Pythagoras' mentor, under the following circumstances: According to Herodotus, Thales accompanied King Croesus when he went to war against the Persians under King Cyrus. Croesus lost, and after several dramatic events he was saved from being burned alive on a pyre erected by the victorious Persians. These same events also led him to become a trusted friend and advisor of King Cyrus. This happened in 547 B.C., admittedly late in Thales' life, if not after his death.

Cyrus was one of Persia's great Kings, who went on to capture the marvelous ancient City of Babylon, in 539 B.C. He is the *Cyrus the King* referred to in the Old Testament, who restored the Jews to Palestine and ordered the Temple of Jerusalem to be rebuild. Unfortunately for him, however, he did not rest on his laurels. Instead, he marched with his troops across *the Araxes*, the river now named *Araks* which flows east to the Caspian Sea. He

[1] Some say this, others claim that Pythagoras feared Polycrates, and fled because of him.

went against the *Massagetic queen Tomyris*, she ruled over a kingdom in that area. His advisor Croesus was with him, and the crossing of the Araxes was undertaken on his advice. This was a disastrous move. Tomyris defeated Cyrus, who was slain in a battle 529 B.C. Then his son, Cambyses II, succeeded him on the Persian throne. On his fathers advise, he retained Croesus as an aide and advisor, in spite of the sad outcome of his last service to his father. And Croesus accompanied Cambyses to Egypt. Thus Pythagoras and Cambyses' aide would have some points of contact.

At any rate, the good state of affairs for Pythagoras in Egypt did not last. Cambyses continued his milliary expansion, and now he met with some very serious, humiliating setbacks and defeats. Without going into details, let us just relate that he turned into a paranoid man, suspicious of everything. When he arrived back from one of his ill-fated expeditions, his troops decimated and starved, having been reduced to cannibalism, he unfortunately came just in time for a big celebration in Memphis. Feeling that the people rejoiced because of his own misfortune, he ordered the leading citizen rounded up and executed. The most repulsive incident occured when it was explained to him that the celebration was on occasion of the appearance of a very special calf, the latest incarnation of the God Apis. On his orders the calf was brought into his presence. Cambyses, in a fit of senseless rage, grabbed his sword and dealt the Holy Calf a powerful blow, wounding it in the thigh, in front of all the terrified Egyptians. The Holy Calf fell to the ground, and it died some time afterwards from the infected wound.

He also committed various other acts of sacrilege, like several instances of outrageous profanation of temples, killings of priests, he broke up ancient tombs and examined the bodies, burned them in some cases, and so on.

Matters worsened. Cambyses appears to have gone completely mad. According to Herodotus one of the misdeeds he committed was to have his own brother, *Smerdis*,[2] murdered. Smerdis had been a member of his Egyptian expedition, but Cambyses had sent him back to Persia because of jealousness caused by his brother's physical strength. Some time after Smerdis' return, Cambyses had a dream which caused him great worry: He dreamt that a messenger arrived from Persia, telling him that Smerdis was sitting on the royal throne and that his head was touching the sky. Interpreting this to mean that his brother would kill him and seize the throne of Persia, Cambyses sent his most trusted Persian friend *Prexaspes* back to Persia to do away with Smerdis. Prexaspes dutifully did what he had been ordered. And then he informed the people that His Royal Highness the Prince spent all his time in seclusion at the palace, praying for the success of his brother the King during his campaign abroad. Cambyses later rewarded him for his services by murdering his son in front of his very eyes, in order to prove his marksmanship with bow and arrow and ability to hold his liquor.

[2] According to the Persian sources Cambyses murdered a brother by the name *Bardiya*.

Now Herodotus relates that Cambyses had left the control of his household with a man who belonged to the caste of *the Magis*, his name was *Patizeites.* Patizeites had a brother, named Smerdis, like the prince. This brother also looked a lot like the murdered prince, and as Patizeites knew of Cambyses' foul deed regarding his brother, he hatched a rather obvious plan: He had his own brother usurp the throne, claiming to be Cambyses' brother![3]

The Magis constituted the hereditary caste of priests among the ancient Persians. They interpreted dreams and performed sacred rituals, being devoted to the Gods. In the New Testament the astrologers who divine the birth of the King of the Jews by the appearance of a star in the east are called *Magis.* The priests of Babylonia are also frequently called *Magis*, and of course the term is preserved today in our word *magic*.

Heralds were sent out proclaiming the change of regent, and one of them happened to encounter Cambyses and his men in Ecbatana in Syria. When brought before the rightful, if incompetent, King, the herald was questioned about the situation. Cambyses suspected that Patizeites had double-crossed him, but the latter had the explanation ready: *"I think, my lord, that I know what happened. The rebels are the two Magi brothers you left in charge of your household. One of the brothers is named Smerdis, as you may recall."* Cambyses now realized the true meaning of his dream. The Smerdis on the throne was really Smerdis the Magi! The murder of his brother had served no purpose, in fact it had made the prophesy of the dream come true, rather than preventing it from happening. As sanity started to return, he understood the depths to which he had fallen, and he bitterly lamented the abysmal situation in which he found himself. Finally he resolved to march back to Persia at once, to attack the Magi. But as he leaped into the saddle, the cap fell off the sheath of his sword. The exposed blade cut his tight, at the very spot where he had struck the sacred Egyptian Bull of Apis. Cambyses now felt that he was mortally wounded, and asked his men for the name of the town they were in. Being told that the name was Ecbatana, he realized the true meaning of a prophecy from the oracle at Buto: Namely, that he should die at Ecbatana. He had thought this to be *Median Ecbatabna*, his capital city, and that he should therefore die at home of old age. Now he realized that the oracle meant Ecbatana in Syria. At this point sanity fully returned to Cambyses, and he said no more. After twenty days he called the leading Persians together, and explained the situation to them. In tears he bitterly lamented his cruel fate, and the Persians tore their cloths, crying and

[3] Persian sources give the name of the Magian usurper, or pretender, as *Gaumata.* Thus there is no homonymy in the Persian version of this story, as well as other discrepancies with the account as given by Herodotus. It is generally accepted among historians that Herodotus' version of the story is far from accurate. A political intervention by priests of the temples in the face of a ruler who was obviously incompetent and mentally disturbed, as well as a political rivalry between Medes and Persians with economic and social ramifications, has undoubtedly taken place. But the details are lost today.

groaning. Shortly after, gangrene and mortification of the thigh set in, and Cambyses died.

However, his men really did not believe him. They suspected another malicious lie, to set the country against his brother Smerdis.

Thus no obvious course of action seemed to present itself, and about one year of political strife followed in Persia, with the Magi on the throne. Prexaspes originally decided to side with the Magis, out of fear for punishment and also his bad feelings towards the house of Cyrus and Cambyses. Thus he changed his story about having murdering Smerdis the Prince. The Magi rule ended when a young and ambitious nobleman by the name *Darius*, himself of royal descent, headed a successful *coup d'etat*. Prexaspes, repenting his treason to the Persian cause (the Magi were originally a Median caste), confessed his crime to an assembled crowd from the main tower, and then leaped to his death.[4] Darius then assumed power, to become the famous Darius I, *Darius the Great.*

The story of the false Smerdis, the usurpation of power by the Magis and finally the accession of Darius plays an important role in the history of mathematics, at least indirectly. In fact, the Persian version of it, as told to us by Darius himself, forms part of the inscription at Behistun, described in section 2.1, and thus provided the basis for Rawlinson's decipherment of the cuneiform script. This again led to our present insights into the mathematics in Mesopotamia, of the Sumerians, Assyrians and the Babylonians. As already noted, the inscription by Darius himself differs considerably from the tale as told by Herodotus. For more details, see note 25 on page 571 in [17].

Pythagoras, however, had been brought to Babylon by Cambyses' troops. At least so the story goes. The political situation in the Persian Empire being somewhat murky, he sought refuge in the Temple, where he was once more initiated into the Priesthood. *Iamblichus* writes as follows in [22], in the fourth Century A.D.:

"Here the Magi instructed him in their venerable knowledge and he arrived at the summit of arithmetic, music and other disciplines. After twelve years he returned to Samos, being then about fifty six years of age."

There are some ancient busts claiming to show what Pythagoras may have looked like. One is a bronze copy of an original believed to be from the fourth century B.C., which is displayed at Villa dei Papiri in Herculaneum, Museo Nazionale, Neapels. Here Pythagoras is shown wearing turban and oriental dress, absolutely compatible with our story. A photo of the bust is shown in [21] and in [35].

Iamblichus has Pythagoras' stay in Egypt to last for 22 years, plus 12 years in Babylon, altogether 34 years abroad. At any rate he spent many years in Egypt and in Babylon, working and learning in the temples.

[4] Still according to Herodotus, the Persian story runs differently. There is no character by the name *Prexaspes* in that version.

Cambyses had died in 522 B.C., and Polycrates, the tyrant of Samos, was killed by the Persians about the same time. King Darius I took over in 521 B.C., and after Polycrates death Samos came under his rule. Exactly when Pythagoras returned to Samos is uncertain. Some say that he returned at a time when Polycrates was still alive and in power, others assert that he returned at a time when Samos had fallen under Persian rule. In any case, after the fall of the Magi from power, it would seem to make sense for Pythagoras to leave Babylon, since he presumably had close ties with that group.

Iamblichus reports that Pythagoras formed a school in the city of Samos, called *the semicircle*. He also reports that Pythagoras made a cave outside the city, where he did his teaching, and spent both nights and days doing research in mathematics. But then Iamblichus goes on to tell how Pythagoras attempted to employ the same didactical principles he had learned in the temples of Egypt and Babylon, to teaching the Samians. This did not work too well, they found his teachings too abstract and symbolic. Pythagoras did not like such attitudes any better than some present day college professors do, and decided to leave. At least this is the reason Pythagoras himself is supposed to have given for leaving Samos.

