6. Martingale Hazard Process

In Sect. 4.5, we have introduced the concept of the martingale hazard function of a random time and we have examined the connection between this concept and the notion of the hazard function. It appeared, that both notions coincide if and only if the cumulative distribution function of τ, and thus also its hazard function, are continuous (see Proposition 4.5.1). In this sense, the martingale hazard function uniquely characterizes the unconditional probability distribution of a continuously distributed random time. On the other hand, we have shown in Sect. 5.1.3 (see Proposition 5.1.3) that if the $\mathbb{F}$-hazard process is continuous, the process $H_t - \Gamma_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale. The main goal of this chapter is to extend the concept to the case of a non-trivial filtration, and to examine whether a continuous $\mathbb{F}$-martingale hazard process uniquely specifies the $\mathbb{F}$-conditional survival probabilities of a random time.

6.1 Martingale Hazard Process Λ

In this chapter, we assume that (G.1) is valid, so that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. It should be stressed that the case when $\mathbb{H} \subseteq \mathbb{F}$ (i.e., $\mathbb{F} = \mathbb{G}$) is not excluded. This means that the situation when τ is an $\mathbb{F}$-stopping time is also covered by the results of this chapter. The concept of the $(\mathbb{F}, \mathbb{G})$-martingale hazard process is a direct counterpart of the notion of the martingale hazard function of τ (the latter can be seen as the $(\mathbb{F}^0, \mathbb{H})$-martingale hazard process of τ).

Definition 6.1.1. An $\mathbb{F}$-predictable, right-continuous, increasing process Λ is called a $(\mathbb{F}, \mathbb{G})$-*martingale hazard process* of a random time τ if and only if the process $\widetilde{M}_t := H_t - \Lambda_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale. In addition, $\Lambda_0 = 0$. If, in addition, $\Lambda_t = \int_0^t \lambda_u \, du$ the $\mathbb{F}$-progressively measurable non-negative process λ is referred to as the $(\mathbb{F}, \mathbb{G})$-martingale intensity process.

Under (G.1), a random time τ and a reference filtration $\mathbb{F}$ uniquely specify the enlarged filtration $\mathbb{G}$ through $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. Thus, when (G.1) holds, we find it convenient to refer to the $(\mathbb{F}, \mathbb{G})$-martingale hazard process of τ as the $\mathbb{F}$-*martingale hazard process* of τ.

We first examine the case when the $\mathbb{F}$-martingale hazard process Λ can be expressed through a straightforward counterpart of formula (4.26). To this end, we introduce the following condition.

Condition (F.1) For any $t \in \mathbb{R}_+$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$ under $\mathbb{P}$; that is, for any bounded, $\mathcal{F}_\infty$-measurable random variable ξ and any bounded, $\mathcal{H}_t$-measurable random variable η we have

$$\mathbb{E}_{\mathbb{P}}(\xi\,\eta \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t)\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_t).$$

Let us emphasize that Condition (F.1) is satisfied when τ is constructed through the canonical method (see Sect. 6.5 and 8.2.1). Since $\mathcal{F}_t \subseteq \mathcal{F}_\infty$, we may restate Condition (F.1) as follows.

Condition (F.1a) For any $t \in \mathbb{R}_+$ and every $u \leq t$, the following equality holds: $\mathbb{P}\{\tau \leq u \,|\, \mathcal{F}_t\} = \mathbb{P}\{\tau \leq u \,|\, \mathcal{F}_\infty\}$.

The following condition will also be useful.

Condition (F.2) The process $F_t = \mathbb{P}\{\tau \leq t \,|\, \mathcal{F}_t\}$ admits a modification with increasing sample paths.

Under (F.1), we have $F_t = \mathbb{P}\{\tau \leq t \,|\, \mathcal{F}_t\} = \mathbb{P}\{\tau \leq t \,|\, \mathcal{F}_\infty\}$ for any $t \in \mathbb{R}_+$. It is thus clear that in this case F admits a modification with increasing sample paths, so that (F.2) is valid. However, the process F is not necessarily $\mathbb{F}$-predictable (e.g., when τ is an $\mathbb{F}$-stopping time that is not $\mathbb{F}$-predictable).

6.1.1 Martingale Invariance Property

We work in the following abstract set-up: we are given a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with a filtration $\mathbb{G}$; a reference filtration $\mathbb{F}$ is an arbitrary sub-filtration of $\mathbb{G}$. The definition of martingale invariance property is classic. It is important to notice that this property is not necessarily preserved under an equivalent change of the underlying probability measure $\mathbb{P}$.

Definition 6.1.2. A filtration $\mathbb{F}$ has the with respect to a filtration $\mathbb{G}$ if any $\mathbb{F}$-martingale follows also a $\mathbb{G}$-martingale.

Condition (M.1) Filtrations $\mathbb{F}$ and $\mathbb{G}$, with $\mathbb{F} \subseteq \mathbb{G}$, satisfy (M.1) (under $\mathbb{P}$) whenever $\mathbb{F}$ has the martingale invariance property with respect to $\mathbb{G}$.

The following condition appears to be equivalent to (M.1).

Condition (M.2) For any $t \in \mathbb{R}_+$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$ under $\mathbb{P}$.

By the definition of conditional independence of σ-fields, Condition (M.2) means that for any bounded, $\mathcal{F}_\infty$-measurable random variable ξ and any bounded, $\mathcal{G}_t$-measurable random variable η we have

$$\mathbb{E}_{\mathbb{P}}(\xi\,\eta \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t)\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_t).$$

Notice that Condition (M.2) can also be re-expressed in the following way.

Condition (M.2a) For any $t \in \mathbb{R}_+$, and any $s \geq t$ the σ-fields $\mathcal{F}_s$ and $\mathcal{G}_t$ are conditionally independent given the σ-field $\mathcal{F}_t$.

Since $\mathcal{F}_t \subseteq \mathcal{G}_t$ and $\mathcal{F}_t \subseteq \mathcal{F}_\infty$, each of the following two conditions is also equivalent to (M.2).

Condition (M.2b) For any $t \in \mathbb{R}_+$ and any bounded, $\mathcal{F}_\infty$-measurable random variable ξ we have $\mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t)$.

Condition (M.2c) For any $t \in \mathbb{R}_+$, and any bounded, $\mathcal{G}_t$-measurable random variable η we have $\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_\infty) = \mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_t)$.

Lemma 6.1.1. *A filtration $\mathbb{F}$ has the martingale invariance property with respect to a filtration $\mathbb{G}$ if and only if Condition* (M.2) *is satisfied. Put another way, the conditions* (M.1) *and* (M.2) *are equivalent.*

Proof. Suppose first that (M.2) holds. Let M be an arbitrary $\mathbb{F}$-martingale. Then for any $t \le s$ we have (the first equality below follows from (M.2b))

$$\mathbb{E}_{\mathbb{P}}(M_s \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(M_s \,|\, \mathcal{F}_t) = M_t,$$

and thus M is a $\mathbb{G}$-martingale. Conversely, suppose that every $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. We shall check that this implies (M.2b). To this end, for any fixed $t \le s$ we consider an arbitrary set $A \in \mathcal{F}_\infty$. We introduce the $\mathbb{F}$-martingale $M_u := \mathbb{P}\{A \,|\, \mathcal{F}_t\}$, $t \in \mathbb{R}_+$. Since M is also a $\mathbb{G}$-martingale, we obtain

$$\mathbb{P}\{A \,|\, \mathcal{G}_t\} = M_t = \mathbb{P}\{A \,|\, \mathcal{F}_t\}.$$

By standard arguments, this shows that (M.2b) is valid. □

Assume now that Condition (G.1) holds – that is, we have $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ for some filtration $\mathbb{H}$. Let us recall that we have also introduced Condition (F.1). Since $\mathcal{H}_t \subseteq \mathcal{G}_t$, it is apparent that (M.2) is stronger than (F.1). It appears that both conditions are in fact equivalent.

Lemma 6.1.2. *Conditions* (F.1) *and* (M.1) *are equivalent.*

Proof. We already know that conditions (M.1) and (M.2) are equivalent, and (M.2) is stronger than (F.1). It is enough to check that (F.1) implies (M.2). Condition (F.1) is equivalent to the following condition: for any bounded, $\mathcal{F}_\infty$-measurable random variable ξ we have $\mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{H}_t \vee \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t)$. Since $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$, this immediately gives (M.2b). □

6.1.2 Evaluation of Λ: Special Case

In this section, we assume that (G.1) and (F.2) hold.

Proposition 6.1.1. *Assume that F is an increasing, $\mathbb{F}$-predictable process. Then the process Λ given by the formula*

$$\Lambda_t = \int_{]0,t\,]} \frac{dF_u}{1 - F_{u-}} = \int_{]0,t\,]} \frac{d\mathbb{P}\{\tau \le u \,|\, \mathcal{F}_u\}}{1 - \mathbb{P}\{\tau < u \,|\, \mathcal{F}_u\}} \tag{6.1}$$

is the $\mathbb{F}$-martingale hazard process of τ.

Proof. We need to check that the compensated process $H_t - \Lambda_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale, where $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. Using (5.3), for $t < s$ we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{G}_t) &= \mathbb{P}\{t < \tau \le s \,|\, \mathcal{G}_t\} = 1\!\!1_{\{\tau > t\}} \frac{\mathbb{P}\{t < \tau \le s \,|\, \mathcal{F}_t\}}{\mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\}} \\ &= 1\!\!1_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{P}}(F_s \,|\, \mathcal{F}_t) - F_t}{1 - F_t}.\end{aligned}$$

On the other hand, for the process Λ given by (6.1) we obtain

$$\mathbb{E}_{\mathbb{P}}\big(\Lambda_{s\wedge\tau} - \Lambda_{t\wedge\tau} \,|\, \mathcal{G}_t\big) = \mathbb{E}_{\mathbb{P}}\Big(\int_{]t\wedge\tau, s\wedge\tau]} \frac{dF_u}{1 - F_{u-}} \,\Big|\, \mathcal{G}_t\Big) = \mathbb{E}_{\mathbb{P}}(1\!\!1_{\{\tau > t\}}\, Y \,|\, \mathcal{G}_t),$$

where

$$Y := \int_{]t, s\wedge\tau]} \frac{dF_u}{1 - F_{u-}} = 1\!\!1_{\{\tau > t\}} Y. \tag{6.2}$$

Furthermore, using (5.11), we get

$$\mathbb{E}_{\mathbb{P}}(1\!\!1_{\{\tau > t\}} Y \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{P}}(1\!\!1_{\{\tau > t\}} Y \,|\, \mathcal{F}_t)}{\mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\}}\,.$$

It is thus enough to verify that for $I := \mathbb{E}_{\mathbb{P}}(1\!\!1_{\{\tau > t\}} Y \,|\, \mathcal{F}_t)$ we have:

$$I = \mathbb{E}_{\mathbb{P}}\Big(\int_{]t, s\wedge\tau]} \frac{dF_u}{1 - F_{u-}} \,\Big|\, \mathcal{F}_t\Big) = \mathbb{E}_{\mathbb{P}}(F_s - F_t \,|\, \mathcal{F}_t). \tag{6.3}$$

To this end, notice that

$$\begin{aligned} I &= \mathbb{E}_{\mathbb{P}}\Big(1\!\!1_{\{\tau > s\}} \int_{]t,s]} \frac{dF_u}{1 - F_{u-}} + 1\!\!1_{\{t < \tau \le s\}} \int_{]t, s\wedge\tau]} \frac{dF_u}{1 - F_{u-}} \,\Big|\, \mathcal{F}_t\Big) \\ &= \mathbb{E}_{\mathbb{P}}\Big(\mathbb{E}_{\mathbb{P}}\Big(1\!\!1_{\{\tau > s\}} \int_{]t,s]} \frac{dF_u}{1 - F_{u-}} \,\Big|\, \mathcal{F}_s\Big) + 1\!\!1_{\{t < \tau \le s\}} \int_{]t, s\wedge\tau]} \frac{dF_u}{1 - F_{u-}} \,\Big|\, \mathcal{F}_t\Big) \\ &= \mathbb{E}_{\mathbb{P}}\Big((1 - F_s) \int_{]t,s]} \frac{dF_u}{1 - F_{u-}} + \int_{]t,s]} \int_{]t,u]} \frac{dF_v}{1 - F_{v-}}\, dF_u \,\Big|\, \mathcal{F}_t\Big) \\ &= \mathbb{E}_{\mathbb{P}}\Big((1 - F_s)(\Lambda_s - \Lambda_t) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u \,\Big|\, \mathcal{F}_t\Big), \end{aligned}$$

where the third equality is a consequence of (5.19) applied to the $\mathbb{F}$-predictable process $Z_s = \int_{]t,s]} (1 - F_{u-})^{-1} dF_u$. To conclude the proof, one may now argue along similar lines as in the proof of part (i) in Proposition 4.5.1. Under the present assumptions, Λ and F are processes of finite variation, so that their continuous martingale parts vanish identically. The product rule (cf. (4.29)):

$$\int_{]t,s]} \Lambda_u\, dF_u = \Lambda_s F_s - \Lambda_t F_t - \int_{]t,s]} F_{u-}\, d\Lambda_u \tag{6.4}$$

is thus the path-by-path version of the deterministic integration by parts formula of Lemma 4.2.2. □

Remarks. Alternatively, to evaluate the conditional expectation

$$K := \mathbb{E}_{\mathbb{P}}\big(\Lambda_{s\wedge\tau} - \Lambda_{t\wedge\tau} \,|\, \mathcal{G}_t\big),$$

we can directly apply formula (5.18) of Corollary 5.1.3. It is enough to notice that

$$K = \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau>s\}}(\Lambda_s - \Lambda_t) \,|\, \mathcal{G}_t\big) + \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{t<\tau\le s\}}\, \tilde{\Lambda}_\tau \,|\, \mathcal{G}_t\big),$$

where, for a fixed t, we write $\tilde{\Lambda}_u = (\Lambda_u - \Lambda_t)\mathbb{1}_{]t,\infty[}(u)$ (so that $\tilde{\Lambda}$ follows an $\mathbb{F}$-predictable process). Therefore, an application of (5.18) yields

$$\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{t<\tau\le s\}}\, \tilde{\Lambda}_\tau \,|\, \mathcal{G}_t\big) = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u \,\Big|\, \mathcal{F}_t\Big).$$

On the other hand, (5.11) gives

$$\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau>s\}}(\Lambda_s - \Lambda_t) \,|\, \mathcal{G}_t\big) = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}}\, (\Lambda_s - \Lambda_t) \,|\, \mathcal{F}_t).$$

Combining the above formulae, we obtain

$$\begin{aligned} K &= \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_{\{\tau>s\}}\, (\Lambda_s - \Lambda_t) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u \,\Big|\, \mathcal{F}_t\Big) \\ &= \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{P}}\Big((1-F_s)(\Lambda_s - \Lambda_t) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u \,\Big|\, \mathcal{F}_t\Big), \end{aligned}$$

where the last equality is derived by conditioning first with respect to the σ-field $\mathcal{F}_s$.

