VI

Expansion of Filtrations

1 Introduction

By an expansion of the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, we mean that we expand the filtration $\mathbb{F}$ to get a new filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ which satisfies the usual hypotheses and $\mathcal{F}_t \subset \mathcal{H}_t$, each $t \geq 0$. There are three questions we wish to address: (1) when does a specific, given semimartingale remain a semimartingale in the enlarged filtration; (2) when do all semimartingales remain semimartingales in the enlarged filtration; (3) what is a new decomposition of the semimartingale for the new filtration.

The subject of the expansion of filtrations began with a seminal paper of K. Itô in 1976 (published in [104] in 1978), when he showed that if B is a standard Brownian motion, then one can expand the natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ of B by adding the σ-algebra generated by the random variable B_1 to all $\mathcal{F}_t$ of the filtration, including of course $\mathcal{F}_0$. He showed that B remains a semimartingale for the expanded filtration, he calculated its decomposition explicitly, and he showed that one has the intuitive formula

$$B_1 \int_0^t H_s dB_s = \int_0^t B_1 H_s dB_s$$

for any bounded $\mathbb{F}$ predictable process H where the integral on the left is computed with the original filtration, and the integral on the right is computed using the expanded filtration. Obviously such a result is of interest only for $0 \leq t \leq 1$. We will establish this formula more generally for Lévy processes in Sect. 2.

The second advance for the theory of the expansion of filtrations was a flurry of papers in 1978: partly inspired by a question posed by P.A. Meyer, M. Yor in France wrote [264], where he began the study of progressive expansion of filtrations. This was quickly followed by two other papers in collaboration with Th. Jeulin [121] and [122], while at the same time in England M. Barlow [9] was developing answers to similar questions posed by David Williams.

They considered the problem that if L is a positive random variable, and one expands the filtration in a minimal way to make L a stopping time (this type of expansion is called *progressive expansion*), what conditions ensure that semimartingales remain semimartingales for the expanded filtration? This type of question is the topic of Sect. 3.

Th. Jeulin, in his thesis (1979), later expanded in Springer Lecture Notes in Math. 833 [119], systematically studied both initial and progressive expansions. Later Th. Jeulin and M. Yor, as editors of Springer Lecture Notes in Math. 1118 [124], gathered many papers about the subject, including a number of applications to path decompositions of Markov processes in particular. See [125] for an overview.

2 Initial Expansions

Throughout this section we assume given an underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ which satisfies the usual hypotheses. As in previous chapters, for convenience we denote the filtration $(\mathcal{F}_t)_{t\geq 0}$ by the symbol $\mathbb{F}$.

The most elementary result on the expansion of filtrations is due to Jacod and was established in Chap. II (Theorem 5). We recall it here.

Theorem 1 (Jacod's Countable Expansion). *Let $\mathcal{A}$ be a collection of events in $\mathcal{F}$ such that if A_α, $A_\beta \in \mathcal{A}$ then $A_\alpha \cap A_\beta = \phi$, $\alpha \neq \beta$. Let $\mathcal{H}_t$ be the filtration generated by $\mathcal{F}_t$ and $\mathcal{A}$. Then every $((\mathcal{F}_t)_{t\geq 0}, P)$ semimartingale is an $((\mathcal{H}_t)_{t\geq 0}, P)$ semimartingale also.*

We also record a trivial observation as a second elementary theorem.

Theorem 2. *Let X be a semimartingale with decomposition $X = M + A$ and let $\mathcal{G}$ be a σ-algebra independent of the local martingale term M. Let $\mathbb{H}$ denote the filtration obtained by expanding $\mathbb{F}$ with the one σ-algebra $\mathcal{G}$ (that is, $\mathcal{H}_t^0 = \mathcal{F}_t \vee \mathcal{G}$, each $t \geq 0$ and $\mathcal{H}_t = \mathcal{H}_{t+}^0$). Then X is an $\mathbb{H}$ semimartingale with the same decomposition.*

Proof. Since the local martingale M remains a local martingale under $\mathbb{H}$, the theorem follows. □

We now turn to Lévy processes and an extension of Itô's first theorem. Let Z be a given Lévy process on our underlying space, and define $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ to be the smallest filtration satisfying the usual hypotheses, such that Z_1 is $\mathcal{H}_0$ measurable and $\mathcal{F}_t \subset \mathcal{H}_t$ for all $t \geq 0$.

Theorem 3 (Itô's Theorem extended to Lévy Processes). *The Lévy process Z is an $\mathbb{H}$ semimartingale. If moreover $E\{|Z_t|\} < \infty$, all $t \geq 0$, then the process*

$$M_t = Z_t - \int_0^{t \wedge 1} \frac{Z_1 - Z_s}{1 - s} ds$$

is an $\mathbb{H}$ martingale on $[0, \infty)$.

Proof. We begin by assuming $E\{Z_t^2\} < \infty$, each $t > 0$. Without loss of generality we can further assume $E\{Z_t\} = 0$. Since Z has independent increments, we know Z is an $\mathbb{F}$ martingale. Let $0 \leq s < t \leq 1$ be rationals with $s = j/n$ and $t = k/n$. We set

$$Y_i = Z_{\frac{i+1}{n}} - Z_{\frac{i}{n}}.$$

Then $Z_1 - Z_s = \sum_{i=j}^{n-1} Y_i$ and $Z_t - Z_s = \sum_{i=j}^{k-1} Y_i$. The random variables Y_i are i.i.d. and integrable. Therefore

$$E\{Z_t - Z_s | Z_1 - Z_s\} = E\{\sum_{i=j}^{k-1} Y_i | \sum_{i=j}^{n-1} Y_i\} = \frac{k-j}{n-j} \sum_{i=j}^{n-1} Y_i$$

$$= \frac{t-s}{1-s}(Z_1 - Z_s).$$

The independence of the increments of Z yields $E\{Z_t - Z_s | \mathcal{H}_s\} = E\{Z_t - Z_s | Z_1 - Z_s\}$; therefore $E\{Z_t - Z_s | \mathcal{H}_s\} = \frac{t-s}{1-s}(Z_1 - Z_s)$ for all rationals, $0 \leq s < t \leq 1$. Since Z is an $\mathbb{F}$ martingale, the random variables $(Z_t)_{0 \leq t \leq 1}$ are uniformly integrable, and since the paths of Z are right continuous, we deduce $E\{Z_t - Z_s | \mathcal{H}_s\} = \frac{t-s}{1-s}(Z_1 - Z_s)$ for all reals, $0 \leq s < t \leq 1$. By Fubini's Theorem for conditional expectations the above gives

$$E\{M_t - M_s | \mathcal{H}_s\} = E\{Z_t - Z_s | \mathcal{H}_s\} - \int_s^t \frac{1}{1-u} E\{Z_1 - Z_u | \mathcal{H}_s\} du$$

$$= \frac{t-s}{1-s}(Z_1 - Z_s) - \int_s^t \frac{1}{1-u} \frac{1-u}{1-s}(Z_1 - Z_s) du$$

$$= 0.$$

There is a potential problem at $t = 1$ because of the possibility of an explosion. Indeed this is typical of initial enlargements. However if we can show $E\{\int_0^1 \frac{|Z_1 - Z_s|}{1-s} ds\} < \infty$ this will suffice to rule out explosions. By the stationarity and independence of the increments of Z we have $E\{|Z_1 - Z_s|\} \leq E\{(Z_1 - Z_s)^2\}^{\frac{1}{2}} \leq a(1-s)^{\frac{1}{2}}$ for some constant a and for all s, $0 \leq s \leq 1$. Therefore $E\{\int_0^1 \frac{|Z_1 - Z_s|}{1-s} ds\} \leq a \int_0^1 \frac{\sqrt{1-s}}{1-s} ds < \infty$. Note that if $t > 1$ then $\mathcal{F}_t = \mathcal{H}_t$, and it follows that M is a martingale. Since $Z_t = M_t + \int_0^1 \frac{Z_1 - Z_s}{1-s} ds$ we have that Z is a semimartingale.

