Preface to the first edition

Space and time are the two most fundamental concepts in our world because all else is unimaginable without assuming that space (or time) exists. It is therefore not surprising that the sophisticated Euclidean model of space already existed more than 2000 years. For centuries it was a common belief by scientists and philosophers alike that the Euclidean structure of space was one of the very few eternal truths. It was only at the beginning of the 20th century that this belief was shattered with the introduction of Albert Einstein's theories of special and general relativity. Today, Einstein's theory of general relativity is completely established, and there are many textbooks which explain it at all levels of mathematical sophistication. What is missing, however, is a modern textbook on general relativity for mathematicians and mathematical physicists with emphasis on the physical justification of the mathematical framework. This book aims to fill this gap.

Knowledge of physics is *not* assumed. While physical and heuristic arguments are given, they are not used as substitutes for any proofs. The book is also suitable as an introduction to pseudo-Riemannian geometry with emphasis on the intuition for geometrical concepts.

The physical theme of the book

Modern textbooks on general relativity typically start with a more or less formal introduction to pseudo-Riemannian geometry. In such textbooks some knowledge of special relativity is usually assumed, and the reader is expected to accept the geometrical framework presented on trust. This approach is very economical but obscures the extent to which classical general relativity succeeds in describing our universe, and also where it may fail. This is a point that is of particular relevance to those attempting to quantise gravity. From a physical point of view it is important to realise which parts of the theory reflect genuine physical insights, and which are dispensible. One way this can be achieved is through a critical introduction that stresses foundational matters. There are no modern textbooks taking this approach, and I hope to fill this gap with my book.

One of the most exciting aspects of general relativity is the prediction of black holes and the Big Bang. Such predictions gained weight through the singularity theorems pioneered by Penrose. In various textbooks on general relativity singularity theorems are presented and then used to argue that black holes exist and that the universe started with a big bang. To date what has been lacking is a critical analysis of what these theorems really predict.[1] We give a proof of a typical singularity theorem and use this theorem to illustrate problems arising through the possibilities of "causality violations" and very weak "shell crossing singularities". These problems add weight to the point of view that the singularity theorems alone are not sufficient to predict the existence of physical singularities.

The mathematical theme of the book

In order to gain both a solid understanding of and good intuition for any mathematical theory, one should try to realise it as a model of a familiar non-mathematical concept. Physical theories have had an especially important impact on the development of mathematics, and conversely various modern physical theories require rather sophisticated mathematics for their formulation. Today, both physics and mathematics are so complex that it is often very difficult to master the theories in both subjects. However, in the case of pseudo-Riemannian differential geometry or general relativity the relationship between physics and mathematics is especially close, and it is therefore possible to profit from an interdisciplinary approach.

Euclidean geometry had its origins as the description of shapes in *physical* space. It is generally considered a mathematical discipline rather than a physical theory, because it is possible to derive it from a small set of physical *postulates*, which can alternatively be viewed as mathematical *axioms*. Since the concept of space is basic to our everyday experience, Euclidean geometry combines mathematical rigor with intuitiveness — a combination which has proved to be extremely fruitful for both mathematics and physics. Riemannian geometry is abstracted from the study of surfaces in Euclidean space and inherits much of the intuitiveness of Euclidean geometry. Hence Riemannian geometry is very well developed, and a growing number of geometers have branched out to Lorentzian or even pseudo-Riemannian geometry. In my experience, these fields (and

[1] Since I had written this passage a review article (Senovilla 1998) which has a very similar theme has been pointed out to me. This article provides many very illuminating examples of spacetimes as well as discussions which reinforce our sceptical approach towards the physical interpretation of singularity theorems.

even Riemannian geometry) appear quite abstract to the majority of students.

A careful analysis of space, time, and free fall — the most fundamental (classical) physical concepts — leads almost automatically to Lorentzian geometry. With respect to Lorentzian geometry, we are therefore in a similar situation as ancient geometers were with respect to Euclidean geometry. What's more, virtually no physical background is required for this approach. Since Riemannian geometry comes to play in the study of submanifolds representing an instant in time, it is completely straightforward to extrapolate pseudo-Riemannian geometry from the special and physically motivated cases of Lorentzian and Riemannian geometry.