Actually, the Samians were by no means ignorant of geometry. Herodotus relates how they constructed, at the order of Polycrates, an aqueduct for bringing drinking water to the capital city by the same name as the island. They had to dig a tunnel through a mountain, and started to dig at both ends simultaneously. And in fact, they met in the middle of the mountain with remarkable accuracy! The direction of the tunnel had to be found by reasoning with similar triangles. Also a fairly sophisticated use of a diopter had to be employed. *Heron of Alexandria* explains the method in his work *Dioptra*, about 600 years later, around 60 A.D. For details, see [21] or [35]. The engineers, some of them quite possibly being *slaves*, who worked on the tunnel at Samos certainly knew quite sophisticated geometry. But this knowledge was part of their practical work in the field, not necessarily as an object of the *"refined contemplation"* considered worthy of *free men*.

So Pythagoras left for Croton, a Greek city on the coast of southern Italy. Here he formed his school or brotherhood, *The Pythagoreans*. The society consisted of an *inner circle*, whose members were called *mathematikoi*, and an outer circle whose members were known as *the akousmatics*.

The mathematikoi lived permanently with the Society, they had no personal belongings, were vegetarians and practiced celibacy, did not eat *beans*, and did not wear cloths made of animal skin. Presumably this was the way of life Pythagoras had picked up at the temples in Egypt, although *Herodotus* does report on ample supplies of meat and wine for the Egyptian priests.

It should be noted that there are marked similarities between the practices of the Pythagoreans and those associated with *the Orphic Cult*. Orpheus of Thrace was the founder of this cult. He played so divinely on the lyre that all nature stopped to listen. When his wife *Eurydice* died, he went to the nether world, to *Hades*, to bring her back. By the music from his lyre he succeeded in obtaining her release, but on the condition that he would not look at her until they were clear of the world of death. However, he could not bear to refrain from looking, and she had to return to Hades for good.

The akousmatics, however, were allowed to live normal lives. Both men and women were allowed to be Pythagoreans, and there are some reports of women Pythagoreans who became well known mathematicians and philosophers.

There are accounts to the effect that *Pythagoras had a wife*. Her existence would seem to contradict the claimed practice of celibacy, but this particular kind of contradiction should not disturb historians too much. Her name was *Theano*, and she had three daughters with Pythagoras. Together with them she is said to have continued Pythagoras' school after his death. Her most important mathematical work is supposed to have been a treatise on *the Golden Section.* We refer to [39], [23] and to [36]. As far as this author's information goes, this is the first known, or claimed, individual name of a woman mathematician. Pythagoras' three daughters also were Pythagoreans. *Damo* is said to have been entrusted the responsibility for her fathers works, which she refused to sell and therefore had to live in poverty. The two other daughters *Arignote* and *Miyia* were also Pythagoreans, and are credited with several works on a variety of subjects. Other women Pythagoreans were *Themistoclea*, priestess of Apollo at Delphi and said to be Pythagoras' sister, and *Melissa*, thought to have been one of the very first Pythagoreans.

The Pythagoreans were in opposition to the *democratic* movement in Greece. The followers of the philosophical school of the *Sophists* were democrats, while the Pythagoreans believed in *oligarchy*, the rule by a small political elite. Some of the Greek geometers did in fact belong to the democrats. They did not get along too well with the main stream Pythagoreans, who were very influential. Thus for example *Hippasus of Metapontium*, who was a Pythagorean, and nevertheless democrat, made known the findings that not all line segments have a common measure, that there are *incommensurable* line segments. We say more about this below, but the Pythagoreans did not take lightly to this breach of secrecy! In fact, he was severely denounced for having described the *Sphere of the Twelve Pentagons*, in other words the *dodecahedron* and for having revealed *the nature of the non-mensurable* to *the Unworthy*.

To the Pythagoreans the regular pentagon with the inscribed pentagram, the 5-pointed star formed by all the diagonals, was a sacred symbol. There is a story about a Pythagorean who became seriously ill while traveling, far from home. The keeper of the inn where he stayed was a compassionate man, and had his servants nurse him as best they could. The money of our traveling Pythagorean expended, he was reduced to nothing: Seriously ill, at the mercy of foreigners, far from home. Nevertheless the inn-keeper stood by him, providing for him at his own expense. As the unfortunate Pythagorean realized that his *Earthly Goal* for the present incarnation was approaching, he called for his benefactor. Not being able to leave behind any significant earthly values, he told him to paint the symbol of the pentagon with the inscribed pentagram on his door, *but to paint it right, not upside down.* If ever a Pythagorean came this way again, he would generously return the favor. And so the man did, after his foreign guest had passed away. Not that he had much belief in the benefits to be reaped from this undertaking. But years later a rich Pythagorean traveled through the area, saw the pentagon with the inscribed pentagram, and did indeed repay the local good Samaritan generously.

Returning to Hippasus, his treasonous publication may have happened towards the end of Pythagoras' life, maybe after his death. Hippasus was expelled from the Brotherhood, and one version of what happened afterwards is this: The Pythagoreans made a grave monument for him, as he was to be considered dead. Soon afterwards he perished at sea, and this was seen as punishment from the Gods: *He died as a godless person at sea.* Another version of the story is that he was murdered by Pythagoreans, who threw him overboard from a ship at sea. Be this as it may, during this time the opposition to the Pythagoreans grew, Pythagoras himself had to move from Croton to Metapontium. A prominent citizen of Croton by the name *Cylon* is said to have been refused entry into the Pythagorean Brotherhood by Pythagoras, presumably because he was lacking in the spiritual qualities required, and as a result the same Cylon mobilized his followers against Pythagoras and the Pythagoreans. Others report the events differently, but at any rate Pythagoras had to move to Metapontium, not too far from Croton, as the situation became difficult. He died in Metapontium soon afterwards. According to some accounts he was murdered, killed by arson at the house of his daughter Damo.

In Croton the Pythagoreans continued to exist as an organization, but increasingly surrounded by controversy. Finally mobs emanating from the democratic party killed a large number of Pythagoreans when they set fire to the house in which they were assembled, the house of an athlete named *Milo*, a famous wrestler. As many as 50 or 60 Pythagoreans are said to have been killed at that time. The surviving Pythagoreans fled from Croton, and thus, ironically, the ideas of Pythagoras were spread more widely in the Greek domain. Later still the Pythagoreans reappeared in the area, the last important of them being *Archytas of Tarentum,* 438 - 365 B.C. His best known work is probably an ingenious 3-dimensional construction which accomplishes the doubling of the cube. We shall explain this in Section 3.11.

Again we should reiterate the warning that the story of Pythagoras' life which we have told here is regarded by some as being highly unreliable. Contradicting ones are in circulation as well. The indisputable fact, however, is that these stories and legends about him do exist, and have been told for 2500 years.

3.3 The Geometry of the Pythagoreans

No work by Pythagoras is extant, and in fact the practice of the early Pythagoreans was to ascribe all their findings to the master himself, to Pythagoras. But it is well documented from later sources that the Pythagoreans viewed mathematics as basic to the very fabric of reality, and that certain fundamental doctrines were important to their thinking and teaching. One such doctrine was that *numbers*, that is to say, the natural numbers, formed the basic organizing principle for everything. The motion of the planets could be expressed by ratios of numbers. Musical harmonies could be expressed so as well. The right angle was fixed by ratios like 3:4:5, as a triangle with sides in these proportions is a right triangle.

This takes us to the *geometry* of the Pythagoreans. Several discoveries have traditionally been attributed to the Pythagoreans, but at least some of them are without question of a much earlier origin. We reproduce a list of such discoveries in geometry, together with some comments. See [15] and [37].

1. The Pythagoreans knew that the sum of the angles of a triangle is equal to two right angles. They also knew the generalization to any polygon, namely, that in any n-gon the sum of all the interior angles is equal to $2n-4$ right angles, while the sum of all exterior angles is equal to four right angles.

The last assertion may be viewed as completely obvious, as far as the mathematical realities are concerned. As for the first, that the sum of the angles in a triangle equals two right angles, Egyptian, Babylonian, Chinese and Indian geometers knew well the properties of similar triangles. It is, therefore, hard to believe that the realities behind such properties of triangles were not known before the Pythagoreans. However, the precise formulation as a mathematical proposition, as well as a formal proof may well have been first supplied by them.

2. The Pythagoreans knew that in a right triangle the square on the hypothenuse is equal to the sum of the squares on the two sides containing the right angle.

This theorem, the so-called *Pythagorean Theorem*, was certainly known to the Babylonians at least 1000 years before Pythagoras. As we have seen, not only did the Babylonians know this, they also knew how to generate *all the so called Pythagorean triples*, namely triples (a, b, c) of integers such that $a^2 + b^2 = c^2$. Whether the Babylonians also knew proofs of the Pythagorean

Theorem is more hypothetical. But proofs based on a simple figure combined with some algebraic manipulation could well have been known to the Babylonians, who were superb algebraists.

> 3. The Pythagoreans knew several types of constructions by straightedge and compass of figures of a given area. They also solved what we would call algebraic problems by *geometric means.*

Again, much of this would be known long before the Pythagoreans. Thus for instance *the Sulva Sutra*, the oldest source of Indian mathematics, contains rules for constructing altars of a given area. Typical assignments would include the following:

1. Construct a square altar table, the area of which is twice that of a given square alter table. Solution: Use the diagonal of the given one as the length of the sides of the new one. We will return to this assignment in Section 3.8.

2. Given a rectangular altar table. Construct a square one of the same area. Solution: Let the sides in the rectangular table be a and b, the unknown side of the square be x. Then $x^2 = ab$, thus $a : x = x : b$, in other words, x is the *mean proportional* of a and b. We then draw a half circle of diameter $a + b$, erect a line normal to this diagonal where a is joined to b, and find x as the half-cord. See Figure 3.2.