Next suppose only that $E\{|Z_t|\} < \infty$, $t \geq 0$ instead of Z being in L^2 as we assumed earlier. We define

$$J_t^1 = \sum_{0 < s \leq t} \Delta Z_s 1_{\{\Delta Z_s > 1\}} \quad \text{and} \quad J_t^2 = \sum_{0 < s \leq t} \Delta Z_s 1_{\{\Delta Z_s < -1\}}.$$

(Since Z has càdlàg paths a.s. for each ω there are at most a finite number of jumps bigger than a fixed size on each compact time set; hence each J_t^i is finite a.s.) By the results of Chap. I on Lévy processes we have that $Y_t = Z_t - J_t^1 + J_t^2$

is a Lévy process with bounded jumps, and hence it is square integrable. Additionally, the processes Y, J^1, and J^2 are jointly independent.

Combining these facts with the preceding proof yields

$$N_t = Y_t - \int_0^t \frac{Y_1 - Y_s}{1 - s} ds$$

is an $\mathbb{H}'$ martingale, where $\mathbb{H}'$ is the smallest right continuous filtration obtained by expanding $\mathbb{F}$ with (Y_1, J_1^1, J_1^2). Note that $\mathbb{H} \subset \mathbb{H}'$. The same argument, plus the observation that since J^i does not jump at time $t = 1$ and is therefore constant in a (random) neighborhood of 1, which in turn yields that $\int_0^1 \frac{|J_1^i - J_s^i|}{1-s} ds < \infty$ a.s., shows that $(J_t^i - \int_0^t \frac{J_1^i - J_s^i}{1-s} ds)_{t \geq 0}$ is a local martingale for $\mathbb{H}'$. Moreover it is a martingale for $\mathbb{H}'$ as soon as $E\{\int_0^1 \frac{|J_1^i - J_s^i|}{1-s} ds\} < \infty$. But the function $t \mapsto E\{J_t^i\} = a_i t$ for all i by the stationarity of the increments. Hence

$$E\{\int_0^1 \frac{|J_1^i - J_s^i|}{1 - s} ds\} = |E\{\int_0^1 \frac{J_1^i - J_s^i}{1 - s} ds\}| = |\int_0^1 E\{\frac{J_1^i - J_s^i}{1 - s}\} ds|$$

$$= |a_i| \int_0^1 \frac{1 - s}{1 - s} ds$$

$$= |a_i| < \infty.$$

Since Y, J^1, and J^2 are all independent, we conclude that M is an $\mathbb{H}'$ martingale. Since M is adapted to $\mathbb{H}$, by Stricker's Theorem it is also an $\mathbb{H}$ martingale, and thus Z is an $\mathbb{H}$ semimartingale.

Finally we drop all integrability assumptions on Z. We let

$$J_t^1 = \sum_{0 < s \leq t} \Delta Z_s 1_{\{|\Delta Z_s| > 1\}} \quad \text{and also} \quad X_t = Z_t - J_t^1.$$

Then X is also a Lévy process, and since X has bounded jumps it is in L^p for all $p \geq 1$, and in particular $E\{X_t^2\} < \infty$, each $t \geq 0$. Let $\mathbb{H}(X_1)$ denote $\mathbb{F}$ expanded by the adjunction of the random variable X_1. Let $\mathbb{K} = \mathbb{H}(X_1) \vee \mathbb{H}(J_1^1)$, the filtration generated by $\mathbb{H}(X_1)$ and $\mathbb{H}(J_1^1)$. Then $\mathbb{H}(Z_1) \subset \mathbb{K}$. But X is a semimartingale on $\mathbb{H}(X_1)$, and since J^1 is independent of X we have by Theorem 2 that X is a $(\mathbb{K}, P)$ semimartingale. Therefore by Stricker's Theorem, X is an $(\mathbb{H}, P)$ semimartingale; and since J_t^1 is a finite variation process adapted to $\mathbb{H}$ we have that Z is an $(\mathbb{H}, P)$ semimartingale as well. $\square$

The most important example of the above theorem is that of Brownian motion, which was Itô's original formula. In this case let $\mathbb{H} = \mathbb{H}(B_1)$, and we have as a special case the $\mathbb{H}$ decomposition of Brownian motion:

$$B_t = (B_t - \int_0^t \frac{B_1 - B_s}{1 - s} ds) + \int_0^t \frac{B_1 - B_s}{1 - s} ds = \beta_t + \int_0^t \frac{B_1 - B_s}{1 - s} ds.$$

Note that the martingale β in the decomposition is continuous and has $[\beta, \beta]_t = t$, which by Lévy's theorem gives that β is also a Brownian motion. A simple calculation gives Itô's original formula, for a process H which is $\mathbb{F}$ predictable:

$$B_1 \int_0^t H_s dB_s = B_1 \int_0^t H_s d\beta_s + B_1 \int_0^t H_s \frac{B_1 - B_s}{1 - s} ds$$

$$= \int_0^t B_1 H_s d\beta_s + \int_0^t B_1 H_s \frac{B_1 - B_s}{1 - s} ds$$

$$= \int_0^t B_1 H_s dB_s$$

where since the random variable B_1 is $\mathcal{H}_0$ measurable, it can be moved inside the stochastic integral. We can extend this theorem with a simple iteration; we omit the fairly obvious proof.

Corollary. Let Z be a given Lévy process with respect to a filtration $\mathbb{F}$, and let $0 = t_0 < t_1 < \cdots < t_n < \infty$. Let $\mathbb{H}$ denote the smallest filtration satisfying the usual hypotheses containing $\mathbb{F}$ and such that the random variables $Z_{t_1}, \ldots, Z_{t_n}$ are all $\mathcal{H}_0$ measurable. Then Z is an $\mathbb{H}$ semimartingale. If we have a countable sequence $0 = t_0 < t_1 < \cdots < t_n < \cdots$, we let $\tau = \sup_n t_n$, with $\mathbb{H}$ the corresponding filtration. Then Z is an $\mathbb{H}$ semimartingale on $[0, \tau)$.

We next give a general criterion (Theorem 5) to have a local martingale remain a semimartingale in an expanded filtration. (Note that a finite variation process automatically remains one in the larger filtration, so the whole issue is what happens to the local martingales.) We then combine this theorem with a lemma due to Jeulin to show how one can expand the Brownian filtration. Before we begin let us recall that a process X is *locally integrable* if there exist a sequence of stopping times $(T_n)_{n \geq 1}$ increasing to ∞ a.s. such that $E\{|X_{T_n} 1_{\{T_n > 0\}}|\} < \infty$ for each n. Of course, if $X_0 = 0$ this reduces to the condition $E\{|X_{T_n}|\} < \infty$ for each n.

Theorem 4. *Let M be an $\mathbb{F}$ local martingale and suppose M is a semimartingale in an expanded filtration $\mathbb{H}$. Then M is a special semimartingale in $\mathbb{H}$.*

Proof. First recall that any local martingale is a special semimartingale. In particular the process $M_t^* = \sup_{s \leq t} |M_s|$ is locally integrable (see Theorem 37 of Chap. III), and this of course remains locally integrable in the expanded filtration $\mathbb{H}$, since stopping times remain stopping times in an expanded filtration. Since M is an $\mathbb{H}$ semimartingale by hypothesis, it is special because M_t^* is locally integrable (see Theorem 38 of Chap. III). $\qquad\square$

Theorem 5. *Let M be an $\mathbb{F}$ local martingale, and let H be predictable such that $\int_0^t H_s^2 d[M, M]_s$ is locally integrable. Suppose $\mathbb{H}$ is an expansion of $\mathbb{F}$ such that M is an $\mathbb{H}$ semimartingale. Then M is a special semimartingale in $\mathbb{H}$*

and let $M = N + A$ denote its canonical decomposition. The stochastic integral process $(\int_0^t H_s dM_s)_{t \geq 0}$ is an $\mathbb{H}$ semimartingale if and only if the process $(\int_0^t H_s dA_s)_{t \geq 0}$ exists as a path-by-path Lebesgue-Stieltjes integral a.s.