6.1.3 Evaluation of Λ: General Case

We maintain the assumption that (G.1) holds. On the other hand, we assume that either (F.2) is not satisfied (so that the process F is not increasing) or (F.2) is valid, but the increasing process F is not $\mathbb{F}$-predictable.

Example 6.1.1. For instance, a random time τ can be an $\mathbb{F}$-stopping time, which is not $\mathbb{F}$-predictable. If τ is an $\mathbb{F}$-stopping time, we have $F = H$, and the process H is not $\mathbb{F}$-predictable, unless a stopping time τ is $\mathbb{F}$-predictable.

As the next result shows, the $\mathbb{F}$-martingale hazard process Λ can still be found through a suitable modification of formula (6.1). In the next result, we do not need to assume that (F.2) holds. We shall write $\tilde{F}$ to denote the $\mathbb{F}$-*compensator* of the bounded $\mathbb{F}$-submartingale F. This means that $\tilde{F}$ is the unique $\mathbb{F}$-predictable, increasing process, with $\tilde{F}_0 = 0$, and such that the *compensated process* $U = F - \tilde{F}$ follows an $\mathbb{F}$-martingale. Let us recall that the existence and uniqueness of $\tilde{F}$ is an immediate consequence of the Doob-Meyer decomposition theorem.

Remarks. In some applications, the $\mathbb{F}$-stopping time τ is assumed to be *totally inaccessible* (cf. Dellacherie (1972)). In this case, the compensator $\tilde{F}$ of the increasing process $F = H$ is known to follow an $\mathbb{F}$-adapted process with continuous increasing sample paths.

Lemma 6.1.3. *Let Z be a bounded, $\mathbb{F}$-predictable process. Then for any $t \le s$*

$$\mathbb{E}_{\mathbb{P}}(1\!\!1_{\{t<\tau\le s\}}\, Z_\tau \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} Z_u \, d\tilde{F}_u \,\Big|\, \mathcal{F}_t \Big).$$

Proof. The martingale property of $U = F - \tilde{F}$ yields

$$\mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} Z_u \, d(F_u - \tilde{F}_u) \,\Big|\, \mathcal{F}_t \Big) = \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} Z_u \, dU_u \,\Big|\, \mathcal{F}_t \Big) = 0.$$

It is thus enough to make use of (5.19). □

Proposition 6.1.2. (i) *The $\mathbb{F}$-martingale hazard process of a random time τ is given by the formula*

$$\Lambda_t = \int_{]0,t\,]} \frac{d\tilde{F}_u}{1 - F_{u-}}. \tag{6.5}$$

(ii) *Assume that $\tilde{F}_t = \tilde{F}_{t\wedge\tau}$ for every $t \in \mathbb{R}_+$, i.e., the process $\tilde{F}$ is stopped at a random time τ. Then the equality $\Lambda = \tilde{F}$ is valid.*

Proof. It is clear that the process Λ given by (6.5) is $\mathbb{F}$-predictable. We thus need only to verify that the process $\widetilde{M}_t = H_t - \Lambda_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale. In the first part of the proof, we proceed along the same lines as in the proof of Proposition 6.1.1. We find that, in the present case, it is enough to show that for any $s \ge t$ (cf. (6.3))

$$\tilde{I} := \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s\wedge\tau]} \frac{d\tilde{F}_u}{1 - F_{u-}} \,\Big|\, \mathcal{F}_t \Big) = \mathbb{E}_{\mathbb{P}}(F_s - F_t \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\tilde{F}_s - \tilde{F}_t \,|\, \mathcal{F}_t),$$

where the second equality is a consequence of the definition of $\tilde{F}$. We have

$$\begin{aligned}
\tilde{I} &= \mathbb{E}_{\mathbb{P}}\Big(1\!\!1_{\{\tau>s\}} \int_{]t,s]} \frac{d\tilde{F}_u}{1 - F_{u-}} + 1\!\!1_{\{t<\tau\le s\}} \int_{]t,s\wedge\tau]} \frac{d\tilde{F}_u}{1 - F_{u-}} \,\Big|\, \mathcal{F}_t \Big) \\
&= \mathbb{E}_{\mathbb{P}}\Big(\mathbb{E}_{\mathbb{P}}\Big(1\!\!1_{\{\tau>s\}} \int_{]t,s]} \frac{d\tilde{F}_u}{1 - F_{u-}} \,\Big|\, \mathcal{F}_s \Big) + 1\!\!1_{\{t<\tau\le s\}} \int_{]t,s\wedge\tau]} \frac{d\tilde{F}_u}{1 - F_{u-}} \,\Big|\, \mathcal{F}_t \Big) \\
&= \mathbb{E}_{\mathbb{P}}\Big((1 - F_s) \int_{]t,s]} \frac{d\tilde{F}_u}{1 - F_{u-}} + \int_{]t,s]} \int_{]t,u]} \frac{d\tilde{F}_v}{1 - F_{v-}} \, d\tilde{F}_u \,\Big|\, \mathcal{F}_t \Big) \\
&= \mathbb{E}_{\mathbb{P}}\Big((\Lambda_s - \Lambda_t)(1 - F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t) \, d\tilde{F}_u \,\Big|\, \mathcal{F}_t \Big),
\end{aligned}$$

where the third equality follows from Lemma 6.1.3, combined with equality (5.19). Since Λ is $\mathbb{F}$-predictable and U is an $\mathbb{F}$-martingale, we obtain

$$\mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} (\Lambda_u - \Lambda_t) \, dU_u \,\Big|\, \mathcal{F}_t \Big) = 0,$$

which in turn entails that

$$\begin{aligned}\tilde{I} &= \mathbb{E}_{\mathbb{P}}\Big((\Lambda_s - \Lambda_t)(1 - F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, d\tilde{F}_u \,\Big|\, \mathcal{F}_t\Big) \\ &= \mathbb{E}_{\mathbb{P}}\Big((\Lambda_s - \Lambda_t)(1 - F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, d(F_u - U_u) \,\Big|\, \mathcal{F}_t\Big) \\ &= \mathbb{E}_{\mathbb{P}}\Big((\Lambda_s - \Lambda_t)(1 - F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u \,\Big|\, \mathcal{F}_t\Big).\end{aligned}$$

Our goal is to show that $\tilde{I} = \mathbb{E}_{\mathbb{P}}(\tilde{F}_s - \tilde{F}_t \,|\, \mathcal{F}_t)$. For this purpose, we observe that

$$\int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u = -\Lambda_t(F_s - F_t) + \int_{]t,s]} \Lambda_u\, dF_u.$$

Since Λ is a process of finite variation, Itô's product rule yields

$$\int_{]t,s]} \Lambda_u\, dF_u = \Lambda_s F_s - \Lambda_t F_t - \int_{]t,s]} F_{u-}\, d\Lambda_u. \tag{6.6}$$

Finally, it follows From (6.5) that

$$\int_{]t,s]} F_{u-}\, d\Lambda_u = \Lambda_s - \Lambda_t - \tilde{F}_s + \tilde{F}_t.$$

Combining the above formulae, we conclude that

$$(\Lambda_s - \Lambda_t)(1 - F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u = \tilde{F}_s - \tilde{F}_t. \tag{6.7}$$

This completes the proof of part (i). We shall now prove part (ii). We assume that $\tilde{F}_{t\wedge\tau} = \tilde{F}_t$ for $t \in \mathbb{R}_+$, and thus the process $F_t - \tilde{F}_{t\wedge\tau}$ is an $\mathbb{F}$-martingale. We wish to show that if the process $H_t - \tilde{F}_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale, that is, for any $t \leq s$,

$$\mathbb{E}_{\mathbb{P}}(H_s - \tilde{F}_{s\wedge\tau} \,|\, \mathcal{G}_t) = H_t - \tilde{F}_{t\wedge\tau}$$

or, equivalently,

$$\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(\tilde{F}_{s\wedge\tau} - \tilde{F}_{t\wedge\tau} \,|\, \mathcal{G}_t).$$

By virtue of (5.3), we have

$$\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{G}_t) = (1 - H_t)\, \frac{\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(1 - H_t \,|\, \mathcal{F}_t)}\,. \tag{6.8}$$

On the other hand, for the random variable $\tilde{J} := \mathbb{E}_{\mathbb{P}}(\tilde{F}_{s\wedge\tau} - \tilde{F}_{t\wedge\tau} \,|\, \mathcal{G}_t)$, we obtain

$$\begin{aligned}\tilde{J} &= \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau>t\}}(\tilde{F}_{s\wedge\tau} - \tilde{F}_{t\wedge\tau}) \,\big|\, \mathcal{G}_t\big) = (1 - H_t)\, \frac{\mathbb{E}_{\mathbb{P}}(\tilde{F}_{s\wedge\tau} - \tilde{F}_{t\wedge\tau} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(1 - H_t \,|\, \mathcal{F}_t)} \\ &= (1 - H_t)\, \frac{\mathbb{E}_{\mathbb{P}}(F_s - F_t \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(1 - H_t \,|\, \mathcal{F}_t)} = (1 - H_t)\, \frac{\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(1 - H_t \,|\, \mathcal{F}_t)},\end{aligned}$$

where the second equality follows from (5.2), and the third is a consequence of our assumption that the process $F_t - \tilde{F}_{t\wedge\tau}$ is an $\mathbb{F}$-martingale. □

Under (F.1), the process $\tilde{F}$ is never stopped at τ, unless τ is an $\mathbb{F}$-stopping time. To show this assume, on the contrary, that $\tilde{F}_t = \tilde{F}_{t\wedge\tau}$. Under (F.1), the process $F_t - \tilde{F}_{t\wedge\tau}$ is not only an $\mathbb{F}$-martingale, but also a $\mathbb{G}$-martingale (see Lemmas 6.1.1–6.1.2). Since by virtue of part (ii) in Proposition 6.1.2 the process $H_t - \tilde{F}_{t\wedge\tau}$ is a $\mathbb{G}$-martingale, we conclude that $H - F$ follows a $\mathbb{G}$-martingale. In view of the definition of F, the last property reads, for $t \le s$,

$$\mathbb{E}_{\mathbb{P}}\big(H_s - \mathbb{E}_{\mathbb{P}}(H_s \,|\, \mathcal{F}_s) \,\big|\, \mathcal{G}_t\big) = H_t - \mathbb{E}_{\mathbb{P}}(H_t \,|\, \mathcal{F}_t)$$

or, equivalently,

$$\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(H_s \,|\, \mathcal{F}_s) \,|\, \mathcal{G}_t\big) - \mathbb{E}_{\mathbb{P}}(H_t \,|\, \mathcal{F}_t) = I_1 - I_2. \tag{6.9}$$

Under (F.1), we have (cf. (F.1a))

$$\begin{aligned} I_1 &= \mathbb{E}_{\mathbb{P}}(\mathbb{P}\{\tau \le s \,|\, \mathcal{F}_s\} \,|\, \mathcal{F}_t \vee \mathcal{H}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{P}\{\tau \le s \,|\, \mathcal{F}_\infty\} \,|\, \mathcal{F}_t \vee \mathcal{H}_t) \\ &= \mathbb{E}_{\mathbb{P}}(\mathbb{P}\{\tau \le s \,|\, \mathcal{F}_\infty\} \,|\, \mathcal{F}_t). \end{aligned}$$

The last equality follows from the $\mathcal{F}_\infty$-measurability of the random variable $\mathbb{P}\{\tau \le s \,|\, \mathcal{F}_\infty\}$, combined with the fact that the σ-fields $\mathcal{F}_\infty$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$. Consequently,

$$I_1 = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(H_s \,|\, \mathcal{F}_\infty) \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(H_s \,|\, \mathcal{F}_t).$$

We conclude that (6.9) can be rewritten as follows:

$$\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(H_s \,|\, \mathcal{F}_t) - \mathbb{E}_{\mathbb{P}}(H_t \,|\, \mathcal{F}_t).$$

Finally, by applying (6.8) to the left-hand side of the last equality, we obtain

$$(1 - H_t)\,\frac{\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(1 - H_t \,|\, \mathcal{F}_t)} = \mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{F}_t).$$

Letting s tend to ∞ in the last formula, we obtain $H_t = \mathbb{E}_{\mathbb{P}}(H_t \,|\, \mathcal{F}_t)$ or, more explicitly, $\mathbb{P}\{\tau \le t \,|\, \mathcal{F}_t\} = \mathbb{1}_{\{\tau \le t\}}$ for every $t \in \mathbb{R}_+$. We conclude that a random time τ is indeed an $\mathbb{F}$-stopping time.

6.1.4 Uniqueness of a Martingale Hazard Process Λ

We shall first examine the relationship between the concept of an $\mathbb{F}$-martingale hazard process Λ of τ and the classic notion of the $\mathbb{G}$-*compensator* (that is, the *dual predictable projection*) of the jump process H associated with a random time τ. For convenience, the compensator of the process H is henceforth called the compensator of τ.

Definition 6.1.3. A process A is a $\mathbb{G}$-*compensator* of τ if and only if the following conditions are satisfied: (i) A is a $\mathbb{G}$-predictable, right-continuous, increasing process, with $A_0 = 0$, (ii) the process $H - A$ is a $\mathbb{G}$-martingale.