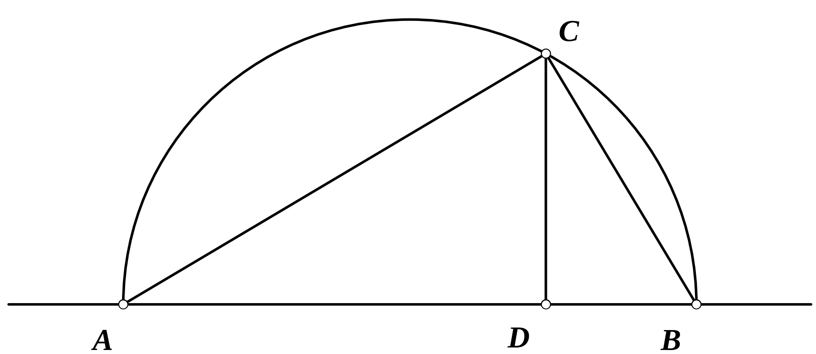

Fig. 3.2. Construction of the mean proportional.

By the way, we may also use 2. to solve 1., of course. But the first method is simpler.

Finally we come to a discovery which is universally credited to the Pythagoreans, if not to Pythagoras himself. There are some who think that the discovery was made by a woman mathematician, *Theano*, who was

Pythagoras' wife. It is arguably one of the most profound piece of mathematics discovered by the Greek classical school, and brought the Greeks almost to the point of discovering the system, or *the field,* of real numbers, as we would say in modern language.

But somehow the decisive last step was never taken, and the discovery of the field of real numbers as a powerful extension of the rationals would have to wait for about 2000 years. Perhaps one of the reasons for this was that the Greeks did not possess any good algebraic notation. Only towards the end of the Hellenistic epoch do we see a movement in this direction, in the work of *Apollonius.* Also, the Greeks were really *true geometers*, and not algebraists. They considered geometry to be a more complete science than algebra, in fact they did their "algebra" in terms of geometry, we would call it *Geometric Algebra*. Perhaps it was this philosophical prejudice which prevented them from taking the last definitive step and discovering the system of real numbers as an extension of the rationals. But even to say that the Greeks worked with rational numbers, is somewhat misleading. To them, what we would understand as the number $\frac{3}{2} = 1.5$ would be the proportion $3 : 2$.

However, when this is said it has to be added that some historians of mathematics seem to have underestimated the sophistication and power of Greek computing abilities. Especially towards the end of the Hellenistic Epoch such abilities to an impressive degree are documented in the work of *Claudius Ptolemy* and others. See Section 4.11.

3.4 The Discovery of Irrational Numbers

Presumably the Pythagoreans would early on work from the assumption that given any two line segments a and b, then their *proportion* $a : b$ would always be equal[5] to the proportion between two *numbers*, i.e., in our present language be equal to a fraction $\frac{r}{s}$ where r and s are positive integers. Arguably, this would be the position taken by Pythagoras himself, at least originally. Of course at this time many Greek philosophers espoused the *atomistic* view of the physical world. According to this idea, all things are made up of *incredibly many*, but a finite number, of *incredibly small*, but of a definite size, *indivisible atoms.* In fact, this *model* for the physical world became generally accepted all the way up to our own times. Some of the early Pythagoreans applied this idea to geometry and mathematics as well. For *numbers* they had the atom in the number 1, from which all other numbers were built.

In accordance with this general way of thinking, *lines* would consist of small chained *line elements*. In particular two line segments a and b would have *a common measure*: There would exist some line segment c such that c would fit exactly an integral number of times, say r, in a, and exactly an

[5] The Greek concept of equality for proportions will be explained below.

integral number of times, say s, in b: Of course this would be true, at the very worst one would have to take *one of the miniscule line elements*, which would work since the two line segments were made up of whole numbers of such line elements. The line element would *always* constitute a common measure, for any two line segments. Now, for convenience one would let c be the largest such *common measure*. This situation is illustrated in Figure 3.3.

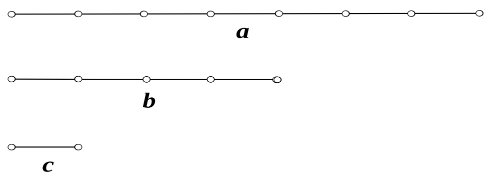

Fig. 3.3. c is the largest common measure of a and b.

How would we go about finding the biggest common measure of two given line segments a and b? The procedure is an ancient method, which the Greeks called *antanairesis*, meaning successive subtractions. Literally, given the two line segments a and b, the smallest is subtracted from the biggest. Of the remaining, the smallest is again subtracted from the biggest. This subtraction-procedure is repeated again and again, until the two segments are *equal in length*. Note that if you believe in the *atomistic nature of lines*, then this will occur sooner or later, at the very worst when you are left with two line-atoms, two line elements discussed above. Then a moment of contemplation will convince you that these two equal line segments are indeed the greatest common measure of the original line segments a and b.

This method of *successive subtractions* was very useful in ancient times. It allowed amazingly exact mensurations of an unknown distance, using only a measuring rod without subdivisions, and a good sized compass. It is no accident that the Master Builder so frequently is depicted with the measuring rod and the compass! He would proceed as follows. Let's say that the measuring rod would be, anachronistically, one meter long. First, as carefully as possible he would count the number of times the whole measuring rod could be subtracted from the unknown distance, i.e., find the number of whole meters. Let's say he gets 50. Then he would take the residue, the left over piece, in his compass, and count the number of times *it* could be subtracted from the length of the *measuring rod* itself. Let's say he gets 2, and a new left over piece, a new residue. He now successively repeats the procedure, counting the number of times the new residue can be subtracted from the previous one, and writing down the numbers. Let us say he repeats this 4 more times, getting 1, 1, 4 and 2, at which point there is nothing left, at least as far as he can see: Then, of course, he has to stop. Denoting the length to be measured by L, the measuring rod (here of 1 meter) by m, the first residue by r_1, the second by r_2, then r_3 and finally, r_4, we obtain

$$\begin{aligned} L &= 50m + r_1 \\ m &= 2r_1 + r_2 \\ r_1 &= r_2 + r_3 \\ r_2 &= r_3 + r_4 \\ r_3 &= 4r_4 + r_5 \\ r_4 &= 2r_5 \end{aligned}$$

To find L in terms of m, we substitute $r_5 = \frac{1}{2}r_4$ from the sixth relation into the fifth relation, obtaining $r_4 = (\frac{1}{4+\frac{1}{2}})r_3$, which substituted into the fourth yields

$$r_3 = (\frac{1}{1 + \frac{1}{4+\frac{1}{2}}})r_2$$

and so on, until we finally get

$$L = (50 + \frac{1}{2 + \frac{1}{1+\frac{1}{1+\frac{1}{4+\frac{1}{2}}}}})m$$

We show the process in Figure 3.4.

As m is supposed to be one meter, we find after some computing of fractions that the length L is $50\frac{20}{51}$ meters, or 50.39 meters, in present days decimal notation. Of course the number is deceptive, as counting the 50 meters to begin with could introduce an error of around 5 cm. But using a longer rod, or a longer string of a known length, this measuring error would be reduced.

Now, it is generally thought that the first "irrational number", discovered by the Pythagoreans, was $\sqrt{2}$. But first of all, the Pythagoreans, as indeed all Greek mathematicians of this time, did not think of this as a *number*. Rather, it was a question about the *proportion* between the lengths of two line segments *not being equal to the proportion of two numbers*, we would say not being a rational number, the fraction of two integers. It is presumed, by some, that the first such pair of line segments found was the diagonal and the side of a square. It is also asserted, frequently, that the so called Pythagorean Theorem should have been essential in realizing this. Others find this questionable. First of all, at the time of Pythagoras proving that two line segments are *incommensurable* would consist in showing that the *process of repeated subtraction applied to these two particular line segments never stops*. Later more sophisticated methods were developed by geometers like Theodorus of Cyrene, (465 – 398 B.C), pupil of Pythagoras and teacher of

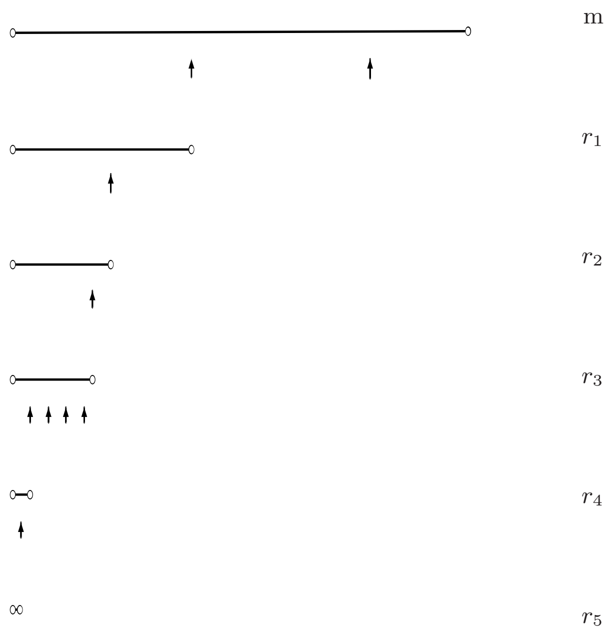

Fig. 3.4. Measuring by repeated subtractions.

Plato, and by Theaetetus. They are the principal characters in two of Plato's famous dialogues, one of them dealing with square roots.

At Pythagoras' time the simplest case to consider would be *the diagonal and the side of the regular pentagon*. This certainly appears surprising, since we would view the regular pentagon as considerably more complicated than a square. But from the point of view of repeated subtraction of the side and the diagonal it is the absolutely simplest figure in existence. A look at Figure 3.5 will explain this.