Proof. First assume that $E\{\int_0^\infty H_s^2 d[M, M]_s\} < \infty$, which implies that $H \cdot M$ (where $H \cdot M$ denotes the stochastic integral process $(\int_0^t H_s dM_s)_{t \geq 0}$) is a square integrable martingale, and hence by Theorem 4 it is a special semimartingale in the $\mathbb{H}$ filtration. Let $M = N + A$ be the canonical $\mathbb{H}$ decomposition of M, and it follows that $H \cdot M = H \cdot N + H \cdot A$ is the canonical $\mathbb{H}$ decomposition of $H \cdot M$. By the lemma following Theorem 23 of Chap. IV we have $E\{\int_0^\infty H_s^2 d[N, N]_s\} \leq E\{\int_0^\infty H_s^2 d[M, M]_s\} < \infty$ and $E\{\int_0^\infty H_s^2 d[A, A]_s\} \leq E\{\int_0^\infty H_s^2 d[M, M]_s\} < \infty$. This allows us to conclude that H is $(\mathcal{H}^2, M)$ integrable, calculated in the $\mathbb{H}$ filtration.

To remove the assumption $E\{\int_0^\infty H_s^2 d[M, M]_s\} < \infty$, we only need to recall that $H \cdot M$ is assumed to be locally square integrable, and thus take M stopped at a stopping time T_n that makes $H \cdot M$ square integrable, and we are reduced to the previous case. $\square$

Theorem 6 (Jeulin's Lemma). *Let R be a positive, measurable stochastic process. Suppose for almost all s, R_s is independent of $\mathcal{F}_s$; and the law (or distribution) of R_s is μ which is independent of s, with $\mu(\{0\}) = 0$ and $\int_0^\infty x\mu(dx) < \infty$. If a is a positive predictable process with $\int_0^t a_s ds < \infty$ a.s. for each t, then the two sets below are equal almost surely:*

$$\{\int_0^\infty R_s a_s ds < \infty\} = \{\int_0^\infty a_s ds < \infty\} \quad a.s.$$

Proof. We first show that $\{\int_0^\infty R_s a_s ds < \infty\} \subset \{\int_0^\infty a_s ds < \infty\}$ a.s. Let A be an event with $P(A) > 0$, and let $J = 1_A$, $J_t = E\{J | \mathcal{F}_t\}$, the càdlàg version of the martingale. Let $j = \inf_t J_t$. Then $j > 0$ on $\{J = 1\}$. We have a.s.

$$E\{1_A R_t | \mathcal{F}_t\} = \int_0^\infty E\{1_A 1_{\{R_t > u\}} | \mathcal{F}_t\} du. \tag{$*$}$$

Consider next

$$E\{1_A 1_{\{R_t > u\}} | \mathcal{F}_t\} = E\{(1_A - 1_{\{R_t \leq u\}})^+ | \mathcal{F}_t\} \geq (E\{(1_A - 1_{\{R_t \leq u\}}) | \mathcal{F}_t\})^+$$

by Jensen's inequality, and this is equal to $(E\{1_A | \mathcal{F}_t\} - \mu(0, u])^+$, where μ is the law of R_t. Continuing with $(*)$ we have

$$E\{1_A R_t | \mathcal{F}_t\} \geq \int_0^\infty (E\{1_A | \mathcal{F}_t\} - \mu(0, u])^+ du = \Phi(E\{1_A | \mathcal{F}_t\}) = \Phi(J_t),$$

where $\Phi(x) = \int_0^\infty (x - \mu(0, u])^+ du$. Note that Φ is increasing and continuous on $[0, 1]$, and $\Phi > 0$ on $(0, 1]$ because $\mu(\{0\}) = 0$.

Choose $A_n = \{\int_0^\infty R_s a_s ds \leq n\}$ for the event A in the foregoing. Then

$$\infty > nP(A_n) \geq E\{1_{A_n} \int_0^\infty R_s a_s ds\} = E\{\int_0^\infty 1_{A_n} R_s a_s ds\}$$

$$= E\{\int_0^\infty E\{1_{A_n} R_s | \mathcal{F}_s\} a_s ds\} \geq E\{\int_0^\infty \Phi(J_s) a_s ds\},$$

which implies that $\int_0^\infty \Phi(J_s) a_s ds < \infty$ a.s. But

$$\infty > \int_0^\infty \Phi(J_s) a_s ds \geq \Phi(j) \int_0^\infty a_s ds,$$

and therefore $\int_0^\infty a_s ds < \infty$ a.s. on $\{J = 1\}$. That is, $\int_0^\infty a_s ds < \infty$ a.s. on A_n. Letting n tend to ∞ through the integers gives $\{\int_0^\infty R_s a_s ds < \infty\} \subset \{\int_0^\infty a_s ds < \infty\}$ a.s. We next show the inverse inclusion. Let $T_n = \inf\{t > 0 : \int_0^t a_s ds > n\}$. Then T_n is a stopping time, and $n \geq E\{\int_0^{T_n} a_s ds\}$. Moreover

$$E\{\int_0^{T_n} R_s a_s ds\} = E\{\int_0^{T_n} E\{R_s | \mathcal{F}_{s \wedge T_n}\} a_s ds\}$$

$$= E\{\int_0^{T_n} E\{E\{R_s | \mathcal{F}_s\} | \mathcal{F}_{s \wedge T_n}\} a_s ds\} = E\{\int_0^{T_n} E\{R_s\} a_s ds\}$$

$$= \alpha E\{\int_0^{T_n} a_s ds\} \leq \alpha n < \infty$$

where α is the expectation of R_s, which is finite and constant by hypothesis. Therefore $\int_0^{T_n} R_s a_s ds < \infty$ a.s. Let ω be in $\{\int_0^\infty a_s ds < \infty\}$. Then there exists an n (depending on ω) such that $T_n(\omega) = \infty$. Therefore we have the inclusion $\{\int_0^\infty R_s a_s ds < \infty\} \supset \{\int_0^\infty a_s ds < \infty\}$ a.s. and the proof is complete.   □

We are now ready to study the Brownian case. The next theorem gives the main result.