It is well known that for any random time τ and any filtration $\mathbb{G}$ such that τ is a $\mathbb{G}$-stopping time there exists a unique $\mathbb{G}$-compensator A of τ. Moreover, $A_t = A_{t\wedge\tau}$, i.e., the increasing process A is in fact stopped at τ. In the next auxiliary result, we shall deal with an arbitrary filtration $\mathbb{F}$, which, when combined with the natural filtration $\mathbb{H}$ of a $\mathbb{G}$-stopping time τ, generates the enlarged filtration $\mathbb{G}$. Since both statements are classic, the proof is omitted.

Lemma 6.1.4. *Let $\mathbb{F}$ be an arbitrary filtration such that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. Then:*
(i) *Let A be a $\mathbb{G}$-predictable right-continuous increasing process satisfying $A_t = A_{t\wedge\tau}$. Then there exists an $\mathbb{F}$-predictable right-continuous increasing process Λ such that $A_t = \Lambda_{t\wedge\tau}$.*
(ii) *Let Λ be an $\mathbb{F}$-predictable right-continuous increasing process. Then $A_t = \Lambda_{t\wedge\tau}$ is a $\mathbb{G}$-predictable right-continuous increasing process.*

The next proposition summarizes the connections between the $\mathbb{G}$-compensator A of τ and the $\mathbb{F}$-martingale hazard process Λ of τ. Once more $\mathbb{F}$ is an arbitrary filtration such that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$.

Proposition 6.1.3. (i) *Let A be the $\mathbb{G}$-compensator of τ. Then there exists an $\mathbb{F}$-martingale hazard process Λ such that $A_t = \Lambda_{t\wedge\tau}$.*
(ii) *Let Λ be an $\mathbb{F}$-martingale hazard process of τ. Then the process $A_t = \Lambda_{t\wedge\tau}$ is the $\mathbb{G}$-compensator of τ.*

Proof. The first (second, resp.) statement follows from part (i) (part (ii), resp.) in Lemma 6.1.4. □

From the uniqueness of the $\mathbb{G}$-compensator, combined with part (ii) in Proposition 6.1.3, it follows that the $\mathbb{F}$-martingale hazard process is unique up to time τ in the following sense: if Λ^1 and Λ^2 are the two $\mathbb{F}$-martingale hazard processes of τ, then the stopped processes coincide: $\Lambda^1_{t\wedge\tau} = \Lambda^2_{t\wedge\tau}$ for every $t \in \mathbb{R}_+$. To ensure some sort of uniqueness after τ of an $\mathbb{F}$-martingale hazard process, one needs to impose some additional restrictions.

Let τ be a $\mathbb{G}$-stopping time τ for some filtration $\mathbb{G}$. Then the sub-filtration $\mathbb{F}$ of $\mathbb{G}$ for which we have $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ is not uniquely specified. Assume that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}^1 = \mathbb{H} \vee \mathbb{F}^2$, and denote by Λ^i an $\mathbb{F}^i$-martingale hazard process of τ. Then $\Lambda^1_{t\wedge\tau} = A_{t\wedge\tau} = \Lambda^2_{t\wedge\tau}$. It seems natural to search for the $\hat{\mathbb{F}}$-martingale hazard process, where $\hat{\mathbb{F}}$ is a 'minimal' filtration for which $\mathbb{G} = \mathbb{H} \vee \hat{\mathbb{F}}$.

6.2 Relationships Between Hazard Processes Γ and Λ

Let us assume that the $\mathbb{F}$-hazard process Γ is well defined (in particular, τ is not an $\mathbb{F}$-stopping time). We already know that under (G.1), for any $\mathcal{F}_s$-measurable random variable Y we have (cf. (5.13))

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}\big(Y e^{\Gamma_t - \Gamma_s} \,|\, \mathcal{F}_t\big). \tag{6.10}$$

The natural question, which arises in this context is: can we substitute Γ with the $\mathbb{F}$-martingale hazard function Λ in the last formula? Of course, the answer is trivial when it is known that the equality $\Lambda = \Gamma$ is satisfied, for instance, when conditions (G.1) and (F.2) are fulfilled and the process F is continuous. We are thus in a position the following result, which corresponds to parts (ii)-(iii) of Proposition 4.5.1.

Proposition 6.2.1. *Under* (G.1) *and* (F.2), *the following statements hold.*
(i) *If the increasing process F is $\mathbb{F}$-predictable, but F is not continuous, then the $\mathbb{F}$-martingale hazard process Λ is also discontinuous process and we have*

$$e^{-\Gamma_t} = e^{-\Lambda_t^c} \prod_{0<u\leq t} (1 - \Delta\Lambda_u),$$

where we write Λ^c to denote the continuous component of Λ. More explicitly, $\Lambda_t^c = \Lambda_t - \sum_{0\leq u\leq t} \Delta\Lambda_u$ for every $t \in \mathbb{R}_+$.
(ii) *If the increasing process F is continuous, then the $\mathbb{F}$-martingale hazard process Λ is also continuous and*

$$\Gamma_t = \Lambda_t = -\ln(1 - F_t), \quad \forall\, t \in \mathbb{R}_+.$$

If, in addition, the process $\Lambda = \Gamma$ is absolutely continuous then for an integrable $\mathcal{F}_s$-measurable random variable Y we get

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}\big(Y e^{-\int_t^s \lambda_u\, du} \,|\, \mathcal{F}_t\big).$$

Proof. The hazard process Γ is discontinuous if and only if the process F admits discontinuities. Under (G.1) and (F.2), by virtue of Proposition 6.1.1, we have

$$G_t = -\int_{]0,t]} G_{u-}\, d\Lambda_u.$$

Since Λ is a process of finite variation, we obtain (cf. (4.24)–(4.25))

$$e^{-\Gamma_t} = G_t = e^{-\Lambda_t^c} \prod_{0<u\leq t} (1 - \Delta\Lambda_u).$$

The second statement is an immediate consequence of part (i). □

The following result is a straightforward consequence of Corollary 5.1.3.

Corollary 6.2.1. *Suppose that* (G.1) *and* (F.2) *hold and F is a continuous process. Let $Y = h(\tau)$ for some bounded, continuous function $h : \mathbb{R}_+ \to \mathbb{R}$. Then*

$$\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau\leq t\}}\, h(\tau) + \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}_{\mathbb{P}}\Big(\int_t^\infty h(u) e^{\Lambda_t - \Lambda_u}\, d\Lambda_u \,\Big|\, \mathcal{F}_t\Big).$$

Let Z be a bounded, $\mathbb{F}$-predictable process. Then for any $t \leq s$

$$\mathbb{E}_{\mathbb{P}}(Z_\tau \mathbb{1}_{\{t<\tau\leq s\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}_{\mathbb{P}}\Big(\int_t^s Z_u e^{\Lambda_t - \Lambda_u}\, d\Lambda_u \,\Big|\, \mathcal{F}_t\Big).$$

In some instances, the $\mathbb{F}$-martingale hazard process of a random time τ can be found through the martingale approach. The following question thus arises: does the continuity of the $\mathbb{F}$-martingale hazard process Λ imply the equality $\Lambda = \Gamma$? The next result furnishes a partial answer to this question.

Proposition 6.2.2. *Under* (G.1) *and* (F.2), *assume that the filtration* $\mathbb{F}$ *supports only continuous martingales. If the* $\mathbb{F}$*-martingale hazard process* Λ *is continuous, then the hazard process* Γ *is also continuous and* $\Gamma = \Lambda$.

Proof. We know that the $\mathbb{F}$-martingale hazard process Λ is given by (6.5). Therefore, if Λ is continuous, then the process $\tilde{F}$ is continuous as well. Since the $\mathbb{F}$-martingale $U = F - \tilde{F}$ is necessarily continuous, it results that $F = U + \tilde{F}$ follows a continuous, increasing process. We conclude that Λ is given by (6.1), so that $\Lambda_t = -\ln(1 - F_t) = \Gamma_t$. □

Let us state the following conjecture.

Conjecture (A). Under assumptions (G.1) and (F.2), if the $\mathbb{F}$-martingale hazard process Λ of τ is continuous, then the equality $\Gamma = \Lambda$ holds.

The following counter-example – borrowed from Elliott et al. (2000) – shows that in general (more specifically, when Condition (F.2) fails to hold) the implication in Conjecture (A) is not valid.

Example 6.2.1. Let W be a standard Brownian motion on $(\Omega, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \mathbb{F}^W$ is the natural filtration of W. We define a random time τ on $(\Omega, \mathcal{F}_1, \mathbb{P})$ by setting: $\tau = \sup\{\, t \le 1 : W_t = 0\,\}$. In words, τ is the last passage time to 0 before time 1 by the Brownian motion W. We set $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. Then the $\mathbb{F}$-hazard process of τ equals $\Gamma_t = -\ln(1 - F_t)$, where

$$F_t = \mathbb{P}\{\tau \le t \,|\, \mathcal{F}_t\} = \tilde{N}\Big(\frac{|W_t|}{\sqrt{1-t}}\Big), \quad \tilde{N}(x) := \sqrt{\frac{2}{\pi}} \int_0^x e^{-u^2/2}\, du.$$

Let L^0 stand for the local time of W at the origin (for the properties of local times, we refer to Karatzas and Shreve (1991) or Revuz and Yor (1991)). We claim that the $\mathbb{F}$-martingale hazard process of τ equals, for $t \in [0,1]$,

$$\Lambda_t = \sqrt{\frac{2}{\pi}} \int_0^t \frac{dL_s^0}{\sqrt{1-s}}\,.$$

To check this, we shall use the following result (see Yor (1997)).

Proposition 6.2.3. *For every* $t \in [0,1)$, *the* $\mathbb{F}$*-hazard process of* τ *equals*

$$F_t = \mathbb{P}\{\tau \le t \,|\, \mathcal{F}_t\} = \tilde{N}\Big(\frac{|\, W_t \,|}{\sqrt{1-t}}\Big). \tag{6.11}$$

The $\mathbb{F}$*-compensator of* F *equals*

$$\tilde{F}_t = \sqrt{\frac{2}{\pi}} \int_0^t \frac{dL_s^0}{\sqrt{1-s}},$$

where L^0 *is the local time at the origin of the Brownian motion* W.

Proof. For any fixed $t < 1$, the event $\{\tau \leq t\}$ coincides with the event $\{d_t > 1\}$, where $d_t = \inf\{\, u \geq t : W_u = 0\}$. Let us quote the following equality (cf. Yor (1997))

$$d_t = t + \inf\{u \geq 0 : W_{u+t} - W_t = -W_t\} = t + \hat{\tau}_{-W_t} \stackrel{d}{=} t + \frac{W_t^2}{G^2}. \qquad (6.12)$$

We denote here $\hat{\tau}_b = \inf\{u \geq 0 : \hat{W}_u = b\}$, where $\hat{W}_u = W_{u+t} - W_t$, $u \geq 0$, is a Brownian motion independent of $\mathcal{F}_t^W$. Also, G has a Gaussian law with mean 0 and variance 1, and G is independent of W_t.

Standard calculations show that, for any $a \in \mathbb{R}$,

$$\mathbb{P}\Big(\frac{a^2}{G^2} > 1 - t\Big) = \tilde{N}\Big(\frac{|\,a\,|}{\sqrt{1-t}}\Big). \qquad (6.13)$$

The Itô-Tanaka formula, combined with the classic identity:

$$x\tilde{N}'(x) + \tilde{N}''(x) = 0,$$

lead to

$$\begin{aligned} \tilde{N}\left(\frac{|\,W_t\,|}{\sqrt{1-t}}\right) &= \int_0^t \tilde{N}'\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) d\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) + \frac{1}{2}\int_0^t \tilde{N}''\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) \frac{ds}{1-s} \\ &= \int_0^t \tilde{N}'\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) \frac{\operatorname{sgn}(W_s)}{\sqrt{1-s}}\, dW_s + \int_0^t \tilde{N}'\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) \frac{dL_s^0}{\sqrt{1-s}}\,. \end{aligned}$$

Let us recall the well-known property of the Brownian local time: if $g : \mathbb{R} \to \mathbb{R}$ is a non-negative, Borel measurable function, then

$$\int_0^t g(W_s)\, dL_s^0 = g(0)L_t^0, \quad \forall\, t \in \mathbb{R}_+.$$

We conclude that

$$\tilde{N}\left(\frac{|\,W_t\,|}{\sqrt{1-t}}\right) = \int_0^t \tilde{N}'\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) \frac{\operatorname{sgn}(W_s)}{\sqrt{1-s}}\, dW_s + \sqrt{\frac{2}{\pi}} \int_0^t \frac{dL_s^0}{\sqrt{1-s}}\,.$$

But, in view of (6.12)–(6.13), we have

$$F_t = \mathbb{P}\{\tau \leq t \,|\, \mathcal{F}_t\} = \mathbb{P}\{d_t > 1 \,|\, \mathcal{F}_t\} = \tilde{N}\left(\frac{|\,W_t\,|}{\sqrt{1-t}}\right),$$

hence, the $\mathbb{F}$-compensator of F equals

$$\tilde{F}_t = \sqrt{\frac{2}{\pi}} \int_0^t \frac{dL_s^0}{\sqrt{1-s}}.$$

This ends the proof of the proposition. □

We continue an analysis of our example. Again using the property of support of the local time, specifically, the equality

$$L_t^0 = \int_0^t \mathbb{1}_{\{W_s=0\}}\, dL_s^0, \quad \forall\, t \in \mathbb{R}_+,$$

we find that $L_t^0 = L_{t\wedge\tau}^0$, and thus $\tilde{F}_t = \tilde{F}_{t\wedge\tau}$. In view of part (ii) in Proposition 6.1.2, the $\mathbb{F}$-martingale hazard process Λ thus equals $\tilde{F}$. Furthermore, both Γ and Λ are continuous processes, but Λ is increasing, while Γ has non-zero continuous martingale part. We conclude that $\Gamma \neq \Lambda$.