Indeed, the diagonal is AC and the side is AB. Now AB = AD, as elementary considerations yield the equality of the angles ∠ABD = ∠ADB. Thus subtracting AB from AC we are left with DC, and the subtraction can only be performed once. In the next step CD is to be subtracted from AB. Now CD = AD' and AB = AD, thus in this next step we may also only subtract once, and the remainder is D'D. But as CD = CE = ED', the third step will be to subtract the side of *the inner pentagon from the diagonal of the inner pentagon*! Thus, magnifying the inner pentagon and turning it upside down, we are back to the starting point. Hence the process evidently repeats itself

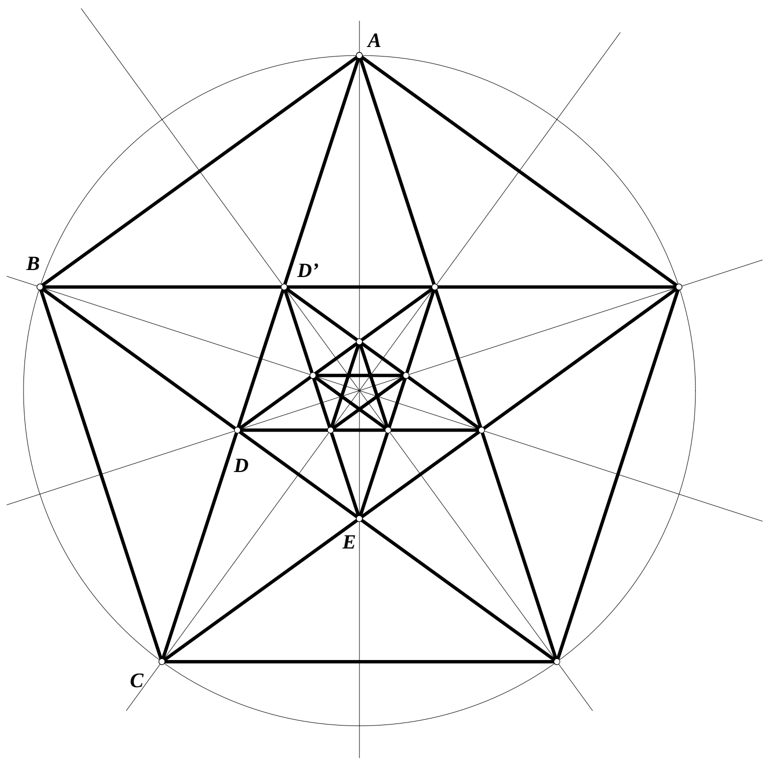

Fig. 3.5. The pentagon and the pentagram.

without ever stopping. Thus the incommensurability of diagonal and side of the regular pentagon is proven.

A similar procedure may be carried out for the diagonal and the side of a square, but it is considerably more complicated. And in view of the special relationship the Pythagoreans had to the regular pentagon, it is a very plausible guess that this is how they arrived at the conclusion that *not all line segments are commensurable.*

A final point to be made is this: If we put $x =$ AC:AB, then we obtain

$$x = 1 + \frac{1}{1 + \frac{1}{1+\dots}} = 1 + \frac{1}{x},$$

which yields the equation

$$x^2 - x - 1 = 0,$$

Indeed, this follows in the same manner as the computation carried out on the basis of Figure 3.4. Hence $x = \frac{1}{2}(1+\sqrt{5}) \approx 1.6180$. This number is often referred to as the *Golden Section*.[6]

3.5 Origin of the Classical Problems

There are three problems occupying a special position in Greek geometry, namely the so-called *classical problems*. They are all insoluble in their strictest interpretation. However, they may be solved by various creative procedures and they have generated an enormous amount of mathematics. Their attraction on mathematical amateurs is perhaps paralleled only by the famous *Fermat Conjecture*, which was finally proven not too many years ago by *Andrew Wiles*. The first of these problems we encounter in the history of mathematics is the

> **Squaring the Circle.** Given any circle. Then construct a square with the same area as the one enclosed by the circle.

The first time we find this problem mentioned, is in connection with the Greek philosopher *Anaxagoras*. Anaxagoras lived at a time when Athens stood at the summit of its power, politically and intellectually.

After Athens and Sparta had won the protracted war against Persian invaders, there followed half a century of peace and prosperity. This was a time of flourishing cultural life in Athens. Many of the expelled Pythagoreans found their way to Athens, and *Socrates* played an important role in the intellectual life of the city state.

Athens had an enlightened leader in *Pericles* for a great part of this time, from about 460 B.C until he died in the great plague in the year 429 B.C., two years after the peace had been broken and the devastating Peloponnesian war with Sparta had broken out. Unfortunately Pericles must bear a large part of the responsibility for this fratricidal struggle. In fact, he transformed the alliance of the Greek cities against the Persians, *the Delian Alliance*, into an instrument for Athenian dominance. This worked fine for Athens, but the Spartans and others were considerably less pleased. Athens now had more than 300000 inhabitants, one third were slaves and about 40000 were male citizens enjoying full rights. The city wall also enclosed the port city of *Piraeus*, and their fleet was the dominating power at sea.

Pericles erected the magnificent buildings at Acropolis, and showed great interest in mathematics and philosophy. He belonged to *the democrats*, from the aristocratic wing of the party. He was succeeded by *Cleon* when he died, also a democrat but from the less aristocratic wing.

[6] Other names include the Golden Mean, the Golden Number and the Golden Ratio.

Pericles' teacher and close friend was *Anaxagoras.* Anaxagoras was born about 500 B.C., in Clazomenae (now Izmir), in Ionia, presently Turkey. He died 428 B.C. in Lampsacus in the Troad, where he had sought refuge for persecution by his enemies in Athens, who continued to press charges for *impiety* against him.

He was more a natural philosopher than a mathematician. Nevertheless he played an important role in Greek geometry, and indeed in the development of mathematics, since he was, apparently, the first to be tied to one of the great problems of antiquity, *the Squaring of the Circle.*

In his teachings, he had denied that the heavenly bodies were divinities. Instead, he explained them as stones torn from the earth, the sun being red hot from its motion. The sun was as big as all of the Peloponnese, he asserted, and the moon reflected the light from the sun. The moon was an inhabited world, like the earth, according to Anaxagoras.

These ideas were hard to swallow, any right-thinking Athenian would be disgusted at such impiety. Consequently Anaxagoras was incarcerated. According to *Plutarchus* Anaxagoras spent the time in prison by attempting to square the circle.

Pericles had to be cautious, since he had many powerful enemies in Athens. But he also stood by his friends, and he finally managed to get Anaxagoras out of prison. But Athens certainly was not a safe place for him any more, and he therefore moved to Lampsacus where he founded his own Academy. Aristotle speaks highly of the reputation he enjoyed there.

The Peloponnesian War broke out in 431 B.C., and two and a half years later Pericles died in the great plague which had started to ravage Athens. One year later Anaxagoras also died.

The plague had broken out for full in 427 B.C., the presumed year of Plato's birth. The plague weakened Athens considerably, one fourth of its population is said to have perished. According to the legend, the citizens of Athens sent a delegation to the oracle of Apollo at Delos, to ask for advice on how to emerge form the dire circumstances in which they found themselves: A war with Sparta which would be very difficult to win, now that they also had this debilitating pestilence to cope with. The answer delivered by the priestess of Apollo was enigmatic: *The cubic Altar of Apollo should be doubled.* They may also have received other instructions as well, since Athens carried out extensive *purifications* of the island in the year 426 B.C.: Among other things all graves on the island were opened, and the remains which were buried there removed and reburied on the neighboring island of Rheneia. Doubling the cubic altar proved more difficult. The Greek geometers realized of course that the purest, and most pleasing way to Apollo, would be using compass and straightedge. In other words to perform a geometric construction which for a given cube would render another with volume twice the given:

Doubling the Cube, or the Delian Problem. Given any cube, construct with straightedge and compass the side of another cube, the volume of which is twice that of the given one.

It must have been quite intriguing to the geometers in Athens that this problem proved so hard, since the corresponding assignment for *a square* was so easy. More on that below, in Section 3.8.

There are accounts to the effect that *Plato*, when consulted about the problem later, voiced the opinion that Apollo had not offered the oracle because he wanted his altar doubled, but that he had intended to censure the Greeks for having turned their back on mathematics and geometry: By paying more attention to science and philosophy instead of making war, things would start to go better for them.

Greek geometers were fully aware that all circles are similar, as are all squares and cubes. Thus the problems stated above for *any* circle and for *any* cube is equivalent to the same problem stated for *one* circle or for *one* cube: If you can square one circle you can square them all, if you can double one cube, then you can double them all. Not so with the *third problem*, which also circulated in Athens about this time:

Trisecting the Angle. Given any angle, divide it in three equal parts using straightedge and compass.

In this last case the situation is different: There is an infinite number of angles which may be *trisected* using ruler and compass. We show the construction for *a right angle*, that is to say an angle of $\frac{\pi}{2}$ radians or 90°, in Figure 3.6.

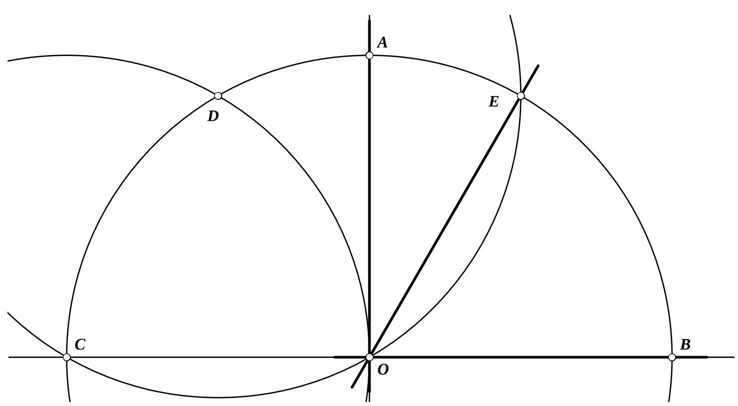

Fig. 3.6. Trisecting a very special angle by straightedge and compass.