Theorem 7. *Let M be a local martingale defined on the standard space of canonical Brownian motion. Let $\mathbb{H}$ be the minimal expanded filtration containing B_1 and satisfying the usual hypotheses. Then M is an $\mathbb{H}$ semimartingale if and only if the integral $\int_0^1 \frac{1}{\sqrt{1-s}} |d[M,B]_s| < \infty$ a.s. In this case $M_t - \int_0^{t \wedge 1} \frac{B_1 - B_s}{1-s} d[M,B]_s$ is an $\mathbb{H}$ local martingale.*

Proof. By the Martingale Representation Theorem we have that every $\mathbb{F}$ local martingale M has a representation $M_t = M_0 + \int_0^t H_s dB_s$, where H is predictable and $\int_0^t H_s^2 ds < \infty$ a.s., each $t > 0$. By Theorem 5 we know that M is an $\mathbb{H}$ semimartingale if and only if $\int_0^t |H_s| \frac{|B_1 - B_s|}{1-s} ds$ is finite a.s., $0 \leq t \leq 1$. We take $a_s = \frac{|H_s|}{\sqrt{1-s}}$, and $R_s = 1_{\{s<1\}} \frac{|B_1 - B_s|}{\sqrt{1-s}}$. Then $\int_0^1 |H_s| \frac{|B_1 - B_s|}{1-s} ds = \int_0^1 a_s R_s ds$, which is finite only if $\int_0^1 a_s ds < \infty$ a.s. by Jeulin's Lemma. Thus it is finite only if $\int_0^1 \frac{|H_s|}{\sqrt{1-s}} ds < \infty$ a.s. But

$$\int_0^1 \frac{|H_s|}{\sqrt{1-s}}\,ds = \int_0^t \frac{1}{\sqrt{1-s}}\,|H_s|\,d[B,B]_s = \int_0^t \frac{1}{\sqrt{1-s}}\,d[|H|\cdot B,B]_s$$

$$= \int_0^t \frac{1}{\sqrt{1-s}}\,|d[H\cdot B,B]_s| = \int_0^t \frac{1}{\sqrt{1-s}}\,|d[M,B]_s|.$$

and this completes the proof. $\square$

As an example, let $1/2 < \alpha < 1$, and define

$$H_s = \frac{1}{\sqrt{1-s}}(-\ln(1-s))^{-\alpha}1_{\{\frac{1}{2}<s<1\}}.$$

Then H is trivially predictable and also $\int_0^1 H_s^2 ds < \infty$. However $\int_0^1 H_s \frac{1}{\sqrt{1-s}}ds$ is divergent. Therefore $M = H\cdot B$ is an $\mathbb{F}$ local martingale which is *not* an $\mathbb{H}$ semimartingale, by Theorem 7, where of course $\mathbb{H} = \mathbb{H}(B_1)$. Thus we conclude that not all $\mathbb{F}$ local martingales (and hence *a fortiori* not all semimartingales) remain semimartingales in the $\mathbb{H}$ filtration.

We now turn to a general criterion that allows the expansion of filtration such that all semimartingales remain semimartingales in the expanded filtration. It is due to Jacod, and it is Theorem 10. The idea is surprisingly simple: recall that for a càdlàg adapted process X to be a semimartingale, if H^n is a sequence of simple predictable processes tending uniformly in (t,ω) to zero, then we must have also that the stochastic integrals $H^n \cdot X$ tend to zero in probability. If we expand the filtration by adding a σ-algebra generated by a random variable L to the $\mathbb{F}$ filtration at time 0 (that is, $\sigma\{L\}$ is added to $\mathcal{F}_0$), then we obtain more simple predictable processes, and it is harder for X to stay a semimartingale. We will find a simple condition on the random variable L which ensures that this condition is not violated. This approach is inherently simpler than trying to show there is a new decomposition in the expanded filtration.

We assume that L is an $(\mathbb{E},\mathcal{E})$-valued random variable, where E is a *standard Borel space*[1] and $\mathcal{E}$ are its Borel sets, and we let $\mathbb{H}(L)$ denote the smallest filtration satisfying the usual hypotheses and containing both L and the original filtration $\mathbb{F}$. When there is no possibility of confusion, we will write $\mathbb{H}$ in place of $\mathbb{H}(L)$. Note that if $Y \in \mathcal{H}_t^0 = \mathcal{F}_t \vee \sigma\{L\}$, then Y can be written $Y(\omega) = G(\omega, L(\omega))$, where $(\omega, x) \mapsto G(\omega, x)$ is an $\mathcal{F}_t \otimes \mathcal{E}$ measurable function. We next recall two standard theorems from elementary probability theory.

Theorem 8. *Let X^n be a sequence of real-valued random variables. Then X^n converges to 0 in probability if and only if $\lim_{n\to\infty} E\{\min(1,|X^n|)\} = 0$.*

A proof of Theorem 8 can be found in textbooks on probability (see for example [114]).

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[1] $(\mathbb{E},\mathcal{E})$ is a *standard Borel space* if there is a set $\Gamma \in \mathcal{B}$, where $\mathcal{B}$ are the Borel subsets of $\mathbb{R}$, and an injective mapping $\phi : \mathbb{E} \to \Gamma$ such that ϕ is $\mathcal{E}$ measurable and ϕ^{-1} is $\mathcal{B}$ measurable. Note that $(\mathbb{R}^n, \mathcal{B}^n)$ are standard Borel spaces, $1 \le n \le \infty$.

We write $1 \wedge |X^n|$ for $\min(1, |X^n|)$. Also, given a random variable L, we let $Q_t(\omega, dx)$ denote the **regular conditional distribution of** L with respect to $\mathcal{F}_t$, each $t \geq 0$. That is, for any $\Lambda \in \mathcal{E}$ fixed, $Q_t(\cdot, \Lambda)$ is a version of $E\{1_{\{L \in \Lambda\}} | \mathcal{F}_t\}$, and for any fixed ω, $Q_t(\omega, dx)$ is a probability on $\mathcal{E}$. A second standard elementary result is the following.

Theorem 9. *Let L be a random variable with values in a standard Borel space. Then there exists a regular conditional distribution $Q_t(\omega, dx)$ which is a version of $E\{1_{\{L \in dx\}} | \mathcal{F}_t\}$.*

For a proof of Theorem 9 the reader can see, for example, Breiman [25, page 79].

Theorem 10 (**Jacod's Criterion**). *Let L be a random variable with values in a standard Borel space $(\mathbb{E}, \mathcal{E})$, and let $Q_t(\omega, dx)$ denote the regular conditional distribution of L given $\mathcal{F}_t$, each $t \geq 0$. Suppose that for each t there exists a positive σ-finite measure η_t on $(\mathbb{E}, \mathcal{E})$ such that $Q_t(\omega, dx) \ll \eta_t(dx)$ a.s. Then every $\mathbb{F}$ semimartingale X is also an $\mathbb{H}(L)$ semimartingale.*

Proof. Without loss of generality we assume $Q_t(\omega, dx) \ll \eta_t(dx)$ surely. Then by Doob's Theorem on the disintegration of measures there exists an $\mathcal{E} \otimes \mathcal{F}_t$ measurable function $q_t(x, \omega)$ such that $Q_t(\omega, dx) = q_t(x, \omega) \eta_t(dx)$. Moreover since $E\{\int_{\mathbb{E}} Q_t(\cdot, dx)\} = E\{E\{1_{\{Y \in \mathbb{E}\}} | \mathcal{F}_t\}\} = P(Y \in \mathbb{E}) = 1$, we have

$$E\{\int_{\mathbb{E}} Q_t(\cdot, dx)\} = E\{\int_{\mathbb{E}} q_t(x, \omega) \eta_t(dx)\} = \int_{\mathbb{E}} E\{q_t(x, \omega)\} \eta_t(dx) = 1.$$

Hence for almost all x (under $\eta_t(dx)$), we have $q_t(x, \cdot) \in L^1(dP)$.

Let X be an $\mathbb{F}$ semimartingale, and suppose that X is not an $\mathbb{H}(L)$ semimartingale. Then there must exist a $u > 0$ and an $\varepsilon > 0$, and a sequence H^n of simple predictable processes for the $\mathbb{H}$ filtration, tending uniformly to 0 but such that $\inf_n E\{1 \wedge |H^n \cdot X|\} \geq \varepsilon$. Let us suppose that $t_n \leq u$, and

$$H_t^n = \sum_{i=0}^{n-1} J_i^n 1_{(t_i, t_{i+1}]}(t)$$

with $J_i^n \in \mathcal{F}_{t_i} \vee \sigma\{L\}$. Hence J_i^n has the form $g_i(\omega, L(\omega))$, where $(\omega, x) \mapsto g_i(\omega, x)$ is $\mathcal{F}_{t_i} \otimes \mathcal{E}$ measurable. Since H^n is tending uniformly to 0, we can take without loss $|H^n| \leq 1/n$, and thus we can also assume that $|g_i| \leq 1/n$.