Note that Condition (F.2) is not satisfied in the present set-up (hence (F.1) does not hold either). We conclude that when (F.2) fails to hold, the continuity of Γ and Λ does not necessarily imply the equality $\Gamma = \Lambda$. Notice that the $\mathbb{G}$-compensator A of H, which satisfies $A_t = \Lambda_{t\wedge\tau}$, is also equal to $\tilde{F}$ since

$$\tilde{F}_t = \sqrt{\frac{2}{\pi}} \int_0^{t\wedge\tau} \frac{dL_s^0}{\sqrt{1-s}} = \Lambda_{t\wedge\tau}.$$

Finally, let us notice that the $\mathbb{F}$-martingale hazard process Λ represents at the same time the $\hat{\mathbb{F}}$-martingale hazard process of τ, where $\hat{\mathbb{F}}$ stands for the natural filtration of the process $|W_t|$ (it is well known that $\hat{\mathbb{F}}$ is a strict sub-filtration of $\mathbb{F}$). Likewise, for every t we have

$$F_t = \mathbb{P}\{\tau \le t \,|\, \mathcal{F}_t\} = \mathbb{P}\{\tau \le t \,|\, \hat{\mathcal{F}}_t\} = \hat{F}_t,$$

so that $\Gamma = \hat{\Gamma}_t$.

6.3 Martingale Representation Theorem

We consider the following set-up: we are given a reference filtration $\mathbb{F}$ and the enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where the filtration $\mathbb{H}$ is generated by a random time τ. In addition, we assume that the assumptions of Proposition 6.1.2 are valid so that F follows an increasing $\mathbb{F}$-predictable process, and the $\mathbb{F}$-martingale hazard process Λ of a random time τ is given by the formula

$$\Lambda_t = \int_{]0,t]} \frac{dF_u}{1-F_{u-}}\,. \tag{6.14}$$

By virtue of the definition of the $\mathbb{F}$-martingale hazard process, the compensated process $\tilde{M}_t = H_t - \Lambda_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale. In Lemma 5.1.7, we have proved that the process L, given by the formula

$$L_t := \mathbb{1}_{\{\tau>t\}}(1-F_t)^{-1} = \frac{1-H_t}{1-F_t}\,,$$

follows a $\mathbb{G}$-martingale. We shall now check that the following equality is valid:

$$dL_t = -(1-F_t)^{-1}\, d\tilde{M}_t. \tag{6.15}$$

To this end, we observe that

$$A_t := \Lambda_{t\wedge\tau} = \int_{]0,t]} \frac{1-H_{u-}}{1-F_{u-}}\, dF_u = \int_{]0,t]} L_{u-}\, dF_u. \tag{6.16}$$

Moreover, since

$$1 - H_t = L_t(1-F_t),$$

using Itô's lemma, we obtain (notice that L is a process of finite variation)

$$dH_t = L_{t-}\, dF_t - (1-F_t)\, dL_t = A_t - (1-F_t)\, dL_t$$

The following result is a counterpart of Proposition 5.2.1.

Proposition 6.3.1. *Let Z be an $\mathbb{F}$-predictable process such that the random variable Z_τ is integrable. Then the $\mathbb{G}$-martingale $M_t^Z := \mathbb{E}_{\mathbb{P}}(Z_\tau \,|\, \mathcal{G}_t)$ admits the following integral representation*

$$M_t^Z = m_0 + \int_{]0,t]} L_{t-} dm_u + \int_{]0,t]} (Z_u - D_u)\, d\tilde{M}_u, \tag{6.17}$$

where m stands for an $\mathbb{F}$-martingale, given by the formula

$$m_t = \mathbb{E}_{\mathbb{P}}\Big(\int_0^\infty Z_u\, dF_u \,\Big|\, \mathcal{F}_t \Big),$$

and

$$D_t = (1-F_t)^{-1} \mathbb{E}_{\mathbb{P}}\Big(\int_t^\infty Z_u\, dF_u \,\Big|\, \mathcal{F}_t \Big).$$

Proof. By virtue of Proposition 5.1.1, we have (cf. (5.18))

$$M_t^Z = \mathbb{E}_{\mathbb{P}}(Z_\tau \,|\, \mathcal{G}_t) = H_t Z_\tau + (1-H_t) D_t = H_t Z_\tau + \hat{D}_t,$$

where

$$\hat{D}_t := (1-H_t) D_t = L_t \Big(m_t - \int_{]0,t]} Z_u\, dF_u \Big).$$

Since L is a process of finite variation, we obtain

$$\begin{aligned} d\hat{D}_t &= L_{t-}(dm_t - Z_t\, dF_t) + D_t(1-F_t)\, dL_t \\ &= L_{t-}\, dm_t - Z_t\, dD_t + D_t(1-F_t)\, dL_t \\ &= L_{t-}\, dm_t - Z_t\, dD_t - D_t\, d\tilde{M}_t, \end{aligned}$$

where we have used (6.15) and (6.16). Consequently,

$$dM_t^Z = Z_t\, dH_t + d\hat{D}_t = L_{t-}\, dm_t + (Z_t - D_t)\, d\tilde{M}_t.$$

This gives the desired expression (6.17). □

It is clear that $m_0 = M_0^Z$. Notice also that equality (6.17) can be rewritten as follows:

$$M_t^Z = m_0 + \int_{]0,t\wedge\tau]} (1-F_{t-})^{-1}\, dm_u + \int_{]0,t]} (Z_u - D_u)\, d\tilde{M}_u. \tag{6.18}$$

6.4 Case of the Martingale Invariance Property

In the next result, we shall work directly with the $\mathbb{F}$-martingale hazard process. Therefore, Proposition 6.4.1 also covers the case when the $\mathbb{F}$-hazard process Γ does not exist (for example, when τ is an $\mathbb{F}$-stopping time). It appears that a counterpart of formula (6.10), with Γ replaced by Λ, is valid. However, we need to impose here a suitable continuity condition. The following proposition is essentially due to Duffie et al. (1996).

Proposition 6.4.1. *Let $\mathbb{H} \vee \mathbb{F} \subseteq \mathbb{G}$. Assume that Condition* (M.1) *is valid, and the $\mathbb{F}$-martingale hazard process Λ of a random time τ is continuous. For a fixed $s > 0$, let Y stand for an $\mathcal{F}_s$-measurable, integrable random variable.*
(i) *If the* (*right-continuous*) *process V, given by the formula*

$$V_t = \mathbb{E}_{\mathbb{P}}\big(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\big), \quad \forall\, t \in [0, s], \tag{6.19}$$

is continuous at τ, i.e., if $\Delta V_{s\wedge\tau} = V_{s\wedge\tau} - V_{(s\wedge\tau)-} = 0$, then for any $t < s$ we have

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}}\, Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}_{\mathbb{P}}\big(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\big).$$

(ii) *If the process V, given by the formula*

$$V_t = \mathbb{E}_{\mathbb{P}}\big(e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\big), \quad \forall\, t \in [0, s], \tag{6.20}$$

is continuous at τ then for any $t \leq s$ we have

$$\mathbb{P}\{\tau > s \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}\big(e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\big). \tag{6.21}$$

Proof. It is clear that it suffices to prove part (i). We shall first check that

$$U_t := \mathbb{1}_{\{\tau>t\}} V_t = \mathbb{E}_{\mathbb{P}}\big(\Delta V_\tau \mathbb{1}_{\{t<\tau\leq s\}} + \mathbb{1}_{\{\tau>s\}} Y \,\big|\, \mathcal{G}_t\big) \tag{6.22}$$

or, equivalently,

$$U_t = \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} \Delta V_u \, dH_u + \mathbb{1}_{\{\tau>s\}} Y \,\Big|\, \mathcal{G}_t \Big). \tag{6.23}$$

In view of (6.19), we have $V_t = e^{\Lambda_t} m_t$, where m denotes an $\mathbb{F}$-martingale: $m_t := \mathbb{E}_{\mathbb{P}}\big(Y e^{-\Lambda_s} \,|\, \mathcal{F}_t\big)$ for $t \in [0, s]$. In view of our assumptions, m also follows a $\mathbb{G}$-martingale. Using Itô's product rule, we obtain

$$dV_t = m_{t-}\, de^{\Lambda_t} + e^{\Lambda_t}\, dm_t = V_{t-} e^{-\Lambda_t}\, de^{\Lambda_t} + e^{\Lambda_t}\, dm_t. \tag{6.24}$$

On the other hand, another application of Itô's product rule yields

$$dU_t = (1 - H_{t-})\, dV_t - V_{t-}\, dH_t - \Delta V_t \Delta H_t.$$

Combining the last equality with (6.24), we obtain

$$dU_t = (1 - H_{t-})\big(V_{t-} e^{-\Lambda_t}\, de^{\Lambda_t} + e^{\Lambda_t}\, dm_t\big) - V_{t-}\, dH_t - \Delta V_t\, dH_t,$$

so that, after rearranging,

$$dU_t = -\Delta V_t\, dH_t + dC_t. \tag{6.25}$$

In the last formula, we write C to denote a $\mathbb{G}$-martingale

$$dC_t = (1 - H_{t-})e^{\Lambda_t}\, dm_t + dD_t,$$

where in turn the $\mathbb{G}$-martingale D equals

$$dD_t = -V_{t-}\big(\, dH_t - (1 - H_{t-})e^{-\Lambda_t}\, de^{\Lambda_t}\big) = -V_{t-}\, d(H_t - \Lambda_{t\wedge\tau}) = -V_{t-}\, d\tilde{M}_t.$$

Since obviously $U_s = \mathbb{1}_{\{\tau>s\}}Y$, equality (6.25) implies (6.23). If the process V is continuous at τ, then (6.22) yields

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}}\, Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}V_t = \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}_{\mathbb{P}}\big(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\big).$$

This completes the proof. □

6.4.1 Valuation of Defaultable Claims

Let an $\mathbb{F}$-adapted process B be given by the formula

$$B_t = \exp\Big(\int_0^t r_u\, du\Big), \quad \forall\, t \in \mathbb{R}_+,$$

for some $\mathbb{F}$-progressively measurable, integrable (short-term rate) process r. It is clear that B, referred to as the (default-free) savings account, follows a continuous process of finite variation. For a $\mathbb{G}$-predictable process Z, and a $\mathcal{G}_T$-measurable random variable X, we define the value process S by setting

$$S_t = B_t\, \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,T]} B_u^{-1} Z_u\, dH_u + B_T^{-1} X\, \mathbb{1}_{\{T<\tau\}} \,\Big|\, \mathcal{G}_t\Big), \tag{6.26}$$

where Z and X satisfy suitable integrability conditions. The next result, borrowed from Duffie et al. (1996), is a suitable extension of Proposition 6.4.1. For convenience, we postpone the proof of this result to Sect. 8.3.

Proposition 6.4.2. *Assume that Condition* (M.1) *is fulfilled, and a random time τ admits an absolutely continuous $\mathbb{F}$-martingale hazard function Λ. For an $\mathbb{F}$-predictable process Z and an $\mathcal{F}_T$-measurable random variable X, we define the process V_t, $t \in [0,T]$, by setting*

$$V_t = \tilde{B}_t\, \mathbb{E}_{\mathbb{P}}\Big(\int_t^T \tilde{B}_u^{-1} Z_u \lambda_u\, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t\Big), \tag{6.27}$$

where $\tilde{B}$ is the 'savings account' corresponding to the default-risk-adjusted short-term rate $R_t = r_t + \lambda_t$, specifically,

$$\tilde{B}_t = \exp\Big(\int_0^t (r_u + \lambda_u)\, du\Big).$$

Then

$$\mathbb{1}_{\{\tau>t\}}V_t = B_t\, \mathbb{E}_{\mathbb{P}}\Big(B_\tau^{-1}(Z_\tau + \Delta V_\tau)\mathbb{1}_{\{t<\tau\le T\}} + B_T^{-1} X\, \mathbb{1}_{\{T<\tau\}} \,\Big|\, \mathcal{G}_t\Big).$$

Corollary 6.4.1. *Let the processes S and V be defined by formulae* (6.26) *and* (6.27), *respectively. Then*

$$S_t = \mathbb{1}_{\{\tau>t\}}\Big(V_t - B_t\,\mathbb{E}_{\mathbb{P}}\big(B_\tau^{-1}\mathbb{1}_{\{\tau\le T\}}\Delta V_\tau \,\big|\, \mathcal{G}_t\big)\Big).$$

If, in addition, $\Delta V_\tau = 0$, then $S_t = \mathbb{1}_{\{\tau>t\}}V_t$ for every $t\in[0,T]$.

Conjecture (B). Under (G.1) and (F.1), if Z follows an $\mathbb{F}$-predictable process and X is an $\mathcal{F}_T$-measurable random variable, then the continuity condition $\Delta V_\tau = 0$ is satisfied.

Let us observe that instead of verifying conjecture (B), to establish the equality $S_t = \mathbb{1}_{\{\tau>t\}}V_t$, which is a handy form of the valuation formula (6.26), it suffices to show that $\Lambda=\Gamma$ and to make use of the following result.