We start with the right $\angle AOB$, and draw a circle with O as center passing through B. Producing BO we find the point C. With C as center draw the circle passing through O. The latter circle intersects the former in D. With D as center draw the circle passing through O, this circle intersects the one about O in E. Then $3 \times \angle AOE = \angle AOB$. Thus there are angles which may be trisected by compass and straightedge, and there are infinitely many such angles: Namely, we may by continued bisection divide $\angle AOB$ in 2^n equal parts for any n, and the resulting small angle may then be trisected by similarly bisecting $\angle AON$ in 2^n equal parts. Of course these are not all, there are several other kinds of angles which may be trisected by compass and straightedge as well.

3.6 Constructions by Compass and Straightedge

Another remark to be made concerning the little construction in Figure 3.6 is this: The construction illustrates the *legal use of compass and straightedge*. The legal use of compass and straightedge is tied to what later was codified as *Euclid's axioms*. Many complex constructions may be performed under these rules, but the three classical problems are not soluble in this way. This led Greek geometers to introduce other methods, like the use of *conic sections*, also *curves of higher degrees*, even *transcendental curves*, as we would say in modern language: The transition from elementary to *higher geometry* was initiated as a consequence of the struggle with the *classical problems*. The transition is not as unnatural as one might think, since employing conic sections or higher curves is equivalent to solving the problem *by an infinite number of steps* using ruler and straightedge, at each stage in a completely legal manner, according to the rules. We now state these rules.

> **Legal Use of Compass and Straightedge.** A finite set of points is given. A point is constructed if it is a point of intersection between two lines, two circles or a line and a circle as produced according to 1. and 2. below:
> 1. The straightedge may be used to draw a line passing through two given or previously constructed points, and to produce it arbitrarily in both directions.
> 2. The compass may be used to draw a circle with a given or already constructed point as center, passing through a given or already constructed point.

We note that according to 2. above, the compass may *not* be used to move a distance. A compass which may only be used in this restricted way, is frequently referred to as a *Euclidian compass.* We may imagine that the compass *collapses immediately when either end is lifted from the paper*.

Using these two procedures is also referred to as *constructing by the Euclidian tools*. By Euclidian tools we may easily perform tasks like *dividing any angle in two equal parts,* drop the normal to a given line from a given point or erect the normal at a given point on a given line. This is shown in the three top constructions in Figure 3.7.

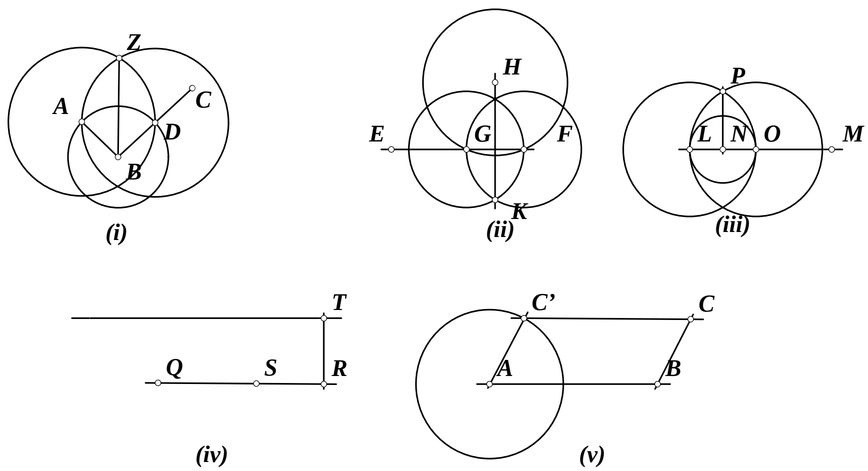

Fig. 3.7. Simple but essential constructions which may be carried out using straightedge and the Euclidian compass.

An angle is given by the points A, B and C. We wish to bisect $\angle ABC$. Draw the circle with B as center through A, D is the point of intersection between this circle and the line (possibly produced) BC. Then circles are drawn with A and D as centers, passing through, respectively, D and A. These circles intersect in a point Z such that a line AZ bisects the angle in two equal parts. Next, the line EF is given, as well as the point H outside it. To drop the perpendicular from H to EF, a circle through F is drawn with H as center, intersecting EF in another point G. With F and G as centers, circles are drawn through G and F, respectively, intersecting in K. Then HK is perpendicular to EF, its *foot* is the point of intersection with EF. Finally, we erect the perpendicular to a line LM in the point N. We leave the explanation of this construction to the reader.

In the lower part of the figure, we show how to construct a parallel to a given line QS through a given point T, by first dropping the perpendicular from T to QS (produced), its foot being R, then erecting the perpendicular to RT at T.

We now find a pattern, similar to *proving complex theorems from simpler propositions or axioms*: The construction in (iv) is obtained by appealing to the two previous ones in (ii) and (iii), without having to start from scratch. This becomes even more striking by including construction (v).

Namely, if we allow the compass to be used to draw a circle about a given or constructed point with radius equal to the distance between two other points in the construction, then this is strictly speaking is not allowed according to the rules above. But actually, we may nevertheless do this, since we have the construction (v). Here the points A, B and C are given, and we wish to draw a circle with A as center and radius BC. Proceed as follows: Draw the line BC. Through C construct the parallel to AB, and through A the parallel to BC. They intersect in C'. Now the length of BC equals the length of AC', so draw the circle with center A passing through C'.

The parallel to AB through C is unique since we are in the Euclidian world. The possibility and the uniqueness of the constructions thus hinge on the Fifth Postulate of Euclid. It might be interesting to contemplate what constructions would be like in a non-Euclidian plane.

But to mark off a distance on *the straightedge* is prohibited. By such illegal use of the straightedge one may indeed trisect any angle in three equal parts, as we shall see in Section 3.9, and a cube may be doubled, as we shall see in Section 4.6. In fact, constructions with compass and a marked straightedge is equivalent to including among the Start Data one single higher curve, namely the *Conchoid of Nicomedes*, which we treat in detail in Section 4.6. We also refer to Section 16.8.

We now turn to some specifics on the three problems. Even though ideally they should be solved with ruler and straightedge, Greek geometers of course soon realized that this would be very difficult. So they came up with a variety of solutions, ranging from rather simple but effective mechanical schemes, in some cases constructing various kinds of instruments, to very sophisticated geometric constructions like *Archytas'* famous three-dimensional construction for the doubling of the cube, using a cylinder, a cone and a torus. Also employed were a variety of higher algebraic, as well as transcendental, curves in the plane. We shall give some glimpses of these prodigious efforts in the following three sections.

3.7 Squaring the Circle

We have already mentioned that if you can square one circle, then you can square them all. In fact, suppose that a circle of the fixed radius r may be squared, that is to say that we may construct a square of side s such that its area equals that of the circle. The situation is shown in Figure 3.8.

Here we have a fixed circle, together with a fixed square with side KQ, known to have the same area as the area enclosed by the circle. These two being given, we may square *any* circle as follows: We construct a right triangle VWX, where the side VW is equal to the diameter of the given circle, while WX is equal to the side KQ of the given square. VW and WX are the sides containing the right angle. Now consider an arbitrary, new circle, shown in the lower left corner. Mark off VY on VW equal to its diameter, and let YZ

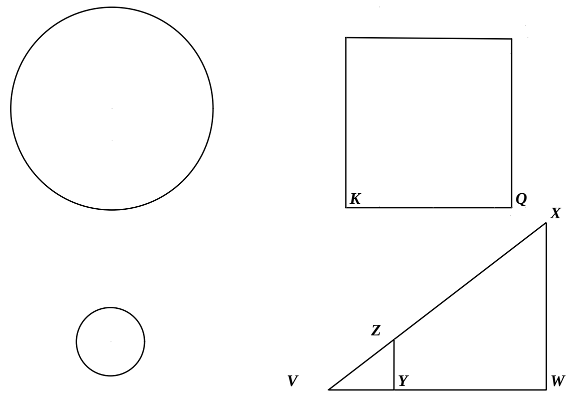

Fig. 3.8. If you can square one circle, you can square them all.

be parallel to WX, Z falling on XV. Then YZ is the side of the square of area equal to the that of the new circle.

3.8 Doubling the Cube

We first look at the much simpler problem of *doubling the square* by straightedge and compass. This construction is shown in Figure 3.9.

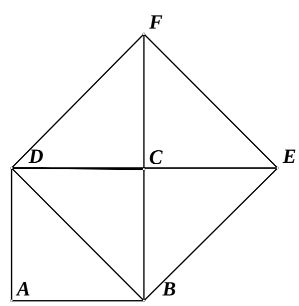

Fig. 3.9. Doubling the square by straightedge and compass.

Here we have the square ABCD. We now perform the *doubling of the square* in a way very much in the spirit of Greek geometry as follows: Produce the line DC, and mark the point E such that DC = CE. Similarly produce BC and mark F such that BC = CF. Then the square BEFD will have twice the area of ABCD. Indeed, the former consists of four congruent right triangles while the latter only requires two.

But this observation is just the beginning of what led *Hippocrates of Chios* to a most remarkable discovery: Namely, we notice that the triangles ABD and BDF are similar, thus

$$\mathrm{AB} : \mathrm{BD} = \mathrm{BD} : \mathrm{BF}$$

Thus

$$\mathrm{AB} : \mathrm{BD} = \mathrm{BD} : 2\mathrm{AB}$$

so the side of the double square is the mean proportional between the side and the double side of the given square. Thus putting AB = 1 and using modern notation, we find the side x of the double square by

$$1 : x = x : 2 \text{ or } \frac{1}{x} = \frac{x}{2}$$

so $x = \sqrt{2}$. A construction for doubling the cube which has much of the same flavor, while of course not being possible by straightedge and compass, is attributed to Plato and will be explained in the next section.