While some modern textbooks present pseudo-Riemannian geometry (and general relativity) to mathematicians (an example of this is that by O'Neill (1983)), they have not motivated the geometry from basic properties of space and time. Instead they have developed it as an abstract mathematical theory. To ensure that the mathematical description mirrors the physical concepts, all definitions have a justification in this book. *This approach also leads to a careful treatment of the structural aspects of the mathematics.*

How to read this book

This book is not designed so that it is necessary for the reader to start at page 1 and then to read on until she or he arrives at page 424. People who take this approach will very likely give up before they reach page 14! The material is ordered in such a way as to allow the text to be used as a reference source. It is an unfortunate fact that many parts of the theory that logically belong to the preliminaries are not of immediate interest to a reader who is interested in space and time, and so the reader is urged to follow the guides in the margins, which provide a shortcut. As an example, the text in the margin denotes the beginning of a passage belonging to the shortcut: **p. 111** ↓ denotes the page number where the last shortcut passage ended and [↓ **p. 222**] the page number where the present passage will end. Additional explanations in the footnotes are indicated by →2, where 2 refers to the number of the corresponding footnote. The end of shortcut passages is marked similarly. Having understood the material leading to Einstein's equation it is then not difficult to return to the parts that have been skipped on an earlier reading. In addition, hints are given at the beginning of most sections as to what is important and should be read .

p. 111 ↓
→2
[↓ **p. 222**]

[2] Explanations referring to the guide in the margin.

This book, with its 424 pages is meant to cover both general relativity and pseudo-Riemannian differential geometry. It is therefore clear that some important topics had to be omitted.

For mathematicians, the most important omissions are certainly some topics peculiar to Riemannian geometry, such as the Hopf-Rinow theorem (O'Neill 1983, Theorem 5.21) and the Myers theorem (O'Neill 1983, Theorem 10.24). Because these results are contrary to intuition one should obtain for Lorentzian (or general pseudo-Riemannian) geometry and since they are not needed for the description of space and time, they have been omitted from this book.

Physicists may find that the presentation of this book is only loosely linked to other physical theories. This loose linkage is possible since the theory of space and time is fundamental to any other physical theory. The book is therefore accessible to mathematicians and physicists alike. Physicists who are interested in applications to astrophysics may wish to consult the book by Weinberg (1972). Weinberg's approach is opposite to the one used in this book, and personally I believe that it should ideally be read after the reader has a solid knowledge of the conceptional aspects of relativity as presented in this book. Most other books on general relativity also present the "Kerr solution", which is supposed to model the exterior of a rotating black hole. It has been omitted since it is not essential to understanding general relativity. Moreover, it is well described in other books. People interested in this solution should probably first read Chap. 12 of the book by Wald (1984). The purely mathematical aspects of this solutions are clearly presented in O'Neill's book (1995).

Acknowledgements

The reader will undoubtedly notice that this book owes much to excellent text books and survey articles. For the philosophical aspects of this book I wish to mention especially the classic book by Weyl (1923) and the survey article by Ehlers (1973).

I have also freely used material which appears elsewhere (O'Neill 1983; Wald 1984; Beem and Ehrlich 1981; Hawking and Ellis 1973; Sachs and Wu 1979; Karcher 1994; De Felice and Clarke 1990; Abraham and Marsden 1978; Garabedian 1986) without always acknowledging this fact.

I warmly thank Bernd Wegner, who encouraged me and recommended the book project to Springer-Verlag. I wish especially to thank Volker Perlick not only for introducing me to relativity but also for reading through the whole manuscript and for his many important improvements.

This book is dedicated to two Australian relativity students who on their way to gaining their doctorates courageously stood up against the immoral behaviour of their supervisor and the highhandedness of their university.

Göttingen, 19th July 1999 M. Kriele

List of Figures

List of postulates