Proposition 6.4.3. *Assume that the conditions* (G.1) *and* (F.1) *are valid and a random time τ admits an absolutely continuous $\mathbb{F}$-martingale hazard function Λ. Let Z be an $\mathbb{F}$-predictable process, and let X be an $\mathcal{F}_T$-measurable random variable. If $\Gamma=\Lambda$ then $S_t=\mathbb{1}_{\{\tau>t\}}V_t$ for $t\le T$, where the processes S and V are given by expressions* (6.26) *and* (6.27), *respectively.*

Proof. In view of (F.1), we have (for the first equality, see Lemma 5.1.4)

$$\mathbb{P}\{\tau\ge u\,|\,\mathcal{F}_\infty\vee\mathcal{H}_t\} = \mathbb{1}_{\{\tau>t\}}\,\frac{\mathbb{P}\{\tau\ge u\,|\,\mathcal{F}_\infty\}}{\mathbb{P}\{\tau\ge t\,|\,\mathcal{F}_\infty\}} = \mathbb{1}_{\{\tau>t\}}\,\frac{\mathbb{P}\{\tau\ge u\,|\,\mathcal{F}_u\}}{\mathbb{P}\{\tau\ge t\,|\,\mathcal{F}_t\}}$$

for any $u>t$. Put more explicitly,

$$\mathbb{P}\{\tau\ge u\,|\,\mathcal{F}_\infty\vee\mathcal{H}_t\} = \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t-\Gamma_u}.$$

If Z is an $\mathbb{F}$-predictable process and X is an $\mathcal{F}_T$-measurable random variable, using the $\mathbb{G}$-martingale property of the compensated process $H_t-\Lambda_{t\wedge\tau}$, we obtain

$$\begin{aligned}
S_t &= B_t\,\mathbb{E}_{\mathbb{P}}\Big(\int_t^T B_u^{-1}Z_u\lambda_u\mathbb{1}_{\{u\le\tau\}}\,du + B_T^{-1}X\mathbb{1}_{\{T<\tau\}}\,\Big|\,\mathcal{G}_t\Big)\\
&= B_t\,\mathbb{E}_{\mathbb{P}}\Big(\int_t^T B_u^{-1}Z_u\lambda_u\,\mathbb{P}\{\tau\ge u\,|\,\mathcal{F}_\infty\vee\mathcal{H}_t\}\,du\,\Big|\,\mathcal{G}_t\Big)\\
&\quad + B_t\,\mathbb{E}_{\mathbb{P}}\Big(B_T^{-1}X\,\mathbb{P}\{\tau>T\,|\,\mathcal{F}_\infty\vee\mathcal{H}_t\}\,\Big|\,\mathcal{G}_t\Big)\\
&= \mathbb{1}_{\{\tau>t\}}\,B_t\,\mathbb{E}_{\mathbb{P}}\Big(\int_t^T B_u^{-1}Z_u\lambda_u e^{\Gamma_t-\Gamma_u}\,du\,\Big|\,\mathcal{F}_t\vee\mathcal{H}_t\Big)\\
&\quad + B_t\,\mathbb{1}_{\{\tau>t\}}\,\mathbb{E}_{\mathbb{P}}\Big(B_T^{-1}Xe^{\Gamma_t-\Gamma_T}\,\Big|\,\mathcal{F}_t\vee\mathcal{H}_t\Big)\\
&= \mathbb{1}_{\{\tau>t\}}\,B_t\,\mathbb{E}_{\mathbb{P}}\Big(\int_t^T B_u^{-1}Z_u\lambda_u e^{\Lambda_t-\Lambda_u}\,du\,\Big|\,\mathcal{F}_t\Big)\\
&\quad + B_t\,\mathbb{1}_{\{\tau>t\}}\,\mathbb{E}_{\mathbb{P}}\Big(B_T^{-1}Xe^{\Lambda_t-\Lambda_T}\,\Big|\,\mathcal{F}_t\Big),
\end{aligned}$$

where the last equality is an immediate consequence of (F.1) (see, for instance, condition (M.2b)). Since $\tilde{B}_t = B_te^{\Lambda_t}$, the result follows. □

6.4.2 Case of a Stopping Time

In this section, we assume that the random time τ is an $\mathbb{F}$-stopping time. In other words, we postulate that $\mathbb{H} \subseteq \mathbb{F}$ or, equivalently, that $\mathbb{F} = \mathbb{G}$. Then, conditions (F.1), (F.2) and (M.1) are trivially satisfied. On the other hand, it is clear that $F = H$, and thus the $\mathbb{F}$-hazard process Γ of τ is not well defined.

Let us comment very briefly on the classification of stopping times (see, e.g., Dellacherie (1972)). If τ is a $\mathbb{G}$-predictable stopping time, we get the trivial equality $\Lambda = H$, and thus the concept of a $\mathbb{G}$-martingale hazard process of a $\mathbb{G}$-predictable stopping time is of no real use. If τ is a totally inaccessible $\mathbb{G}$-stopping time, the $\mathbb{G}$-compensator of the associated jump process H follows a continuous process (see Theorem V.T40 in Dellacherie (1972)). Recall that the $\mathbb{G}$-compensator of H is always stopped at τ.

From the previous section, we know that the process Λ can be used in the evaluation of certain conditional expectations, provided that a certain continuity condition is fulfilled. The following result, which covers the case of a totally inaccessible $\mathbb{G}$-stopping time, is an immediate consequence of Proposition 6.4.1.

Corollary 6.4.2. *Assume that τ is a $\mathbb{G}$-stopping time and the $\mathbb{G}$-martingale hazard process Λ of τ is continuous. For a fixed $T > 0$, let Y be a $\mathcal{G}_T$-measurable, integrable random variable. If the process V_t, $t \in [0, T]$, given by the formula*

$$V_t = \mathbb{E}_{\mathbb{P}}\big(Y e^{\Lambda_t - \Lambda_T} \,|\, \mathcal{G}_t\big), \tag{6.28}$$

is continuous at τ then for any $t < T$ we have

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > T\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{P}}\big(Y e^{\Lambda_t - \Lambda_T} \,|\, \mathcal{G}_t\big).$$

Example 6.4.1. Let τ be a random time, given on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$, such that the cumulative distribution function F of τ is continuous, and $\mathbb{P}\{\tau > t\} > 0$ for every $t \in \mathbb{R}_+$. Let us take $\mathbb{G} = \mathbb{H}$. Then τ is a totally inaccessible $\mathbb{G}$-stopping time and its $\mathbb{G}$-martingale hazard process Λ equals

$$\Lambda_{t \wedge \tau} = \int_0^{t \wedge \tau} \frac{dF(u)}{1 - F(u)}\,.$$

It is thus clear that we have $\Lambda_t = \Gamma^0(t \wedge \tau) = \Lambda^0(t \wedge \tau)$, where Γ^0 (Λ^0, resp.) is the hazard function (the martingale hazard function, resp.) of τ. Let us set, for $t \in \mathbb{R}_+$,

$$\Lambda_t = \int_0^t \frac{dF(u)}{1 - F(u)} = \Gamma^0(t) = \Lambda^0(t).$$

For Λ given above and any fixed $T > 0$, the process V associated with the random variable $Y = 1$ does not have a discontinuity at τ. Thus, for arbitrary $0 \le t < s$ we have (recall that here $\mathcal{G}_t = \mathcal{H}_t$)

$$\mathbb{P}\{\tau > s \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{P}}(e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{1 - F(s)}{1 - F(t)}\,.$$

6.5 Random Time with a Given Hazard Process

We shall examine the standard construction of a random time τ associated with a given hazard process Φ. It appears that in this method, the process Φ can be considered either as the $\mathbb{F}$-hazard process Γ, or as the $\mathbb{F}$-martingale hazard process Λ. Indeed, we shall show that the following properties are valid:
(i) Φ coincides with the $\mathbb{F}$-hazard process Γ of τ,
(ii) Φ is the $\mathbb{F}$-martingale hazard process of a random time τ,
(iii) Φ is the $\mathbb{G}$-martingale hazard process of a a $\mathbb{G}$-stopping time τ.
Let Φ be an $\mathbb{F}$-adapted, continuous, increasing process given on a filtered probability space $(\tilde{\Omega}, \mathbb{F}, \tilde{\mathbb{P}})$ such that $\Phi_0 = 0$ and $\Phi_\infty = +\infty$. For instance, Φ can be given by the formula

$$\Phi_t = \int_0^t \phi_u \, du, \quad \forall\, t \in \mathbb{R}_+, \tag{6.29}$$

where ϕ is a non-negative, $\mathbb{F}$-progressively measurable process. Our goal is to construct a random time τ, on an enlarged probability space $(\Omega, \mathcal{G}, \mathbb{P})$, in such a way that Φ is an $\mathbb{F}$-(martingale) hazard process of τ. To this end, we assume that ξ is a random variable on some probability space[1] $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, with the uniform probability law on $[0,1]$. We may take the product space $(\Omega = \tilde{\Omega} \times \hat{\Omega},\, \mathcal{G} = \mathcal{F}_\infty \otimes \hat{\mathcal{F}})$ with $\mathbb{P} = \tilde{\mathbb{P}} \otimes \hat{\mathbb{P}}$ as an enlarged probability space. We define $\tau : (\Omega, \mathcal{G}, \mathbb{P}) \to \mathbb{R}_+$ by setting

$$\tau = \inf\{\, t \in \mathbb{R}_+ : e^{-\Phi_t} \le \xi \,\} = \inf\{\, t \in \mathbb{R}_+ : \Phi_t \ge -\ln \xi \,\}.$$

As usual, we set $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for every t, so that Condition (G.1) is satisfied.

Remarks. It is worth stressing that the random time τ constructed above is not a stopping time with respect to the filtration $\mathbb{F}$. Furthermore, τ is a totally inaccessible stopping time with respect to the enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$.

We shall now check that properties (i)-(iii) listed above are satisfied.

Proof of (i). We shall find the process $F_t = \mathbb{P}\{\tau \le t \,|\, \mathcal{F}_t\}$. Since clearly $\{\tau > t\} = \{e^{-\Phi_t} > \xi\}$, we get

$$\mathbb{P}\{\tau > t \,|\, \mathcal{F}_\infty\} = e^{-\Phi_t}.$$

Consequently,

$$1 - F_t = \mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\} = \mathbb{E}_{\mathbb{P}}\big(\mathbb{P}\{\tau > t \,|\, \mathcal{F}_\infty\} \,|\, \mathcal{F}_t\big) = e^{-\Phi_t},$$

and so F is an $\mathbb{F}$-adapted continuous increasing process. In addition,

$$F_t = 1 - e^{-\Phi_t} = \mathbb{P}\{\tau \le t \,|\, \mathcal{F}_\infty\} = \mathbb{P}\{\tau \le t \,|\, \mathcal{F}_t\}. \tag{6.30}$$

We conclude that Φ coincides with the $\mathbb{F}$-hazard process Γ of τ under $\mathbb{P}$.

[1] It is enough to assume that we may define on $(\Omega, \mathcal{G}, \mathbb{P})$ a random variable ξ, which is uniformly distributed on $[0,1]$, and which is independent of the process Φ (we then set $\hat{\mathcal{F}} = \sigma(\xi)$).

Proof of (ii). We shall now check that Φ represents the $\mathbb{F}$-martingale hazard process Λ. This can be done either directly, or by establishing the equality $\Lambda = \Gamma$. Since the process Φ is continuous, to show that $\Lambda = \Gamma$, it is enough to check that Condition (F.1a) (or, equivalently, Condition (F.1)) holds, and to apply Corollary 6.2.1. Let us check that (F.1a) is valid. To this end, we fix t and we consider an arbitrary $u \le t$. Since for any $u \in \mathbb{R}_+$ we have

$$\mathbb{P}\{\tau \le u \,|\, \mathcal{F}_\infty\} = 1 - e^{-\Phi_u}, \tag{6.31}$$

we obtain the desired property:

$$\mathbb{P}\{\tau \le u \,|\, \mathcal{F}_t\} = \mathbb{E}_{\mathbb{P}}\big(\mathbb{P}\{\tau \le u \,|\, \mathcal{F}_\infty\} \,\big|\, \mathcal{F}_t\big) = 1 - e^{-\Phi_u} = \mathbb{P}\{\tau \le u \,|\, \mathcal{F}_\infty\}.$$

Alternatively, we may check directly that (F.1) holds. Since

$$\{\tau \le s\} = \{\Phi_s \ge -\ln \xi\} \in \hat{\mathcal{F}} \vee \mathcal{F}_s,$$

it is clear that $\mathcal{F}_t \subseteq \mathcal{H}_t \vee \mathcal{F}_t \subseteq \hat{\mathcal{F}} \vee \mathcal{F}_t$. Thus, for any bounded, $\mathcal{F}_\infty$-measurable random variable ξ we have

$$\mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{H}_t \vee \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \hat{\mathcal{F}} \vee \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t), \tag{6.32}$$

where the second equality is a consequence of the independence of $\hat{\mathcal{F}}$ and $\mathcal{F}_\infty$. This shows that (F.1) holds.

We conclude that the $\mathbb{F}$-martingale hazard process Λ of τ coincides with Γ, so that, by virtue of part (i): $\Phi_t = \Lambda_t = \Gamma_t = -\ln(1 - F_t)$. Furthermore, we know that Condition (F.1) is equivalent to (M.2), and thus, by virtue of Lemma 6.1.1, the martingale invariance property holds, i.e., any $\mathbb{F}$-martingale also follow a martingale with respect to $\mathbb{G}$.

Proof of (iii). Let us now check directly that Φ is an $\mathbb{F}$-martingale hazard process of a random time τ. Since Φ is a $\mathbb{F}$-predictable process (and thus a $\mathbb{G}$-predictable process), we will show at the same time that Φ is also the $\mathbb{G}$-martingale hazard process of a $\mathbb{G}$-stopping time τ. We need to verify that the compensated process $H_t - \Phi_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale. Since, for any $t \le s$,

$$\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\le s\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\le s\}} \,|\, \mathcal{G}_t),$$

by virtue of Lemma 5.1.2, we have

$$\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{P}\{t < \tau \le s \,|\, \mathcal{F}_t\}}{\mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\}}\,.$$

Using (6.30), we obtain

$$\mathbb{P}\{t < \tau \le s \,|\, \mathcal{F}_t\} = \mathbb{E}_{\mathbb{P}}(F_s \,|\, \mathcal{F}_t) - F_t,$$

and this in turn shows that

$$\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}(F_s \,|\, \mathcal{F}_t) - F_t}{1 - F_t}. \tag{6.33}$$

On the other hand, if we set $Y = \Phi_{s\wedge\tau} - \Phi_{t\wedge\tau}$, then, in view of part (i), we get (cf. (6.2))

$$Y = \mathbb{1}_{\{\tau>t\}} Y = \ln\Big(\frac{1 - F_{s\wedge\tau}}{1 - F_{t\wedge\tau}}\Big) = \int_{]t,s\wedge\tau]} \frac{dF_u}{1 - F_u}.$$

Using again (5.2), we obtain (for the last equality in the formula below, see (6.3))

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\big(Y \,|\, \mathcal{G}_t\big) &= \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{F}_t)}{\mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\}} = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}\big(\int_{]t,s\wedge\tau]} (1 - F_u)^{-1} dF_u \,|\, \mathcal{F}_t\big)}{1 - F_t} \\ &= \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}(F_s \,|\, \mathcal{F}_t) - F_t}{1 - F_t}.\end{aligned}$$

We conclude that the process $H_t - \Phi_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale.