It was Hippocrates who realized that the enigma of doubling the cube was but one very special case of a much more general and much more interesting problem: Namely that of *constructing a continued proportionality:*

> **Construction of a continued proportionality.** Let a and b be two line segments. For a given integer n, construct n line segments $x, y, z, \ldots, u, v, w$ such that
>
> $$a : x = x : y = y : z = \cdots = u : v = v : w = w : b$$
>
> x, y z etc. are referred to as the *mean proportionals* of the continued proportionality. A double mean proportionality is one with two mean proportionals, a triple has three, etc.

He saw that doubling a cube of side a is equivalent to constructing *a double continued proportionality* between a and $2a$: To construct x and y such that

$$a : x = x : y = y : 2a$$

We check this with modern notation. We have

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}$$

This gives

$$ay = x^2 \text{ and } 2ax = y^2$$

Squaring the former and substituting y^2 from the latter yields $2a^3x = x^4$, i.e. $x = a\sqrt[3]{2}$.

Recall the following construction of the mean proportional between two line segments $a \geq b$. We refer to Figure 3.2: First draw a semicircle with diameter AB $= a$, then mark the point D such that AD $= b$.[7] We then have similar triangles ABC and ACD, thus

$$\text{AB:AC} = \text{AC:AD}$$

and so AC $= x$ is the mean proportional.

There is a continuation of this construction to a double continued proportionality, and indeed to any continued proportionality. In fact, from D in Figure 3.2 we construct a line perpendicular to AC, see Figure 3.10.

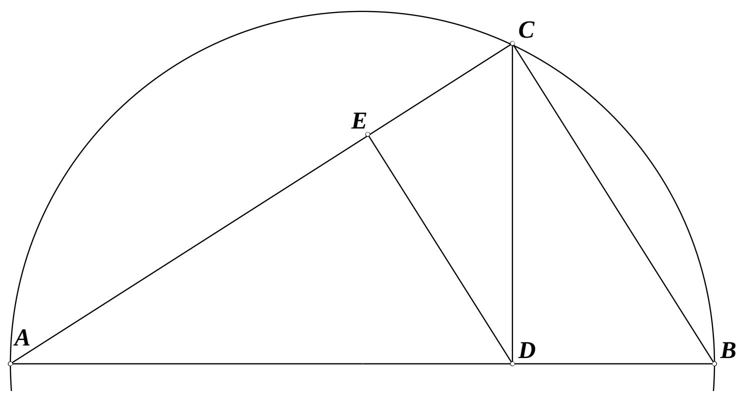

Fig. 3.10. Construction of a double continued proportionality.

Letting $\sim$ denote the relation of being similar triangles, we have

$$\varDelta\text{ADE} \sim \varDelta\text{ACD} \sim \varDelta\text{ABC}$$

from which it follows that

$$\text{AB} : \text{AC} = \text{AC} : \text{AD} = \text{AD} : \text{AE}$$

Thus if we wish to construct the double continued proportionality between the line segments $a \geq b$,

[7] Note that this is a slightly different construction from the one explained when we first encountered Figure 3.2.

$$a : x = x : y = y : b$$

then first draw the semicircle with diameter AB $= a$, and then observe what happens as the point C on the semicircle moves from B to A: In the right ΔABC draw the perpendicular to AB through C, meeting AB in D. Then through D draw the perpendicular to AC, meeting it at E. As now C moves, starting with the degenerate case of C $=$ B where AE $= a$, AE will decrease to 0 when the other degenerate case of C=A is reached. Therefore at some unique location for C on the semicircle, AE $= b$. There we take AC $= x$ and AD $= y$, which solves our problem.

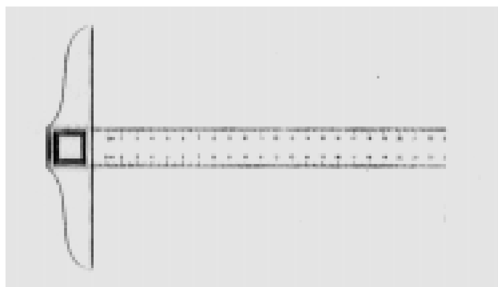

Fig. 3.11. A handy straightedge for finding a double proportionality.

This is the good news, the *bad news* being that this location for C *can not be found using straightedge and compass only*, in an allowable manner. But by "cheating" using two of the convenient tools displayed in Figure 3.11, it becomes simple.

We proceed as shown in Figure 3.12: First draw the semicircle with diameter AB $= a$. Then mark the points A' and E' on one of the rulers as shown, so that A'E' $= b$. Now position the rulers as shown in the figure, so that the vertical, unmarked, straightedge meets the marked one in a point on the line AB, where A' coincides with A, and C is found as the point where the marked straightedge crosses the semicircle. E' on the marked straightedge gives us the point E in our figure. We then have the construction from Figure 3.10.

This construction is of course completely illegal as a construction with straightedge and compass. In fact, it is even illegal as a version of the already illegal *insertion principle*, which we will explain in the next section. However, in its pure form the insertion principle was much used by Greek geometers, this is also known as a *verging construction.*

3.9 Trisecting any Angle

The construction of *bisecting any angle* was, as we have seen, very simple. And subdividing a line segment in any number of equal pieces is also a very

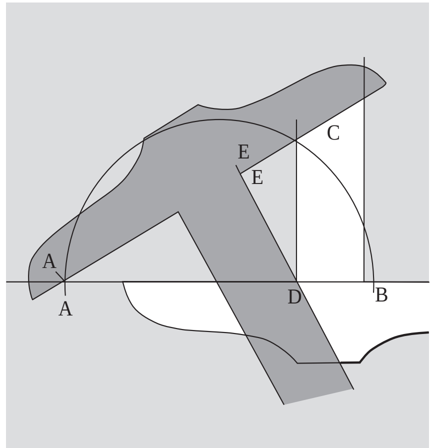

Fig. 3.12. Two rulers, one of them marked, used for finding a double proportionality.

simple construction. To Greek geometers it must therefore have been a source of frustration and bewilderment that the problem of *dividing any angle into three equal pieces* turned out to be so difficult.

This problem began to attract attention at about the same time as the problem of Doubling the Cube. Some special angles could easily be trisected, as the construction we display in Figure 3.6.

Greek geometers found solutions to the trisection-problem by solving what they referred to as a *Verging Problem*. We shall not attempt to give a general definition of this concept, but in Figure 3.13 we present the solution to the trisection problem as being reduced to one variety of such a Verging Problem. Another kind is represented by the famous construction of the regular *7-gon* found by Archimedes, treated in Section 4.4. Of course, neither the trisection problem nor the construction of the regular 7-gon are possible by legal use of compass and straightedge.

Now for Figure 3.13. To the left we have the angle $v = \angle ABC$, we draw the circle about B through C, and then we find the point E on that circle such that the line EC produced meets AB produced in a point D such that the segment DE is equal in length to the radius BC. This is the verging-part of the construction, it is possible by marking off the length BC on the straightedge. Denote the angle at D by u. Then $\angle CEB = 2u = \angle ECB$, thus $v = 3u$. To the right we have the same construction, essentially, but we do not use the circle, nor a marked straightedge, to find the point E such that $AB = BE = ED$. There are simple mechanical devises which may be used, however, based on the construction we have given here.

There are various algebraic curves of degrees higher than 2, so called *Higher Curves*, by means of which the verging problem may be solved. The

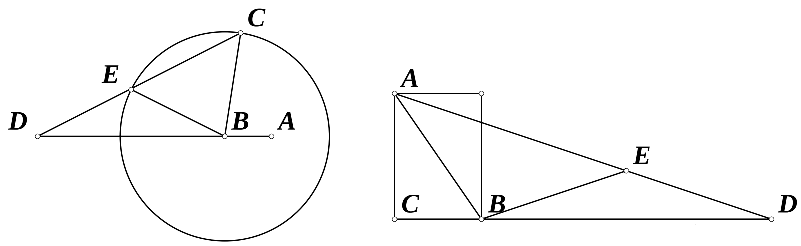

Fig. 3.13. Two Verging Constructions solving the Trisection Problem.

most famous of these are probably the *Conchoid of Nicomedes*, which we treat in detail in Section 4.6.

There is also another famous curve which may be used to trisect any angle, *and* to square the circle as well, in fact it may be used to divide any angle in any number of equal parts and to construct a regular $n-$gon for *any* number n. A truly marvelous curve! It is the *Quadratrix of Hippias*, treated here in Section 4.6 and explained in Figure 4.24. This is not an *algebraic curve*, however. Like the Archimedian Spiral, it is what we call a *transcendental curve*.

3.10 Plato and the Platonic Solids

Plato was born in 427 B.C. in Athens and died there in 347. Although he made no original contribution to geometry himself, he has had an immense influence on the subject. In 387 B.C. he founded the *Academy* in Athens, devoted to philosophy and geometry as well as other sciences. Plato had been engaged in the Peloponnesian war as a young man, and he saw his esteemed teacher and friend Socrates condemned and executed. He felt that one reason why the Greek civilization in general, and the one in Athens in particular, was in decline, had to be sought in the disregard of philosophy and geometry. To Plato the problem of *Doubling the Cube*, for example, was a question of developing insights into geometry. Thus it was not a question of finding some practical means for carrying out the physical labor involved, like devising some mechanical instruments or “cheating” with the straightedge. Instead it was a question of understanding the mathematics involved. Therefore Plato would regard highly the doubling-constructions involving higher curves or space-geometric constructions, even if these were of lesser practical value in the actual work of doubling any given cubical altar!