We write

$$H_t^{n,x}(\omega) = \sum_{i=0}^{n-1} g_i(\omega, x) 1_{(t_i, t_{i+1}]}(t),$$

and therefore $(x, \omega) \mapsto H_t^{n,x}(\omega)$ and $(x, \omega) \mapsto (H^{n,x} \cdot X)_t(\omega)$ are each $\mathcal{E} \otimes \mathcal{F}_u$ measurable, $0 \leq t \leq u$. Moreover one has clearly $H^n \cdot X = H^{n,L} \cdot X$. Combining the preceding, we have

$$E\{1 \wedge |H^n \cdot X_u|\} = E\{\int_{\mathbb{E}} (1 \wedge |H^{n,x} \cdot X_u|) Q_u(\cdot, dx)\}$$

$$= E\{\int_{\mathbb{E}} (1 \wedge |H^{n,x} \cdot X_u|) q_u(\cdot, dx) \eta_u(dx)\}$$

$$= \int_{\mathbb{E}} E\{(1 \wedge |H^{n,x} \cdot X_u|) q_u(\cdot, dx)\} \eta_u(dx)$$

where we have used Fubini's Theorem to obtain the last equality. However the function $h_n(x) = E\{(1 \wedge |H^{n,x} \cdot X_u|) q_u(\cdot, x)\} \leq E\{q_u(\cdot, x)\} \in L^1(d\eta_u)$, and since h_n is non-negative, we have

$$\lim_{n \to \infty} E\{1 \wedge |H^n \cdot X_u|\} = \lim_{n \to \infty} \int_{\mathbb{E}} E\{(1 \wedge |H^n \cdot X_u|) q_u(\cdot, x)\} \eta_u(dx)$$

$$= \int_{\mathbb{E}} \lim_{n \to \infty} E\{(1 \wedge |H^n \cdot X_u|) q_u(\cdot, x)\} \eta_u(dx) \qquad (*)$$

by Lebesgue's Dominated Convergence Theorem. However $q_u(\cdot, x) \in L^1(dP)$ for a.a. x (under $(d\eta_u)$), and if we define $dR = c q_u(\cdot, x) dP$ to be another probability, then convergence in P-probability implies convergence in R-probability, since $R \ll P$. Therefore $\lim_{n \to \infty} E_R\{(1 \wedge |H^{n,x} \cdot X_u|)\} = 0$ as well, which implies

$$0 = \lim_{n \to \infty} \frac{1}{c} E_R\{(1 \wedge |H^{n,x} \cdot X_u|)\}$$
$$= E_P\{(1 \wedge |H^{n,x} \cdot X_u|) q_u(\cdot, x)\}$$

for a.a. x (under $(d\eta_u)$) such that $q_u(\cdot, x) \in L^1(dP)$. Therefore the limit of the integrand in $(*)$ is zero for a.a. x (under $(d\eta_u)$), and we conclude

$$\lim_{n \to \infty} E\{(1 \wedge |H^n \cdot X_u|)\} = 0,$$

which is a contradiction. Hence X must be a semimartingale for the filtration $\mathbb{H}^0$, where $\mathcal{H}_t^0 = \mathcal{F}_t \vee \sigma\{L\}$. Let $X = M + A$ be a decomposition of X under $\mathbb{H}^0$. Since $\mathbb{H}^0$ need not be right continuous, the local martingale M need not be right continuous. However if we define $\tilde{M}_t = M_t$ if t is rational; and $\lim_{u \searrow t, u \in \mathbb{Q}} \tilde{M}_u$ if t is not rational; then $\tilde{M}_t$ is a right continuous martingale for the filtration $\mathbb{H}$ where $\mathcal{H}_t = \bigcap_{u > t} \mathcal{H}_u^0$. Letting $\tilde{A}_t = X_t - \tilde{M}_t$, we have that $X_t = \tilde{M}_t + \tilde{A}_t$ is an $\mathbb{H}$ decomposition of X, and thus X is an $\mathbb{H}$ semimartingale. $\qquad \square$

A simple but useful refinement of Jacod's Theorem is the following where we are able to replace the family of measures η_t be a single measure η.

Theorem 11. *Let L be a random variable with values in a standard Borel space $(\mathbb{E}, \mathcal{E})$, and let $Q_t(\omega, dx)$ denote the regular conditional distribution of L given $\mathcal{F}_t$, each $t \geq 0$. Then there exists for each t a positive σ-finite measure η_t on $(\mathbb{E}, \mathcal{E})$ such that $Q_t(\omega, dx) \ll \eta_t(dx)$ a.s. if and only if there exists **one** positive σ-finite measure $\eta(dx)$ such that $Q_t(\omega, dx) \ll \eta(dx)$ for all ω, each $t > 0$. In this case η can be taken to be the distribution of L.*

Proof. It suffices to show that the existence of η_t for each $t > 0$ implies the existence of η with the right properties; we will show that the distribution measure of L is such an η. As in the proof of Theorem 10 let $(x, \omega) \mapsto q_t(x, \omega)$ be $\mathcal{E} \otimes \mathcal{F}_t$ measurable such that $Q_t(\omega, dx) = q_t(x, \omega)\eta_t(dx)$. Let $a_t(x) = E\{q_t(x, \omega)\}$, and define

$$r_t(x, \omega) = \begin{cases} \frac{q_t(x, \omega)}{a_t(x)}, & \text{if } a_t(x) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $a_t(x) = 0$ implies $q_t(x, \cdot) = 0$ a.s. Hence, $q_t(x, \omega) = r_t(x, \omega)a_t(x)$ a.s.; whence $r_t(x, \omega)a_t(x)\eta_t(dx)$ is also a version of $Q_t(\omega, dx)$.

Let η be the law of L. Then for every positive $\mathcal{E}$ measurable function g we have

$$\int g(x)\eta(dx) = E\{g(L)\} = E\{\int_{\mathbb{E}} g(x)Q_t(\cdot, dx)\}$$
$$= E\{\int_{\mathbb{E}} g(x)q_t(x, \cdot)\eta_t(dx)\}$$
$$= \int_{\mathbb{E}} g(x)E\{q_t(x, \cdot)\}\eta_t(dx)$$
$$= \int_{\mathbb{E}} g(x)a_t(x)\eta_t(dx)$$

from which we conclude that $a_t(x)\eta_t(dx) = \eta(dx)$. Hence, $Q_t(\omega, dx) = r_t(\omega, x)\eta(dx)$, and the theorem is proved. $\qquad\square$

We are now able to re-prove some of the previous theorems, which can be seen as corollaries of Theorem 11.

Corollary 1 (**Independence**). Let L be independent of the filtration $\mathbb{F}$. Then every $\mathbb{F}$ semimartingale is also an $\mathbb{H}(L)$ semimartingale.

Proof. Since L is independent of $\mathcal{F}_t$, $E\{g(L)|\mathcal{F}_t\} = E\{g(L)\}$ for any bounded, Borel function g. Therefore

$$E\{g(L)|\mathcal{F}_t\} = \int_{\mathbb{E}} Q_t(\omega, dx)g(x) = \int_{\mathbb{E}} \eta(dx)g(x) = E\{g(L)\},$$

from which we deduce $Q_t(\omega, dx) = \eta(dx)$, and in particular $Q_t(\omega, dx) \ll \eta(dx)$ a.s., and the result follows from Theorem 11. $\qquad\square$

Corollary 2 (**Countably-valued random variables**). Let L be a random variable taking on only a countable number of values. Then every $\mathbb{F}$ semimartingale is also an $\mathbb{H}(L)$ semimartingale.