Let us analyze the differences between statements (i) and (iii). In part (i), we consider Φ as an $\mathbb{F}$-hazard process of τ, then using Corollary 5.1.1 we deduce that for any $\mathcal{F}_s$-measurable random variable Y

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}\big(Y e^{\Phi_t - \Phi_s} \,|\, \mathcal{F}_t\big). \tag{6.34}$$

In part (iii), Φ is considered as the $\mathbb{G}$-martingale hazard process then, in view of Corollary 6.4.2, for any $\mathcal{G}_s$-measurable random variable Y such that the associated process V is continuous at τ we obtain

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}\big(Y e^{\Phi_t - \Phi_s} \,|\, \mathcal{G}_t\big). \tag{6.35}$$

If Y is actually $\mathcal{F}_s$-measurable then we have (see (6.32))

$$\mathbb{E}_{\mathbb{P}}\big(Y e^{\Phi_t - \Phi_s} \,|\, \mathcal{G}_t\big) = \mathbb{E}_{\mathbb{P}}\big(Y e^{\Phi_t - \Phi_s} \,|\, \mathcal{F}_t \vee \mathcal{H}_t\big) = \mathbb{E}_{\mathbb{P}}\big(Y e^{\Phi_t - \Phi_s} \,|\, \mathcal{F}_t\big).$$

It follows that the associated process V is necessarily continuous at τ, and formulae (6.34) and (6.35) coincide.

Remarks. Assume that the process Φ is absolutely continuous, it satisfies (6.29) for some process ϕ. Then equality (6.33) can be rewritten as follows:

$$\mathbb{P}\{t < \tau \le s \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}\Big(1 - e^{-\int_t^s \phi_u \, du} \,\Big|\, \mathcal{F}_t\Big). \tag{6.36}$$

Using (6.31), we find that the cumulative distribution function of a random time τ under $\mathbb{P}$ equals

$$F(t) = \mathbb{P}\{\tau \le t\} = 1 - \mathbb{E}_{\mathbb{P}}\Big(e^{-\int_0^t \phi_u \, du}\Big) = 1 - e^{-\int_0^t \gamma^0(u) \, du},$$

where we write γ^0 to denote the unique $\mathbb{F}^0$-intensity (that is, the intensity function) of τ.

Let us conclude this section by mentioning that the construction of a random time described above can be extended to the case of a finite family of $\mathbb{F}$-conditionally independent random times (see Sect. 9.1.2).

6.6 Poisson Process and Conditional Poisson Process

Until now, we have focused our attention on the case of a single random time and the associated jump process. In some financial applications, we need to model a sequence of successive random times. Almost invariably, this is done by making use of the so-called $\mathbb{F}$-*conditional Poisson process,* also known as the *doubly stochastic Poisson process.* The general idea is quite similar to the canonical construction of a single random time, which was examined in the previous section. We start by assuming that we are given a stochastic process Φ, to be interpreted as the *hazard process,* and we construct a jump process, with unit jump size, such that the probabilistic features of consecutive jump times are governed by the hazard process Φ.

Poisson process with constant intensity. Let us first recall the definition and the basic properties of the (time-homogeneous) Poisson process N with constant intensity $\lambda > 0$.

Definition 6.6.1. A process N defined on a probability space $(\Omega, \mathbb{G}, \mathbb{P})$ is called the *Poisson process* with intensity λ with respect to $\mathbb{G}$ if $N_0 = 0$ and for any $0 \le s < t$ the following two conditions are satisfied:
(i) the increment $N_t - N_s$ is independent of the σ-field $\mathcal{G}_s$,
(ii)the increment $N_t - N_s$ has the Poisson law with parameter $\lambda(t-s)$; specifically, for any $k = 0, 1, \ldots$ we have

$$\mathbb{P}\{N_t - N_s = k \,|\, \mathcal{G}_s\} = \mathbb{P}\{N_t - N_s = k\} = \frac{\lambda^k (t-s)^k}{k!}\, e^{-\lambda(t-s)}.$$

The Poisson process of Definition 6.6.1 is termed *time-homogeneous,* since the probability law of the increment $N_{t+h} - N_{s+h}$ is invariant with respect to the shift $h \ge -s$. In particular, for arbitrary $s < t$ the probability law of the increment $N_t - N_s$ coincides with the law of the random variable N_{t-s}. Let us finally observe that, for every $0 \le s < t$,

$$\mathbb{E}_{\mathbb{P}}(N_t - N_s \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(N_t - N_s) = \lambda(t-s). \tag{6.37}$$

We take a version of the Poisson process whose sample paths are, with probability 1, right-continuous stepwise functions with all jumps of size 1. Let us set $\tau_0 = 0$, and let us denote by $\tau_1, \tau_2, \ldots$ the $\mathbb{G}$-stopping times given as the random moments of the successive jumps of N. For any $k = 0, 1, \ldots$

$$\tau_{k+1} = \inf\{t > \tau_k : N_t \ne N_{\tau_k}\} = \inf\{t > \tau_k : N_t - N_{\tau_k} = 1\}.$$

One shows without difficulties that $\mathbb{P}\{\lim_{k\to\infty} \tau_k = \infty\} = 1$. It is convenient to introduce the sequence ξ_k, $k \in \mathbb{N}$ of non-negative random variables, where $\xi_k = \tau_k - \tau_{k-1}$ for every $k \in \mathbb{N}$. Let us quote the following well known result.

Proposition 6.6.1. *The random variables ξ_k, $k \in \mathbb{N}$ are mutually independent and identically distributed, with the exponential law with parameter λ, that is, for every $k \in \mathbb{N}$ we have*

$$\mathbb{P}\{\xi_k \le t\} = \mathbb{P}\{\tau_k - \tau_k \le t\} = 1 - e^{-\lambda t}, \quad \forall\, t \in \mathbb{R}_+.$$

Proposition 6.6.1 suggests a simple construction of a process N, which follows a time-homogeneous Poisson process with respect to its natural filtration $\mathbb{F}^N$. Suppose that the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is large enough to support a family of mutually independent random variables ξ_k, $k \in \mathbb{N}$ with the common exponential law with parameter $\lambda > 0$. We define the process N on $(\Omega, \mathcal{G}, \mathbb{P})$ by setting: $N_t = 0$ if $\{t < \xi_1\}$ and, for any natural k,

$$N_t = k \quad \text{if and only if} \quad \sum_{i=1}^{k} \xi_i \le t < \sum_{i=1}^{k+1} \xi_i.$$

It can checked that the process N defined in this way is indeed a Poisson process with parameter λ, with respect to its natural filtration $\mathbb{F}^N$. The jump times of N are, of course, the random times $\tau_k = \sum_{i=1}^{k} \xi_i$, $k \in \mathbb{N}$.

Let us recall some useful equalities that are not hard to establish through elementary calculations involving the Poisson law. For any $a \in \mathbb{R}$ and $0 \le s < t$ we have

$$\mathbb{E}_{\mathbb{P}}\big(e^{ia(N_t - N_s)} \,\big|\, \mathcal{G}_s\big) = \mathbb{E}_{\mathbb{P}}\big(e^{ia(N_t - N_s)}\big) = e^{\lambda(t-s)(e^{ia}-1)},$$

and

$$\mathbb{E}_{\mathbb{P}}\big(e^{a(N_t - N_s)} \,\big|\, \mathcal{G}_s\big) = \mathbb{E}_{\mathbb{P}}\big(e^{a(N_t - N_s)}\big) = e^{\lambda(t-s)(e^{a}-1)}.$$

The next result is an easy consequence of (6.37) and the above formulae. The proof of the proposition is thus left to the reader.

Proposition 6.6.2. *The following stochastic processes follow $\mathbb{G}$-martingales.*
(i) *The compensated Poisson process $\hat{N}$ defined as*

$$\hat{N}_t := N_t - \lambda t.$$

(ii) *For any $k \in \mathbb{N}$, the compensated Poisson process stopped at τ_k*

$$\hat{M}^k_t := N_{t \wedge \tau_k} - \lambda (t \wedge \tau_k).$$

(iii) *For any $a \in \mathbb{R}$, the exponential martingale M^a given by the formula*

$$M^a_t := e^{aN_t - \lambda t (e^a - 1)} = e^{a\hat{N}_t - \lambda t (e^a - a - 1)}.$$

(iv) *For any fixed $a \in \mathbb{R}$, the exponential martingale K^a given by the formula*

$$K^a_t := e^{iaN_t - \lambda t (e^{ia} - 1)} = e^{ia\hat{N}_t - \lambda t (e^{ia} - ia - 1)}.$$

Remarks. (i) For any $\mathbb{G}$-martingale M, defined on some filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$, and an arbitrary $\mathbb{G}$-stopping time τ, the stopped process $M^\tau_t = M_{t \wedge \tau}$ necessarily follows a $\mathbb{G}$-martingale. Thus, the second statement of the proposition is an immediate consequence of the first, combined with the simple observation that each jump time τ_k is a $\mathbb{G}$-stopping time.
(ii) Consider the random time $\tau = \tau_1$, where τ_1 is the time of the first jump of the Poisson process N. Then $N_{t \wedge \tau} = N_{t \wedge \tau_1} = H_t$, so that the process $\hat{M}^1$ introduced in part (ii) of the proposition coincides with the martingale $\hat{M}$ associated with τ.

(iii) The property described in part (iii) of Proposition 6.6.2 characterizes the Poisson process in the following sense: if $N_0 = 0$ and for every $a \in \mathbb{R}$ the process M^a is a $\mathbb{G}$-martingale, then N follows the Poisson process with parameter λ. Indeed, the martingale property of M^a yields

$$\mathbb{E}_{\mathbb{P}}\big(e^{a(N_t - N_s)} \,\big|\, \mathcal{G}_s\big) = e^{\lambda(t-s)(e^a - 1)}, \quad \forall\, 0 \le s < t.$$

By standard arguments, this implies that the random variable $N_t - N_s$ is independent of the σ-field $\mathcal{G}_s$, and has the Poisson law with parameter $\lambda(t-s)$. A similar remark applies to property (iv) in Proposition 6.6.2.

Let us consider the case of a Brownian motion W and a Poisson process N that are defined on a common filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$. In particular, for every $0 \le s < t$, the increment $W_t - W_s$ is independent of the σ-field $\mathcal{G}_s$, and has the Gaussian law $N(0, t-s)$. It might be useful to recall that for any real number b the following processes follow martingales with respect to $\mathbb{G}$:

$$\hat{W}_t = W_t - t, \quad m_t^b = e^{bW_t - \frac{1}{2}b^2 t}, \quad k_t^b = e^{ibW_t + \frac{1}{2}b^2 t}.$$

The next result shows that a Brownian motion W and a Poisson process N, with respect to a common filtration $\mathbb{G}$, are necessarily mutually independent.

Proposition 6.6.3. *Let a Brownian motion W and a Poisson process N be defined on a common filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$. Then the two processes W and N are mutually independent.*

Proof. Let us sketch the proof. For a fixed $a \in \mathbb{R}$ and any $t > 0$, we have

$$\begin{aligned} e^{iaN_t} &= 1 + \sum_{0<u\le t} (e^{iaN_t} - e^{iaN_{t-}}) = 1 + \int_{]0,t]} (e^{ia} - 1) e^{iaN_{u-}}\, dN_u, \\ &= 1 + \int_{]0,t]} (e^{ia} - 1) e^{iaN_{u-}}\, d\hat{N}_u + \lambda \int_0^t (e^{ia} - 1) e^{iaN_{u-}}\, du. \end{aligned}$$

On the other hand, for any $b \in \mathbb{R}$, the Itô formula yields

$$e^{ibW_t} = 1 + ib \int_0^t e^{ibW_u}\, dW_u - \frac{1}{2} b^2 \int_0^t e^{ibW_u}\, du.$$

The continuous martingale part of the compensated Poisson process $\hat{N}$ is identically equal to 0 (since $\hat{N}$ is a process of finite variation), and obviously the processes $\hat{N}$ and W have no common jumps. Thus, using the Itô product rule for semimartingales, we obtain

$$\begin{aligned} e^{i(aN_t + bW_t)} &= 1 + ib \int_0^t e^{i(aN_u + bW_u)}\, dW_u - \frac{1}{2} b^2 \int_0^t e^{i(aN_u + bW_u)}\, du \\ &\quad + \int_{]0,t]} (e^{ia} - 1) e^{i(aN_{u-} + bW_u)}\, d\hat{N}_u + \lambda \int_0^t (e^{ia} - 1) e^{i(aN_u + bW_u)}\, du. \end{aligned}$$

Let us denote $f_{a,b}(t) = \mathbb{E}_{\mathbb{P}}(e^{i(aN_t+bW_t)})$. By taking the expectations of both sides of the last equality, we get

$$f_{a,b}(t) = 1 + \lambda \int_0^t (e^{ia} - 1) f_{a,b}(u)\, du - \frac{1}{2} b^2 \int_0^t f_{a,b}(u)\, du.$$

By solving the last equation, we obtain, for arbitrary $a, b \in \mathbb{R}$,

$$\mathbb{E}_{\mathbb{P}}\big(e^{i(aN_t+bW_t)}\big) = f_{a,b}(t) = e^{\lambda t(e^{ia}-1)} e^{-\frac{1}{2}b^2 t} = \mathbb{E}_{\mathbb{P}}\big(e^{iaN_t}\big)\mathbb{E}_{\mathbb{P}}\big(e^{ibW_t}\big).$$

We conclude that for any $t \in \mathbb{R}_+$ the random variables W_t and N_t are mutually independent under $\mathbb{P}$.