Of course this is exactly how we enjoy this problems today, as well as the one of trisecting any angle or squaring any circle. We understand them in terms of properties of *algebraic numbers*. We return to this in Chapter 16.

To Plato geometry was part of the ideal world, whereas the physical world would only represent imperfect approximations. He ascribed a special significance to the *regular convex polyhedra*, as symbolizing *the four elements Earth, Fire, Air* and *Water.* The fifth one, namely the dodecahedron, stood for the whole *Universe*.

In our modern language a *polyhedron* is a surface enclosing a solid figure composed of (plane) polygons. These are called the *faces* of the polyhedron. The sides of the polygons are called the *edges*, and the corners where the edges meet, are called the *vertices.* At each vertex there a configuration as shown in Figure 3.14: The vertex P from which the edges a, b, c and e emanate.

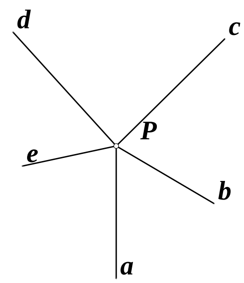

Fig. 3.14. A polyhedral angle.

This configuration is referred to as *the polyhedral angle at* P, so a polyhedral angle is a point in space with a certain number of half lines emanating from it.

A convex polyhedron is one where a plane containing any face does not cut the other ones. See Figure 3.15 for an illustration of the property of *convexity*.

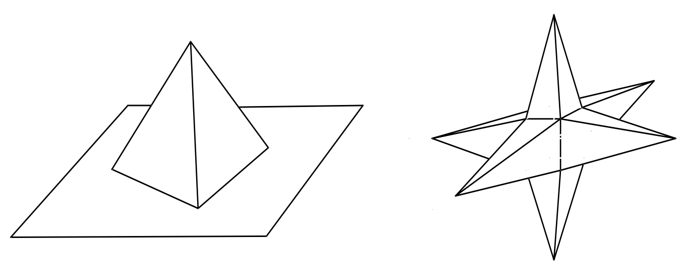

Fig. 3.15. To the left a convex polyhedron. Any plane containing one of the faces, does not cut any other. To the right evidently this property does not hold.

We say that a polyhedron is *regular* if all the edges are of equal length, and all polyhedral angles are *congruent*, that is to say that all the configurations of rays at the vertices are the same. We also require that the faces are *regular polygons of the same kind*, i.e., all are equilateral triangles, all are squares etc. Finally, we require that it be *convex*. Then there are exactly *five such polyhedra*, they are shown in Figure 3.16.

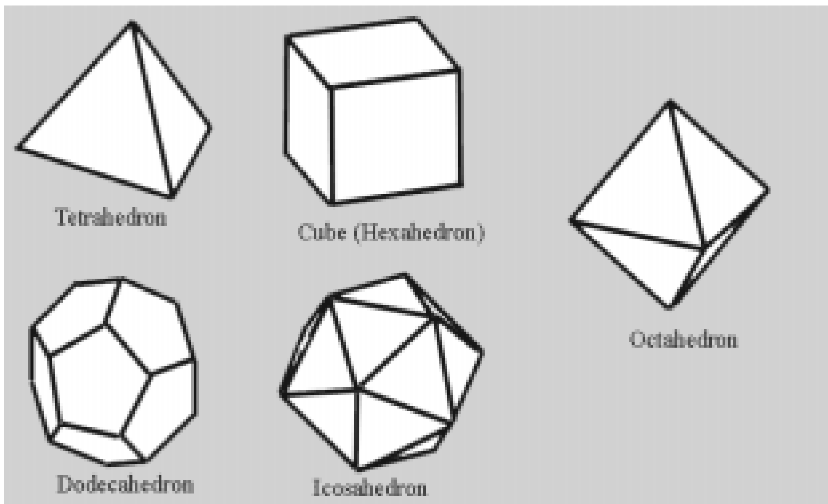

Fig. 3.16. The five Platonic Polyhedra, or as they are also known, the *Platonic Solids*.

We can show that these are the only such polyhedra as follows. Let P be a polyhedron of this type, consisting of regular n-gons. Let v be the angle at each vertex. For any convex n-gon, in particular any regular one, the sum of the angles contained by adjacent sides is $(n-2)\pi$. This is easily seen by subdividing it into $n-2$ triangles. Thus $v = \frac{n-2}{n}\pi$. On the other hand the sum of the angles constituting the polyhedral angle must be mv, m being the number of edges meeting at each vertex. Thus we have

$$m(\frac{n-2}{n})\pi < 2\pi$$

and so

$$m(n-2) < 2n$$

For $n = 3$ this leaves the possibilities $m = 3, 4$ or 5, $n = 4$ leaves only $m = 3$, as does $n = 5$. For $n \geq 6$ no value for m is possible. The values for m listed above are indeed realized, and yield the five Platonic Solids.

3.11 Archytas and the Doubling of the Cube

Archytas of Tarentum was born 428 B.C. and he died in 365 in a shipwreck near his home city of Tarentum. Tarentum is located not far from Croton and Metapontium. After the events when the Pythagoreans had been driven out of Italy, things had quieted down to the effect that they had been able to reestablish themselves in the area. He is considered the last great Pythagorean, and in fact Book VIII of Euclid's Elements is generally attributed to him.

He had been a student of another Pythagorean, namely *Philolaus of Tarentum.* Philolaus had studied with some of the expelled Pythagoreans, and he was interested in number magic and mysticism. But he had been allowed to write about the ideas of the Pythagoreans, and the book he wrote is supposed to have been Plato's source of information on the mathematics of Pythagoras and the Pythagoreans.

Archytas made it to the top of Tarentum's politics, he was elected admiral, never lost a battle, and became the ruler of Tarentum with unlimited power. But he is supposed to have been an enlightened ruler, who had a deeply rooted belief in the virtues of philosophy and rationality in politics. He thought that these forces would lead to enlightenment and social justice.

In spite of his political and military work, he also managed to pay attention to mathematics in general and geometry in particular. He lectured extensively, Plato studied under his direction in Tarentum.

Another important Greek geometer who studied under Archytas' direction was *Eudoxus of Cnidus.* Eudoxus had ideas which were precursors to fundamental concepts in our calculus and analysis of today. He probably did the work contained in Euclid's Elements, Book V. See Section 4.1.

Archytas' significant contributions to the didactics of mathematics include its division into four subjects: *Arithmetics* constitute the numbers at rest, *Geometry* is the magnitudes at rest, *Music* is the numbers in movement and *Astronomy* is the magnitudes in movement. Later *the mathematical quadrivium* was seen as constituting the *seven free arts,* jointly with a *trivium* which consisted of the subjects *Grammar, Rhetoric and Dialectic.* These ideas were important in didactical practice up to our times.

It is told that Plato once became a prisoner of the notorious tyrant of Syracuse, *Dionysus I,* who ruled with an iron fist, while at the same time writing poems and tragedies. Archytas, who was concerned about the safety of his student and friend, sent a letter to his colleague in Syracuse. In it, he explained to Dionysus that Plato was one of his students and also a dear friend, and that he, Archytas of Tarentum who had never yet lost a single battle, would not like it if his friend should come to harm.[8] This saved Plato's life. A quite significant contribution to philosophy from the admiral in Tarentum.

[8] Others say that Archytas sent a warship to Syracuse.

Archytas solved the problem of *doubling the cube* by a general construction of the *second continued proportionality* between $a > b$, applied to the case $a = 2b$. His marvelous construction uses an analogy to constructions with straightedge and compass, in the form of finding points in space as intersections of *tori, cylinders and cones.* We show the situation in Figure 3.17, with the torus, the cylinder and the cone sketched in the first and the second octant, anachronistically including coordinate axes.

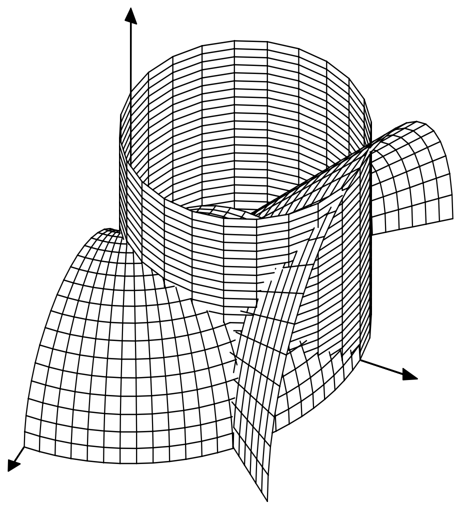

Fig. 3.17. Archytas' setup for the construction of the double continued proportionality, by intersecting a cylinder, a cone and a torus.

We now explain *Archytas' 3-dimensional construction* of the double continued proportionality between $a > b$ in Figure 3.18. The whole point of the construction is to obtain the right triangles in Figure 3.10, without using the extended version of the *insertion principle* we employed with our two rulers in Section 3.8.

One might say that Archytas' construction appears as a clear cut space-geometric generalization of constructions with straightedge and compass, employing higher dimensional versions of the compass.

In order to describe the construction, we introduce, anachronistically, a Cartesian coordinate system with x, y and z axes. We denote the origin by A. The following description is a slightly edited and commented version of the one given by Archytas himself, as related by Proclus in the *Eudemian Summary*. Of course Archytas did not use terms like *"the xy-plane"* and the like. The situation is visualized in Figure 3.17, while Figure 3.18 shows the exact geometry of the construction.