Proof. Let L take on the values $\alpha_1, \alpha_2, \alpha_3, \ldots$. The distribution of L is given by $\eta(dx) = \sum_{i=1}^{\infty} P(L = \alpha_i)\varepsilon_{\alpha_i}(dx)$, where $\varepsilon_{\alpha_i}(dx)$ denotes the point mass at

α_i. With the notation of Theorem 11, we have that the regular conditional distribution of L given $\mathcal{F}_t$, denoted $Q_t(\omega, dx)$, has density with respect to η given by

$$\sum_j \frac{P(L = \alpha_j | \mathcal{F}_t)}{P(L = \alpha_j)} 1_{\{x = \alpha_j\}}.$$

The result now follows from Theorem 11.   □

Corollary 3 (Jacod's Countable Expansion). Let $\mathcal{A} = (A_1, A_2, \dots)$ be a sequence of events such that $A_i \cap A_j = \emptyset$, $i \neq j$, all in $\mathcal{F}$, and such that $\bigcup_{i=1}^{\infty} A_i = \Omega$. Let $\mathbb{H}$ be the filtration generated by $\mathbb{F}$ and $\mathcal{A}$, and satisfying the usual hypotheses. Then every $\mathbb{F}$ semimartingale is an $\mathbb{H}$ semimartingale.

Proof. Define $L = \sum_{i=1}^{\infty} 2^{-i} 1_{A_i}$. Then $\mathbb{H} = \mathbb{H}(L)$ and we need only to apply the preceding corollary.   □

Next we consider several examples.

Example 1 (Itô's example). We first consider the original example of Itô, where in the standard Brownian case we expand the natural filtration $\mathbb{F}$ with $\sigma\{B_1\}$. We let $\mathbb{H}$ denote $\mathbb{H}(B_1)$. We have

$$E\{g(B_1)|\mathcal{F}_t\} = E\{g(B_1 - B_t + B_t)|\mathcal{F}_t\} = \int g(x + B_t)\eta_t(dx)$$

where $\eta_t(dx)$ is the law of $B_1 - B_t$ and where we have used that $B_1 - B_t$ is independent of $\mathcal{F}_t$. Note that $\eta_t(dx)$ is a Gaussian distribution with mean 0 and variance $(1-t)$ and thus has a density with respect to Lebesgue measure. Since Lebesgue measure is translation invariant this implies that $Q_t(\omega, dx) \ll dx$ a.s., each $t < 1$. However at time 1 we have $E\{g(B_1)|\mathcal{F}_1\} = g(B_1)$, which yields $Q_1(\omega, dx) = \varepsilon_{\{B_1(\omega)\}}(dx)$, which is a.s. singular with respect to Lebesgue measure. We conclude that any $\mathbb{F}$ semimartingale is also an $\mathbb{H}(B_1)$ semimartingale, for $0 \leq t < 1$, but not necessarily including 1. This agrees with Theorem 7 which implies that there exist local martingales in $\mathbb{F}$ which are not semimartingales in $\mathbb{H}(B_1)$.

Our next example shows how Jacod's criterion can be used to show a somewhat general, yet specific result on the expansion of filtrations.

Example 2 (Gaussian expansions). Let $\mathbb{F}$ again be the standard Brownian filtration satisfying the usual hypotheses, with B a standard Brownian motion. Let $V = \int_0^{\infty} g(s)dB_s$, where $\int_0^{\infty} g(s)^2 ds < \infty$, g a deterministic function. Let $a = \inf\{t > 0 : \int_t^{\infty} g(s)^2 ds = 0\}$. If h is bounded Borel, then as in the previous example

$$E\{h(V)|\mathcal{F}_t\} = E\{h(\int_0^t g(s)dB_s + \int_t^{\infty} g(s)dB_s)|\mathcal{F}_t\}$$

$$= \int h(\int_0^t g(s)dB_s + x)\,\eta_t(dx),$$

where η_t is the law of the Gaussian random variable $\int_t^\infty g(s)dB_s$. If $a = \infty$, then η_t is non-degenerate for each t, and η_t of course has a density with respect to Lebesgue measure. Since Lebesgue measure is translation invariant, we conclude that the regular conditional distribution of $Q_t(\omega, dx)$ of V given $\mathcal{F}_t$ also has a density, because

$$Q_t(\omega, h) = E\{h(V)|\mathcal{F}_t\} = \int h(\int_0^t g(s)dB_s + x)\eta_t(dx).$$

Hence by Theorem 10 we conclude that every $\mathbb{F}$ semimartingale is an $\mathbb{H}(V)$ semimartingale.

Example 3 (expansion via the end of a stochastic differential equation). Let B be a standard Brownian motion and let X be the unique solution of the stochastic differential equation

$$X_t = X_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds$$

where σ and b are Lipschitz. In addition, assume σ and b are chosen so that for h Borel and bounded,

$$E\{h(X_1)|\mathcal{F}_t\} = \int h(x)\pi(1 - t, X_t, x)dx$$

where $\pi(1 - t, u, x)$ is a deterministic function.[2] Thus $Q_t(\omega, dx) = \pi(1 - t, X_t(\omega), x)dx$, and $Q_t(\omega, dx)$ is a.s. absolutely continuous with respect to Lebesgue measure if $t < 1$. Hence if we expand the Brownian filtration $\mathbb{F}$ by initially adding X_1, we have by Theorem 10 that every $\mathbb{F}$ semimartingale is an $\mathbb{H}(X_1)$ semimartingale, for $0 \leq t < 1$.

The mirror of initial expansions is that of **filtration shrinkage**. This has not been studied to any serious extent. We include one result (Theorem 12 below), which can be thought of as a strengthening of Stricker's Theorem, from Chap. II. Recall that if X is a semimartingale for a filtration $\mathbb{H}$, then it is also a semimartingale for any subfiltration $\mathbb{G}$, provided X is adapted to $\mathbb{G}$, by Stricker's Theorem. But what if a subfiltration $\mathbb{F}$ is so small that X is not adapted to it? This is the problem we address. We will deal with the optional projection Z of X onto $\mathbb{F}$.

Definition. Let $H = (H_t)_{t \geq 0}$ be a bounded measurable process. It can be shown that there exists a unique optional process ^oH, also bounded, such that for any stopping time T one has

$$E\{H_T 1_{\{T < \infty\}}\} = E\{^oH_T 1_{\{T < \infty\}}\}.$$

The process ^oH is called **the optional projection of H**.

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[2] Sufficient conditions are known for this to be true. These conditions involve X_0 having a nice density, and requirements on the differentiability of the coefficients. See, for example, [206] and [209].

We remark that $^oH_t = E\{H_t|\mathcal{F}_t\}$ a.s. for each fixed time t, but the null set depends on t. Therefore were we simply to write as a process $(E\{H_t|\mathcal{F}_t\})_{t\geq0}$, instead of ^oH, it would not be uniquely determined almost surely. This is why we use the optional projection. The uniqueness follows from Meyer's section theorems, which are not treated in this book, so we ask the reader to accept it on faith.

Definition. Let $H = (H_t)_{t\geq0}$ be a bounded measurable process. It can be shown that there exists a predictable process ^pH, also bounded, such that for any predictable stopping time T one has

$$E\{H_T 1_{\{T<\infty\}}\} = E\{^pH_T 1_{\{T<\infty\}}\}.$$

The process ^pH is called **the predictable projection of** H.