In the second step, we fix $0 < t < s$, and we consider the following expectation, for arbitrary real numbers a_1, a_2, b_1 and b_2,

$$f(t,s) := \mathbb{E}_{\mathbb{P}}\big(e^{i(a_1 N_t + a_2 N_s + b_1 W_t + b_2 W_s)}\big).$$

Let us denote $\tilde{a}_1 = a_1 + a_2$ and $\tilde{b}_1 = b_1 + b_2$. Then

$$\begin{aligned}
f(t,s) &= \mathbb{E}_{\mathbb{P}}\big(e^{i(a_1 N_t + a_2 N_s + b_1 W_t + b_2 W_s)}\big) \\
&= \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}\big(e^{i(\tilde{a}_1 N_t + a_2(N_s - N_t) + \tilde{b}_1 W_t + b_2(W_s - W_t))} \,\big|\, \mathcal{G}_t\big)\big) \\
&= \mathbb{E}_{\mathbb{P}}\big(e^{i(\tilde{a}_1 N_t + \tilde{b}_1 W_t)} \mathbb{E}_{\mathbb{P}}\big(e^{i(a_2(N_s - N_t) + b_2(W_s - W_t))} \,\big|\, \mathcal{G}_t\big)\big) \\
&= \mathbb{E}_{\mathbb{P}}\big(e^{i(\tilde{a}_1 N_t + \tilde{b}_1 W_t)} \mathbb{E}_{\mathbb{P}}\big(e^{i(a_2 N_{t-s} + b_2 W_{t-s})}\big)\big) \\
&= f_{a_1,b_1}(t-s)\, \mathbb{E}_{\mathbb{P}}\big(e^{i(\tilde{a}_1 N_t + \tilde{b}_1 W_t)}\big) \\
&= f_{a_1,b_1}(t-s) f_{\tilde{a}_1,\tilde{b}_1}(t),
\end{aligned}$$

where we have used, in particular, the independence of the increment $N_t - N_s$ (and $W_t - W_s$) of the σ-field $\mathcal{G}_t$, and the time-homogeneity of N and W. By setting $b_1 = b_2 = 0$ in the last formula, we obtain

$$\mathbb{E}_{\mathbb{P}}\big(e^{i(a_1 N_t + a_2 N_s)}\big) = f_{a_1,0}(t-s) f_{\tilde{a}_1,0}(t),$$

while the choice of $a_1 = a_2 = 0$ yields

$$\mathbb{E}_{\mathbb{P}}\big(e^{i(b_1 W_t + b_2 W_s)}\big) = f_{0,b_1}(t-s) f_{0,\tilde{b}_1}(t).$$

It is not difficult to check that

$$f_{a_1,b_1}(t-s) f_{\tilde{a}_1,\tilde{b}_1}(t) = f_{a_1,0}(t-s) f_{\tilde{a}_1,0}(t) f_{0,b_1}(t-s) f_{0,\tilde{b}_1}(t).$$

We conclude that for any $0 \le t < s$ and arbitrary $a_1, a_2, b_1, b_2 \in \mathbb{R}$:

$$\mathbb{E}_{\mathbb{P}}\big(e^{i(a_1 N_t + a_2 N_s + b_1 W_t + b_2 W_s)}\big) = \mathbb{E}_{\mathbb{P}}\big(e^{i(a_1 N_t + a_2 N_s)}\big)\mathbb{E}_{\mathbb{P}}\big(e^{i(b_1 W_t + b_2 W_s)}\big).$$

This means that the random variables (N_t, N_s) and (W_t, W_s) are mutually independent. By proceeding along the same lines, one may check that the random variables $(N_{t_1}, \ldots, N_{t_n})$ and $(W_{t_1}, \ldots, W_{t_n})$ are mutually independent for any $n \in \mathbb{N}$ and for any choice of $0 \le t_1 < \ldots < t_n$. □

Let us now examine the behavior of the Poisson process under a specific equivalent change of the underlying probability measure. For a fixed $T > 0$, we introduce a probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{G}_T)$ by setting

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathcal{G}_T} = \eta_T, \quad \mathbb{P}\text{-a.s.}, \tag{6.38}$$

where the Radon-Nikodým density process η_t, $t \in [0, T]$, satisfies

$$d\eta_t = \eta_{t-} \kappa \, d\hat{N}_t, \quad \eta_0 = 1, \tag{6.39}$$

for some constant $\kappa > -1$. Since $Y := \kappa \hat{N}$ is a process of finite variation, we know from Lemma 4.4.1 that (6.39) admits a unique solution, denoted as $\mathcal{E}_t(Y)$ or $\mathcal{E}_t(\kappa \hat{N})$; it can be seen as a special case of the Doléans (or stochastic) exponential. By solving (6.39) path-by-path, we obtain

$$\eta_t = \mathcal{E}_t(\kappa \hat{N}) = e^{Y_t} \prod_{0<u\leq t} (1 + \Delta Y_u) e^{-\Delta Y_u} = e^{Y_t^c} \prod_{0<u\leq t} (1 + \Delta Y_u),$$

where $Y_t^c := Y_t - \sum_{0<u\leq t} \Delta Y_u$ is the path-by-path continuous part of Y. Direct calculations show that

$$\eta_t = e^{-\kappa\lambda t} \prod_{0<u\leq t} (1 + \kappa \Delta N_u) = e^{-\kappa\lambda t} (1+\kappa)^{N_t} = e^{N_t \ln(1+\kappa) - \kappa\lambda t},$$

where the last equality holds if $\kappa > -1$. Upon setting $a = \ln(1+\kappa)$ in part (iii) of Proposition 6.6.2, we get $M^a = \eta$; this confirms that the process η follows a $\mathbb{G}$-martingale under $\mathbb{P}$. We have thus proved the following result.

Lemma 6.6.1. *Assume that $\kappa > -1$. The unique solution η to the SDE* (6.39) *follows an exponential $\mathbb{G}$-martingale under $\mathbb{P}$. Specifically,*

$$\eta_t = e^{N_t \ln(1+\kappa) - \kappa\lambda t} = e^{\hat{N}_t \ln(1+\kappa) - \lambda t(\kappa - \ln(1+\kappa))} = M_t^a, \tag{6.40}$$

where $a = \ln(1+\kappa)$. In particular, the random variable η_T is strictly positive, $\mathbb{P}$a.s. *and* $\mathbb{E}_{\mathbb{P}}(\eta_T) = 1$. *Furthermore, the process M^a solves the following SDE:*

$$dM_t^a = M_{t-}^a (e^a - 1) \, d\hat{N}_t, \quad M_0^a = 1. \tag{6.41}$$

We are in the position to establish the well-known result, which states that under $\mathbb{P}^*$ the process N_t, $t \in [0, T]$, follows a Poisson process with the constant intensity $\lambda^* = (1+\kappa)\lambda$.

Proposition 6.6.4. *Assume that under $\mathbb{P}$ a process N is a Poisson process with intensity λ with respect to the filtration $\mathbb{G}$. Suppose that the probability measure $\mathbb{P}^*$ is defined on $(\Omega, \mathcal{G}_T)$ through* (6.38) *and* (6.39) *for some $\kappa > -1$.*
(i) *The process N_t, $t \in [0, T]$, follows a Poisson process under $\mathbb{P}^*$ with respect to $\mathbb{G}$ with the constant intensity $\lambda^* = (1+\kappa)\lambda$.*
(ii) *The compensated process N_t^*, $t \in [0, T]$, defined as*

$$N_t^* = N_t - \lambda^* t = N_t - (1+\kappa)\lambda t = \hat{N}_t - \kappa\lambda t,$$

follows a $\mathbb{P}^*$-martingale with respect to $\mathbb{G}$.

Proof. From remark (iii) after Proposition 6.6.2, we know that it suffices to find λ^* such that, for any fixed $b \in \mathbb{R}$, the process $\tilde{M}^b$, given as

$$\tilde{M}_t^b := e^{bN_t - \lambda^* t(e^b - 1)}, \quad \forall\, t \in [0, T], \tag{6.42}$$

follows a $\mathbb{G}$-martingale under $\mathbb{P}^*$. By standard arguments, the process $\tilde{M}^b$ is a $\mathbb{P}^*$-martingale if and only if the product $\tilde{M}^b \eta$ is a martingale under the original probability measure $\mathbb{P}$. But in view of (6.40), we have

$$\tilde{M}_t^b \eta_t = \exp\Big(N_t\big(b + \ln(1+\kappa)\big) - t\big(\kappa\lambda + \lambda^*(e^b - 1)\big)\Big).$$

Let us write $a = b + \ln(1+\kappa)$. Since b is an arbitrary real number, so is a. Then, by virtue of part (iii) in Proposition 6.6.2, we necessarily have

$$\kappa\lambda + \lambda^*(e^b - 1) = \lambda(e^a - 1).$$

After simplifications, we conclude that, for any fixed real number b, the process $\tilde{M}^b$ defined by (6.42) is a $\mathbb{G}$-martingale under $\mathbb{P}^*$ if and only if $\lambda^* = (1+\kappa)\lambda$. In other words, the intensity λ^* of N under $\mathbb{P}^*$ satisfies $\lambda^* = (1+\kappa)\lambda$. Also the second statement is clear. □

Remarks. Assume that $\mathbb{G} = \mathbb{F}^N$, i.e., the filtration $\mathbb{G}$ is generated by some Poisson process N. Then any strictly positive $\mathbb{G}$-martingale η under $\mathbb{P}$ is known to satisfy (6.39) for some $\mathbb{G}$-predictable process κ.

Assume that W is a Brownian motion and N follows a Poisson process under $\mathbb{P}$ with respect to $\mathbb{G}$. Let η satisfy

$$d\eta_t = \eta_{t-}\big(\beta_t\, dW_t + \kappa\, d\hat{N}_t\big), \quad \eta_0 = 1, \tag{6.43}$$

for some $\mathbb{G}$-predictable stochastic process β and some constant $\kappa > -1$. A simple application of the Itô's product rule shows that if processes η^1 and η^2 satisfy:

$$d\eta_t^1 = \eta_{t-}^1 \beta_t\, dW_t, \quad d\eta_t^2 = \eta_{t-}^2 \kappa\, d\hat{N}_t,$$

then the product $\eta_t := \eta_t^1 \eta_t^2$ satisfies (6.43). Taking the uniqueness of solutions to the linear SDE (6.43) for granted, we conclude that the unique solution to this SDE is given by the expression:

$$\eta_t = \exp\Big(\int_0^t \beta_u\, dW_u - \frac{1}{2}\int_0^t \beta_u^2\, du\Big) \exp\big(N_t \ln(1+\kappa) - \kappa\lambda t\big). \tag{6.44}$$

The proof of the next result is left to the reader as exercise.

Proposition 6.6.5. *Let the probability $\mathbb{P}^*$ be given by* (6.38) *and* (6.44) *for some constant $\kappa > -1$ and a $\mathbb{G}$-predictable process β, such that $\mathbb{E}_{\mathbb{P}}(\eta_T) = 1$.*
(i) *The process $W_t^* = W_t - \int_0^t \beta_u\, du$, $t \in [0, T]$, follows a Brownian motion under $\mathbb{P}^*$, with respect to the filtration $\mathbb{G}$.*
(ii) *The process N_t, $t \in [0, T]$, follows a Poisson process with the constant intensity $\lambda^* = (1+\kappa)\lambda$ under $\mathbb{P}^*$, with respect to the filtration $\mathbb{G}$.*
(iii) *Processes W^* and N are mutually independent under $\mathbb{P}^*$.*

Poisson process with deterministic intensity. Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be any non-negative, locally integrable function such that $\int_0^\infty \lambda(u)\,du = \infty$. By definition, the process N (with $N_0 = 0$) is the Poisson process with *intensity function* λ if for every $0 \le s < t$ the increment $N_t - N_s$ is independent of the σ-field $\mathcal{G}_s$, and has the Poisson law with parameter $\Lambda(t) - \Lambda(s)$, where the *hazard function* Λ equals $\Lambda(t) = \int_0^t \lambda(u)\,du$.

More generally, let $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a right-continuous, increasing function with $\Lambda(0) = 0$ and $\Lambda(\infty) = \infty$. The Poisson process with the hazard function Λ satisfies, for every $0 \le s < t$ and every $k = 0, 1, \ldots$:

$$\mathbb{P}\{N_t - N_s = k \,|\, \mathcal{G}_s\} = \mathbb{P}\{N_t - N_s = k\} = \frac{(\Lambda(t) - \Lambda(s))^k}{k!}\, e^{-(\Lambda(t)-\Lambda(s))}.$$

Example 6.6.1. The most convenient and widely used method of constructing a Poisson process with a hazard function Λ runs as follows: we take a Poisson process $\tilde{N}$ with the constant intensity $\lambda = 1$, with respect to some filtration $\tilde{\mathbb{G}}$, and we define the time-changed process $N_t := \tilde{N}_{\Lambda(t)}$. The process N is easily seen to follow a Poisson process with the hazard function Λ, with respect to the time-changed filtration $\mathbb{G}$, where $\mathcal{G}_t = \tilde{\mathcal{G}}_{\Lambda(t)}$ for every $t \in \mathbb{R}_+$.

Since for arbitrary $0 \le s < t$

$$\mathbb{E}_{\mathbb{P}}(N_t - N_s \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(N_t - N_s) = \Lambda(t) - \Lambda(s),$$

it is clear that the compensated Poisson process $\hat{N}_t = N_t - \Lambda(t)$ follows a $\mathbb{G}$-martingale under $\mathbb{P}$. A suitable generalization of Proposition 6.6.3 shows that a Poisson process with the hazard function Λ and a Brownian motion with respect to $\mathbb{G}$ follow mutually independent processes under $\mathbb{P}$. The proof of the next lemma relies on a direct application of the Itô formula, and so it is omitted.

Lemma 6.6.2. *Let Z be an arbitrary bounded, $\mathbb{G}$-predictable process. Then the process M^Z, given by the formula*

$$M_t^Z = \exp\Big(\int_{]0,t]} Z_u\, dN_u - \int_0^t (e^{Z_u} - 1)\, d\Lambda(u) \Big),$$

follows a $\mathbb{G}$-martingale under $\mathbb{P}$. Moreover, M^Z is the unique solution to the SDE

$$dM_t^Z = M_{t-}^Z (e^{Z_t} - 1)\, d\hat{N}_t, \quad M_0^Z = 1.$$

In case of a Poisson process with intensity function λ, it can be easily deduced from Lemma 6.6.2 that, for any (Borel measurable) function $\kappa : \mathbb{R}_+ \to (-1, \infty)$, the process

$$\zeta_t = \exp\Big(\int_{]0,t]} \ln(1 + \kappa(u))\, dN_u - \int_0^t \kappa(u)\lambda(u)\, du \Big)$$

is the unique solution to the SDE

$$d\zeta_t = \zeta_{t-}\kappa(t)\, d\hat{N}_t, \quad \eta_0 = 1.$$

Using similar arguments as in the case of constant κ, one can show that the unique solution to the SDE

$$d\eta_t = \eta_{t-}\big(\beta_t \, dW_t + \kappa(t)\, d\hat{N}_t\big), \quad \eta_0 = 1,$$

is given by the following expression:

$$\eta_t = \zeta_t \exp\Big(\int_0^t \beta_u \, dW_u - \frac{1}{2} \int_0^t \beta_u^2 \, du \Big). \tag{6.45}$$

The next result generalizes Proposition 6.6.5. Again, the proof is left to the reader.