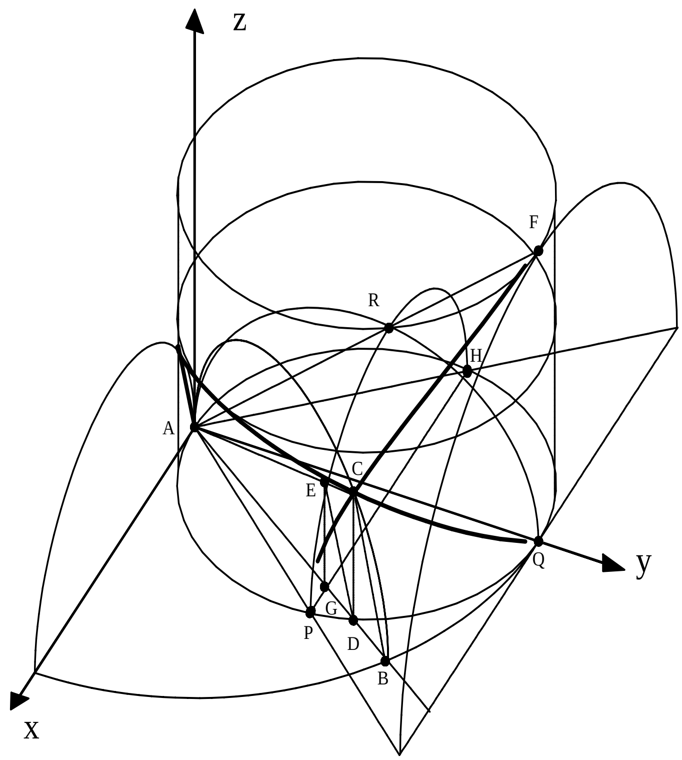

Fig. 3.18. Archytas' construction of the double continued proportionality, by intersecting a cylinder, a cone and a torus. The torus is shown in the third octant only, the cone and the cylinder in the first and the second octants.

Let $a > b$ be the two given line segments, let Q be a the point on the y-axis such that $AQ = a$. Draw a circle with AQ as diameter in the xy-plane and a semicircle with the same diameter in the first quadrant of the yz-plane. Draw a chord AP of length b to the former circle. On this circle also construct a right cylinder above the xy-plane. The semicircle in the yz-plane is now rotated about the z-axis from Q towards P. While being rotated the semicircle meets the cylinder in a moving point which traces out a curve on the cylinder. In Figure 3.18 this curve is indicated from Q to A. (In other words, this is the curve of intersection between the cylinder and the torus produced by rotating the circle.) On the other hand, when the prolongation of the chord AP is rotated about the y-axis, then it also meets the cylinder in a moving point, tracing out a curve, which is indicated in the figure from F through the point C pointing towards P. (This is the curve of intersection between the cone and the cylinder.) Evidently these two curves, one sloping upwards from Q and the other sloping downwards from F, will meet in a unique point. (In other words, the three surfaces, the torus, the cylinder and the cone, have exactly one point in common in the first octant.) In Figure 3.18 this is the point denoted by C. Drop the perpendicular from C to the xy-plane. Denote its foot by D. Now CD of course lies on the cylinder, and thus D lies on the circle in the xy-plane. The moving semicircle through C meets the xy-plane in the point B.

Draw the line PH parallel to the x-axis, it intersects AB in the point G. The line AC meets the circular arc from P to H via R, which P describes as AP is rotated about the y-axis, in a point E. Now EG is perpendicular to the xy-plane: Indeed, it is the intersection of the two planes spanned by ABC and PER, respectively, both of which are perpendicular to the xy-plane. We have now established all points and lines in the figure, and shown their relevant properties. The claim is that we have the double continued proportionality

$$AB : AC = AC : AD = AD : AE$$

which will solve the problem since $AE = AP = b$. From what we already know, it will suffice to show that $\angle AED = \frac{\pi}{4}$, a right angle. In fact, that this suffices was established in the discussion of Figure 3.10 in Section 3.8.

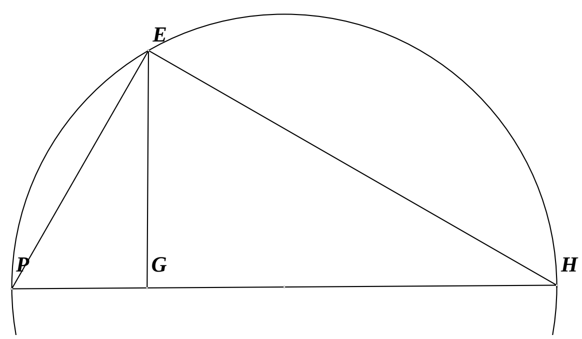

Fig. 3.19. $HG : EG = EG : PG$.

First, from Figure 3.19 we conclude that $HG : EG = EG : PG$. or in other words, $HG \cdot PG = EG^2$.

But from Figure 3.20 we find $HG{\cdot}GP = AG{\cdot}GD$ since $\Delta DPG \sim \Delta HAG$. Thus we conclude that $AG : EG = EG : GD$.

We now finally use this information on the detail from Archytas' construction shown in Figure 3.21.

Indeed, we have that $\Delta AGE \sim \Delta EGD$: They have one angle equal, namely the right angle at G, and the sides containing it are pairwise proportional. Hence in particular $\angle EAD = \angle DEG$. But as the corresponding pair of lines AD and EG of these two angles are perpendicular, so must be the case for the other pair. Thus DE is perpendicular to AC, as claimed.

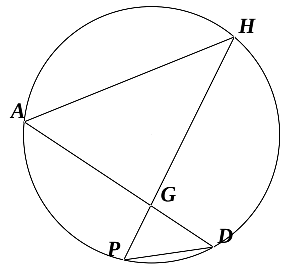

Fig. 3.20. $HG \cdot GP = AG \cdot GD$

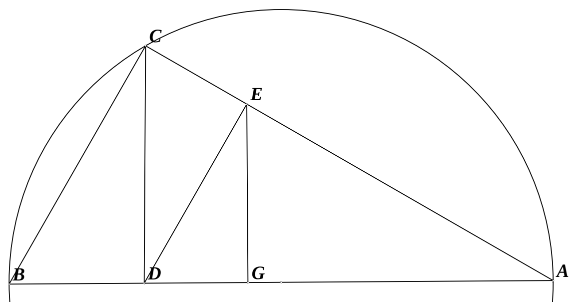

Fig. 3.21. The final argument.

This completes the proof of Archytas' construction of the double continued proportional between $a > b$.

Having completed Archytas' argument, we shall now carry it out by methods which he did not have at his disposal, namely by *algebraic geometry.* Putting the two arguments side by side we are better able to appreciate Archytas' geometric genius, as well as the power and convenience of algebra in geometry. One may even sympathize with those in the beginning of the twentieth century, who resisted the algebraic methods in geometry, feeling that geometry was defaced and destroyed in this way! Plato, incidentally, had similar misgivings about the use of mechanical tools in solving problems like the duplication of the cube. Comparing Archytas' solution to our crude and illegal use of the two rulers, a procedure very probably well known to Archytas, we may safely conclude that these misgivings were shared by Archytas himself.

The equation of the cylinder in Figure 3.18 is

$$x^2 + y^2 = ay,$$

the equation of the torus is obtained by putting $r = \sqrt{x^2 + y^2}$, the equation of this surface is then

$$z^2 + r^2 = ar$$

Thus the equation for the torus is

$$x^2 + y^2 + z^2 = a\sqrt{x^2 + y^2}$$

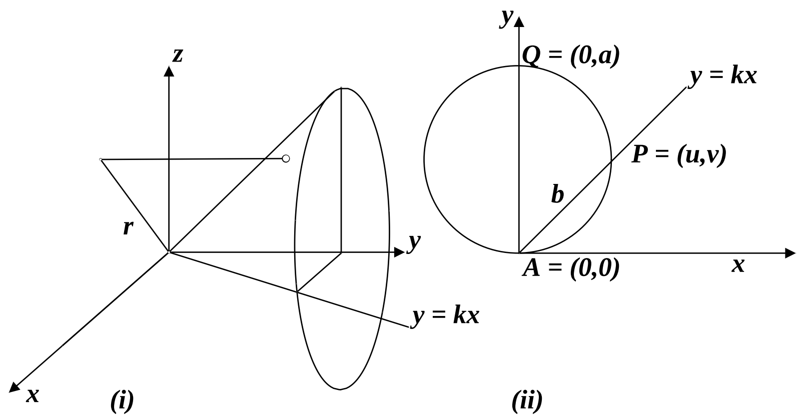

Fig. 3.22. The figure shows how we deduce the equation of the cone in Archytas' construction.

Finally, in (ii) of Figure 3.22 we see how the cone is produced by rotating the line $y = kx$, where $P = (u, v)$, so that $u^2 + v^2 = av$. Thus $b^2 = av$, and hence

$$k = \frac{v}{u} = \frac{b}{\sqrt{a^2 - b^2}}$$

When the line in (ii) is rotated about the y-axis, the cone in (i) is generated. With r given by $r^2 = x^2 + z^2$, the equation of the cone becomes

$$y = kr = k\sqrt{x^2 + z^2}$$

i.e.,

$$y^2 = k^2(x^2 + z^2)$$

and when the expression for k, namely $k = \frac{b}{\sqrt{a^2-b^2}}$, is substituted into this equation for the cone, we finally obtain that the cone is given by

$$x^2 + y^2 + z^2 = \frac{a^2}{b^2}y^2$$

where the only constants occuring are a and b. We are now ready to state the

Claim: *With* $\alpha = AC$ *and* $\beta = AD$ *we have*

$$a : \alpha = \alpha : \beta = \beta : b$$

We put C$= (p, q, r)$, so that

$$\alpha = \sqrt{p^2 + q^2 + r^2} \text{ and } \beta = \sqrt{p^2 + q^2}$$

The equation for the torus yields $\alpha^2 = a\beta$, which gives the first proportionality.

From the equation for the cylinder we have $\beta^2 = aq$, while the equation for the cone yields $\alpha = \frac{a}{b}q$, so that

$$b\alpha = \beta^2,$$

which gives the last proportionality.