It follows that for the optional projection, for each stopping time T we have
$$^oH_T = E\{H_T|\mathcal{F}_T\} \text{ a.s. on } \{T < \infty\}$$
whereas for the predictable projection we have that

$$^pH_T = E\{H_T|\mathcal{F}_{T-}\} \text{ a.s. on } \{T < \infty\}$$

for any *predictable* stopping time T. We can explicitly calculate the optional and predictable projections in a special case. Let K be a bounded random variable, and define $M_t = E\{K|\mathcal{F}_t\}$, the a.s. right continuous (càdlàg) version of the martingale. Let $H_t = h(t)K$, where h is Borel measurable and non-random. Then of course H is not adapted, but it is simple to check that $^oH_t = h(t)M_t$ and $^pH_t = h(t)M_{t-}$. We begin our treatment with two simple lemmas.

Lemma. Let $\mathbb{F} \subset \mathbb{G}$, and let X be a $\mathbb{G}$ martingale but not necessarily adapted to $\mathbb{F}$. Let Z denote the optional projection of X onto $\mathbb{F}$. Then Z is an $\mathbb{F}$ martingale.

Proof. Let $s < t$. Then

$$Z_s = E\{X_s|\mathcal{F}_s\} = E\{E\{X_t|\mathcal{G}_s\}|\mathcal{F}_s\} = E\{E\{X_t|\mathcal{G}_t\}|\mathcal{F}_s\} = E\{Z_t|\mathcal{F}_s\}$$

and the result follows. $\qquad\square$

The Azéma martingale, a projection of Brownian motion onto a subfiltration to which it is not adapted, is an example of the above lemma. The projection of an increasing process, however, is a submartingale but not an increasing process.

Lemma. Let $\mathbb{F} \subset \mathbb{G}$, and let X be a $\mathbb{G}$ supermartingale but not necessarily adapted to $\mathbb{F}$. Let Z denote the optional projection of X onto $\mathbb{F}$. Then Z is an $\mathbb{F}$ supermartingale.

Proof. Let $s < t$. Then

$$Z_s = E\{X_s|\mathcal{F}_s\} \geq E\{E\{X_t|\mathcal{G}_s\}|\mathcal{F}_s\} = E\{X_t|\mathcal{F}_s\} = E\{E\{X_t|\mathcal{F}_t\}|\mathcal{F}_s\}$$
$$= E\{Z_t|\mathcal{F}_s\},$$

and the result follows.   □

The limitation of the two preceding lemmas is the need to require integrability of the random variables X_t for each $t \geq 0$. We can weaken this condition by a localization procedure.

Definition. We say that a $\mathbb{G}$ semimartingale X starting at 0 is an $\mathbb{F}$ **special, $\mathbb{G}$ semimartingale** if there is a sequence $(T_n)_{n \geq 1}$ of $\mathbb{F}$ stopping times increasing a.s. to ∞, and such that the stopped processes X^{T_n} can be written in the form $X^{T_n} = M^n + A^n$ where M^n is a $\mathbb{G}$ martingale with $M_0^n = 0$ and where A^n has integrable variation over each $[0, t]$, each $t > 0$, and with $A_0 = 0$.

Theorem 12 (Filtration Shrinkage). *Let $\mathbb{G}$ be a given filtration and let $\mathbb{F}$ be a subfiltration of $\mathbb{G}$. Let X be an $\mathbb{F}$ special, $\mathbb{G}$ semimartingale. Then the $\mathbb{F}$ optional projection of X, called Z, exists, and it is a special semimartingale for the $\mathbb{F}$ filtration.*

Proof. Without loss of generality we can assume $T_n \leq n$ for each $n \geq 1$. We set $T_0 = 0$ and let $X^n = X^{T_n} - X^{T_{n-1}}$, and $N^n = M^{T_n} - M^{T_{n-1}}$ with $N^0 = 0$. For each n there are two increasing processes C^n and D^n, each starting at 0, with $X^n = N^n + C^n - D^n$, and moreover we can choose this decomposition such that the following holds:

$$E\{C_\infty^n\} + E\{D_\infty^n\} + E\{\sup_t |N_t|\} < \infty,$$

and where $t \leq T_{n-1}$ implies $C_t^n = D_t^n = N_t^n = 0$, and $t \geq T_n$ implies $C_t^n - C_{T_n}^n = D_t^n - D_{T_n}^n = N_t^n - N_{T_n}^n = 0$. The integrability condition implies that the $\mathbb{F}$ optional projections of C^n, D^n, and N^n all exist and have càdlàg versions. By the previous two lemmas the optional projection of N^n is an $\mathbb{F}$ martingale, and those of C^n and D^n are $\mathbb{F}$ submartingales. Therefore letting $^oX^n$, $^oN^n$, $^oC^n$, and $^oD^n$ denote the respective $\mathbb{F}$ optional projections of X^n, N^n, C^n, and D^n, we have that $^oX^n =^o N^n +^o C^n -^o D^n$ exists and is a special $\mathbb{F}$ semimartingale.

Since T_{n-1} and T_n are $\mathbb{F}$ stopping times, we have that also the $\mathbb{F}$ optional projections $^oN^n$, $^oC^n$, and $^oD^n$ and hence $^oX^n$ are all null over the stochastic interval $[0, T_{n-1}]$ and constant over (T_n, ∞). Then $\sum_{n \geq 1} {}^oX^n$ is a càdlàg version of $^oX = Z$ and thus Z is a special $\mathbb{F}$ semimartingale.   □

3 Progressive Expansions

We consider the case where we add a random variable gradually to a filtration in order to create a minimal expanded filtration allowing it to be a stopping time. Note that if the initial filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is given by $\mathcal{F}_t = \{\emptyset, \Omega\}$ for all t, then $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ given by $\mathcal{G}_t = \sigma\{L \wedge s; s \leq t\}$ is the smallest expansion of $\mathbb{F}$ making L a stopping time. Note that $\mathcal{G}_t = \sigma\{L \wedge t\}$ as well. Let L be a strictly positive random variable. Let $\Lambda = \{(t, \omega) : t \leq L(\omega)\}$. Then $L = \sup\{t : (t, \omega) \in \Lambda\}$. In this sense every positive random variable is the end of a random set. Instead however let us begin with a random set $\Lambda \subset \mathbb{R}_+ \times \Omega$ and define L to be the end of the set Λ. That is,

$$L(\omega) = \sup\{t : (t, \omega) \in \Lambda\}$$

where we use the (unusual) convention that $\sup(\emptyset) = 0-$, where $\{0-\}$ is an extra isolated point added to the non-negative reals $[0, \infty]$ and which can be thought of as $0- < 0$. We also define $\mathcal{F}_{0-} = \mathcal{F}_0$. The purpose of $\{0-\}$ is to distinguish between the events $\{\omega : \Lambda(\omega) = \emptyset\}$ and $\{\omega : \Lambda(\omega) = \{0\}\}$, each of which could potentially be added to the expanded filtration.

The smallest filtration expanding $\mathbb{F}$ and making the random variable L a stopping time is $\mathbb{G}^0$ defined by $\mathcal{G}_t^0 = \mathcal{F}_t \vee \sigma\{L \wedge t\}$; but $\mathbb{G}^0$ is not necessarily right continuous. Thus the smallest expanded filtration making L a stopping time and satisfying the usual hypotheses is $\mathbb{G}$ given by $\mathcal{G}_t = \bigcap_{u > t} \mathcal{G}_u^0$. Nevertheless, it turns out that the mathematics is more elegant if we consider expansions slightly more rich than the minimal ones. In order to distinguish progressive expansions from initial ones, we will change the notation for the expanded filtrations. Beginning with a filtered probability space satisfying the usual hypotheses: $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where of course $\mathbb{F}$ denotes the filtration $(\mathcal{F}_t)_{t \geq 0}$, and a random variable L, we define the *expanded filtration* to be $\mathbb{F}^L$ and it is given by

$$\Gamma \in \mathcal{F}_t^L \iff \{\Gamma \in \mathcal{F} \text{ and } \exists\, \Gamma_t \in \mathcal{F}_t \,:\, \Gamma \cap \{L > t\} = \Gamma_t \cap \{L > t\}\}.$$

This filtration is easily seen to satisfy the usual hypotheses, and also it makes L into a stopping time. Thus $\mathbb{G} \subset \mathbb{F}^L$. There are two useful key properties the filtration $\mathbb{F}^L$ enjoys.