Proposition 6.6.6. *Let $\mathbb{P}^*$ be a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, such that the density process η in* (6.38) *is given by* (6.45). *Then, under $\mathbb{P}^*$ and with respect to $\mathbb{G}$:*
(i) *the process $W_t^* = W_t - \int_0^t \beta_u \, du$, $t \in [0,T]$, follows a Brownian motion,*
(ii) *the process N_t, $t \in [0,T]$, follows a Poisson process with the intensity function $\lambda^*(t) = 1 + \kappa(t)\lambda(t)$,*
(iii) *processes W^* and N are mutually independent under $\mathbb{P}^*$.*

Conditional Poisson process. We start by assuming that we are given a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$ and a certain sub-filtration $\mathbb{F}$ of $\mathbb{G}$. Let Φ be an $\mathbb{F}$-adapted, right-continuous, increasing process, with $\Phi_0 = 0$ and $\Phi_\infty = \infty$. We refer to Φ as the *hazard process.* In some cases, we have $\Phi_t = \int_0^t \phi_u \, du$ for some $\mathbb{F}$-progressively measurable process ϕ with locally integrable sample paths. Then the process ϕ is called the *intensity process.* We are in a position to state the definition of the $\mathbb{F}$-conditional Poisson process associated with Φ. Slightly different, but essentially equivalent, definition of a conditional Poisson process (also known as the doubly stochastic Poisson process) can be found in Brémaud (1981) and Last and Brandt (1995).

Definition 6.6.2. A process N defined on a probability space $(\Omega, \mathbb{G}, \mathbb{P})$ is called the $\mathbb{F}$-*conditional Poisson process* with respect to $\mathbb{G}$, associated with the hazard process Φ, if for any $0 \le s < t$ and every $k = 0, 1, \ldots$

$$\mathbb{P}\{N_t - N_s = k \,|\, \mathcal{G}_s \vee \mathcal{F}_\infty\} = \frac{(\Phi_t - \Phi_s)^k}{k!} \, e^{-(\Phi_t - \Phi_s)}, \tag{6.46}$$

where $\mathcal{F}_\infty = \sigma(\mathcal{F}_u : u \in \mathbb{R}_+)$.

At the intuitive level, if a particular sample path $\Phi_\cdot(\omega)$ of the hazard process is known, the process N has exactly the same properties as the Poisson process with respect to $\mathbb{G}$ with the (deterministic) hazard function $\Phi_\cdot(\omega)$. In particular, it follows from (6.46) that

$$\mathbb{P}\{N_t - N_s = k \,|\, \mathcal{G}_s \vee \mathcal{F}_\infty\} = \mathbb{P}\{N_t - N_s = k \,|\, \mathcal{F}_\infty\},$$

i.e., conditionally on the σ-field $\mathcal{F}_\infty$ the increment $N_t - N_s$ is independent of the σ-field $\mathcal{G}_s$.

Similarly, for any $0 \le s < t \le u$ and every $k = 0, 1, \dots$, we have

$$\mathbb{P}\{N_t - N_s = k \,|\, \mathcal{G}_s \vee \mathcal{F}_u\} = \frac{(\Phi_t - \Phi_s)^k}{k!}\, e^{-(\Phi_t - \Phi_s)}. \tag{6.47}$$

In other words, conditionally on the σ-field $\mathcal{F}_u$ the process $N_t,\, t \in [0, u]$, behaves like a Poisson process with the hazard function Φ. Finally, for any $n \in \mathbb{N}$, any non-negative integers $k_1, \dots, k_n$, and arbitrary non-negative real numbers $s_1 < t_1 \le s_2 < t_2 \le \ldots \le s_n < t_n$ we have

$$\mathbb{P}\Big(\bigcap_{i=1}^{n}\{N_{t_i} - N_{s_i} = k_i\}\Big) = \mathbb{E}_{\mathbb{P}}\Big(\prod_{i=1}^{n} \frac{\big(\Phi_{t_i} - \Phi_{s_i}\big)^{k_i}}{k_i!}\, e^{-(\Phi_{t_i} - \Phi_{s_i})}\Big).$$

Let us notice that in all conditional expectations above, the reference filtration $\mathbb{F}$ can be replaced by the filtration $\mathbb{F}^{\Phi}$ generated by the hazard process. In fact, an $\mathbb{F}$-conditional Poisson process with respect to $\mathbb{G}$ follows also a conditional Poisson process with respect to the filtrations: $\mathbb{F}^N \vee \mathbb{F}$ and $\mathbb{F}^N \vee \mathbb{F}^{\Phi}$ (with the same hazard process).

We shall henceforth postulate that $\mathbb{E}_{\mathbb{P}}(\Phi_t) < \infty$ for every $t \in \mathbb{R}_+$.

Lemma 6.6.3. *The compensated process $\hat{N}_t = N_t - \Phi_t$ follows a martingale with respect to $\mathbb{G}$.*

Proof. It is enough to notice that, for arbitrary $0 \le s < t$,

$$\mathbb{E}_{\mathbb{P}}(N_t - \Phi_t \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(N_t - \Phi_t \,|\, \mathcal{G}_s \vee \mathcal{F}_\infty) \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(N_s - \Phi_s \,|\, \mathcal{G}_s) = N_s - \Phi_s,$$

where in the second equality we have used the property of a Poisson process with deterministic hazard function. □

Given the two filtrations $\mathbb{F}$ and $\mathbb{G}$ and the hazard process Φ, it is not obvious whether we may find a process N, which would satisfy Definition 6.6.2. To provide a simple construction of a conditional Poisson process, we assume that the underlying probability space $(\Omega, \mathcal{G}, \mathbb{P})$, endowed with a reference filtration $\mathbb{F}$, is sufficiently large to accommodate for the following stochastic processes: a Poisson process $\tilde{N}$ with the constant intensity $\lambda = 1$ and an $\mathbb{F}$-adapted hazard process Φ. In addition, we postulate that the Poisson process $\tilde{N}$ is independent of the filtration $\mathbb{F}$

Remark. Given a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, it is always possible to enlarge it in such a way that there exists a Poisson process $\tilde{N}$ with $\lambda = 1$, independent of the filtration $\mathbb{F}$, and defined on the enlarged space.

Under the present assumptions, for every $0 \le s < t$, any $u \in \mathbb{R}_+$, and any non-negative integer k, we have

$$\mathbb{P}\{\tilde{N}_t - \tilde{N}_s = k \,|\, \mathcal{F}_\infty\} = \mathbb{P}\{\tilde{N}_t - \tilde{N}_s = k \,|\, \mathcal{F}_u\} = \mathbb{P}\{\tilde{N}_t - \tilde{N}_s = k\}$$

and

$$\mathbb{P}\{\tilde{N}_t - \tilde{N}_s = k \,|\, \mathcal{F}_s^{\tilde{N}} \vee \mathcal{F}_s\} = \mathbb{P}\{\tilde{N}_t - \tilde{N}_s = k\} = \frac{(t-s)^k}{k!}\, e^{-(t-s)}.$$

The next result describes an explicit construction of a conditional Poisson process. This construction is based on a random time change associated with the increasing process Φ.

Proposition 6.6.7. *Let $\tilde{N}$ be a Poisson process with the constant intensity $\lambda = 1$, independent of a reference filtration $\mathbb{F}$, and let Φ be an $\mathbb{F}$-adapted, right-continuous, increasing process. Then the process $N_t = \tilde{N}_{\Phi_t}$, $t \in \mathbb{R}_+$, follows the $\mathbb{F}$-conditional Poisson process with the hazard process Φ with respect to the filtration $\mathbb{G} = \mathbb{F}^N \vee \mathbb{F}$.*

Proof. Since $\mathcal{G}_s \vee \mathcal{F}_\infty = \mathcal{F}^N_s \vee \mathcal{F}_\infty$, it suffices to check that

$$\mathbb{P}\{N_t - N_s = k \,|\, \mathcal{F}^N_s \vee \mathcal{F}_\infty\} = \frac{(\Phi_t - \Phi_s)^k}{k!}\, e^{-(\Phi_t - \Phi_s)}$$

or, equivalently,

$$\mathbb{P}\{\tilde{N}_{\Phi_t} - \tilde{N}_{\Phi_s} = k \,|\, \mathcal{F}^{\tilde{N}}_{\Phi_s} \vee \mathcal{F}_\infty\} = \frac{(\Phi_t - \Phi_s)^k}{k!}\, e^{-(\Phi_t - \Phi_s)}.$$

The last equality follows from the assumed independence of $\tilde{N}$ and $\mathbb{F}$. □

Remark. Within the setting of Proposition 6.6.7, any $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale, so that Condition (M.1) is satisfied.

The total number of jumps of the conditional Poisson process is obviously unbounded with probability 1. In some financial models (see, e.g., Lando (1998) or Duffie and Singleton (1999)), only the properties of the first jump are relevant, though. There exist many ways of constructing the conditional Poisson process, but Condition (F.1) is always satisfied by the first jump of such a process, since it follows directly from Definition 6.6.2. In effect, if we denote $\tau = \tau_1$, then for any $t \in \mathbb{R}_+$ and $u \geq t$ we have (cf. Condition (F1.a) of Sect. 6.1)

$$\mathbb{P}\{\tau \leq t \,|\, \mathcal{F}_u\} = \mathbb{P}\{N_t \geq 1 \,|\, \mathcal{F}_u\} = \mathbb{P}\{N_t - N_0 \geq 1 \,|\, \mathcal{G}_0 \vee \mathcal{F}_u\} = \mathbb{P}\{\tau \leq u \,|\, \mathcal{F}_\infty\},$$

where the last equality follows from (6.47). It is also clear, once more by (6.47), that $\mathbb{P}\{\tau \leq t \,|\, \mathcal{F}_u\} = e^{-\Phi_u}$ for every $0 \leq t \leq u$.

Example 6.6.2. Cox process. In some applications, it is natural to consider a special case of an $\mathbb{F}$-conditional Poisson process, with the filtration $\mathbb{F}$ generated by a certain stochastic process, representing the *state variables.* To be more specific, on considers a conditional Poisson process with the intensity process ϕ given as $\phi_t = g(t, Y_t)$, where Y is an $\mathbb{R}^d$-valued stochastic process independent of the Poisson process $\tilde{N}$, and $g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ is a (continuous) function. The reference filtration $\mathbb{F}$ is typically chosen to be the natural filtration of the process Y; that is, we take $\mathbb{F} = \mathbb{F}^Y$. In such a case, the resulting $\mathbb{F}$-conditional Poisson process is referred[2] to as the *Cox process* associated with the state variables process Y, and the intensity function g.

[2] It should be acknowledged that the terminology in this area is not uniform across various sources.

Our last goal is to examine the behavior of an $\mathbb{F}$-conditional Poisson process N under an equivalent change of a probability measure. Let us assume, for the sake of simplicity, that the hazard process Φ is continuous, and the reference filtration $\mathbb{F}$ is generated by a process W, which follows a Brownian motion with respect to $\mathbb{G}$. For a fixed $T > 0$, we define the probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{G}_T)$ by setting

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathcal{G}_T} = \eta_T, \quad \mathbb{P}\text{-a.s.}, \tag{6.48}$$

where the Radon-Nikodým density process η_t, $t \in [0,T]$, solves the SDE

$$d\eta_t = \eta_{t-}\big(\beta_t \, dW_t + \kappa_t \, d\hat{N}_t\big), \quad \eta_0 = 1, \tag{6.49}$$

for some $\mathbb{G}$-predictable processes β and κ such that $\kappa > -1$ and $\mathbb{E}_{\mathbb{P}}(\eta_T) = 1$. An application of Itô's product rule shows that the unique solution to (6.49) is equal to the product $\nu_t \zeta_t$, where $d\nu_t = \nu_t \beta_t \, dW_t$ and $d\zeta_t = \zeta_{t-} \kappa_t \, d\hat{N}_t$, with $\nu_0 = \zeta_0 = 1$. The solutions to the last two equations are

$$\nu_t = \exp\Big(\int_0^t \beta_u \, dW_u - \frac{1}{2}\int_0^t \beta_u^2 \, du\Big)$$

and

$$\zeta_t = \exp(U_t) \prod_{0<u\le t} (1 + \Delta U_u) \exp(-\Delta U_u),$$

respectively, where we denote $U_t = \int_{]0,t]} \kappa_u \, d\hat{N}_u$. It is useful to observe that ζ admits the following representations:

$$\zeta_t = \exp\Big(-\int_0^t \kappa_u \, d\Phi_u\Big) \prod_{0<u\le t} (1 + \kappa_u \Delta N_u),$$

and

$$\zeta_t = \exp\Big(\int_{]0,t]} \ln(1+\kappa_u)\, dN_u - \int_0^t \kappa_u \, d\Phi_u\Big).$$

The following result is a counterpart of Proposition 5.3.1.

Proposition 6.6.8. *Let the Radon-Nikodým density of $\mathbb{P}^*$ with respect to $\mathbb{P}$ be given by (6.48)–(6.49). Then the process $W_t^* = W_t - \int_0^t \beta_u \, du$, $t \in [0,T]$, follows a Brownian motion with respect to $\mathbb{G}$ under $\mathbb{P}^*$, and the process*

$$N_t^* = \hat{N}_t - \int_0^t \kappa_u \, d\Phi_u = N_t - \int_0^t (1+\kappa_u)\, d\Phi_u, \quad \forall\, t \in [0,T], \tag{6.50}$$

follows a $\mathbb{G}$-martingale under $\mathbb{P}^*$. If, in addition, the process κ is $\mathbb{F}$-adapted, then the process N follows under $\mathbb{P}^*$ an $\mathbb{F}$-conditional Poisson process with respect to $\mathbb{G}$, and the hazard process of N under $\mathbb{P}^*$ equals

$$\Phi_t^* = \int_0^t (1+\kappa_u)\, d\Phi_u.$$