Lemma. If H is a predictable process for $\mathbb{F}^L$ then there exists a process J which is predictable for $\mathbb{F}$ such that $H = J$ on $[0, L]$. Moreover, if T is any stopping time for $\mathbb{F}^L$ then there exists an $\mathbb{F}$ stopping time S such that $S \wedge L = T \wedge L$ a.s.

Proof. Let Γ be an event in $\mathcal{F}_t^L$. Then events of the form $(t, \infty) \times \Gamma$ form a generating set for $\mathcal{P}(\mathbb{F}^L)$, the predictable sets for $\mathbb{F}^L$. Let $H_s = 1_{\{(t,\infty) \times \Gamma\}}(s)$ and then take J to be $J_s = 1_{(t,\infty) \times \Gamma_t}(s)$. The first result follows by an application of the Monotone Class Theorem. For the stopping time T, note that it suffices to take $H = 1_{[0,T]}$, and let J be the $\mathbb{F}$ predictable process guaranteed by the first half of this lemma, and take $S = \inf\{t : J_t = 0\}$. $\qquad\square$

We next define a measure μ^L on $[0, \infty] \times \Omega$ by

$$\mu^L(J) = E\{J_L 1_{\{L > 0-\}}\}$$

for any positive, measurable process J. For such a measure μ^L there exists an increasing process $1_{\{t \geq L\}}$ which is null at $0-$ but which can jump at both 0 and $+\infty$. We will denote $A^L = (A_t^L)_{t \geq 0}$, the **(predictable) compensator of $1_{\{t \geq L\}}$ for the filtration** $\mathbb{F}$. Therefore if J is an $\mathbb{F}$ predictable bounded process we have

$$E\{J_L 1_{\{L > 0-\}}\} = E\{\int_{[0,\infty]} J_s dA_s^L\}.$$

We now define what will prove to be a process fundamental to our analysis. The process Z defined below was first used in this type of analysis by J. Azéma [5]. Recall that if H is a (bounded, or integrable) $\mathbb{F}^L$ process, then its optional projection ^oH onto the filtration $\mathbb{F}$ exists. We define

$$Z_t =^o 1_{\{L > t\}}.$$

Note that $1_{\{L > t\}}$ is decreasing, hence by the lemma preceding Theorem 12 we have that Z is an $\mathbb{F}$ supermartingale. We next prove a needed technical result.

Theorem 13. *The set $\{t : 0 \leq t \leq \infty, Z_{t-} = 0\}$ is contained in the set $(L, \infty]$ and is negligible for the measure dA^L.*

Proof. Let $T(\omega) = \inf\{t \geq 0 : Z_t(\omega) = 0, \text{ or } Z_{t-}(\omega) = 0 \text{ for } t > 0\}$. Then it is a classic result for supermartingales that for almost all ω the function $t \mapsto Z_t(\omega)$ is null on $[T(\omega), \infty]$. (This result is often referred to as "a non-negative supermartingale sticks at zero.") Thus we can write $\{Z = 0\}$ as the stochastic interval $[T, \infty]$, and on $[0, T)$ we have $Z > 0$, $Z_- > 0$. We have $E\{A_\infty^L - A_T^L\} = P(T < L) = E\{Z_T 1_{\{T < \infty\}}\} = 0$, hence dA^L is carried by $[0, T]$. Note that since $d1_{\{L \geq t\}}$ is carried by the graph of L, written $[L]$, we have $L \leq T$, and hence we have $Z > 0$, $Z_- > 0$ on $[0, L)$. Next observe that the set $\{Z_- = 0\}$ is predictable, hence $0 = E\{1_{\{Z_{t-}=0\}} d1_{\{L > t\}}\} = E\{1_{\{Z_{t-}=0\}} dA_t^L\}$ and hence $\{Z_- = 0\}$ is negligible for dA^L. Note that this further implies that $P(Z_{L-} > 0) = 1$, and again that $\{Z_- = 0\} \subset (L, \infty]$. $\square$

We can now give a description of martingales for the filtration $\mathbb{F}^L$, as long as we restrict our attention to processes stopped at the time L. What happens after L is more delicate. For a bounded process J we let ^pJ denote its predictable projection, as defined on page 376. This definition can be extended to **integrable processes**, where by an integrable process we mean a process J such that $E\{|J_T|\} < \infty$ for every bounded stopping time T.

Theorem 14. *Let Y be a random variable with $E\{|Y|\} < \infty$. A right continuous version of the martingale $Y_t = E\{Y | \mathcal{F}_t^L\}$ is given by the formula*

$$Y_t = \frac{1}{Z_t} \,^o(Y1_{\{t<L\}}) + Y1_{\{t \geq L\}}.$$

Moreover the left continuous version Y_- is given by

$$Y_{t-} = \frac{1}{Z_{t-}} \,^p(Y1_{(0,L]}) + Y1_{(L,\infty)}.$$

Proof. Let $\mathcal{O}^L$ denote the optional σ-algebra on $\mathbb{R}_+ \times \Omega$, corresponding to the filtration $\mathbb{F}^L$. On $[0, L)$, $\mathcal{O}^L$ coincides with the trace of $\mathcal{O}$ on $[0, L)$. (By $\mathcal{O}$ we mean of course the optional σ-algebra on $\mathbb{R}_+ \times \Omega$ corresponding to the underlying filtration $\mathbb{F}$.) Moreover on $[L, \infty)$, $\mathcal{O}^L$ coincides with the trace of the σ-algebra $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_\infty$ on $[L, \infty)$. The analogous description of $\mathcal{P}^L$ holds, with $[0, L)$ replaced by $(0, L]$, and with $[L, \infty)$ replaced with (L, ∞). It is then simple to check that the formulas give the *bona fide* conditional expectations, and also the right continuity is easily checked on $[0, L)$ and $[L, \infty)$ separately. The second statement follows since $Z_- > 0$ on $(0, L]$ by Theorem 13.   □

We now make a simplifying assumption for the rest of this paragraph. This assumption is often satisfied in the cases of interesting examples, and it allows us to avoid having to introduce the dual optional projection of the measure $\varepsilon_L 1_{\{L>0\}}$.

Simplifying assumption to hold for the rest of this paragraph. *We assume L avoids all $\mathbb{F}$ stopping times. That is, $P(L = T) = 0$ for all $\mathbb{F}$ stopping times T.*

Definition. The martingale M^L given by $M_t^L = A_t^L + Z_t$ is called the **fundamental L martingale**.

Note that it is trivial to check that M^L is in fact a martingale, since A^L is the compensator of $1 - Z$. Note also that $M_\infty^L = A_\infty^L$, since $Z_\infty = 0$. Last, note that it is easy to check that M^L is a square integrable martingale; indeed, M^L is in BMO since for any $X \in \mathcal{H}^1$, we have $E\{X_L\} = E\{[X, M^L]_\infty\}$, and $|E\{X_L\}| \leq \|X\|_{\mathcal{H}^1}$.

Theorem 15. *Let X be a square integrable martingale for the $\mathbb{F}$ filtration. Then $(X_{t \wedge L})_{t \geq 0}$ is a semimartingale for the filtration $\mathbb{F}^L$. Moreover $X_{t \wedge L} - \int_0^{t \wedge L} \frac{1}{Z_{s-}} d\langle X, M^L \rangle_s$ is a martingale in the $\mathbb{F}^L$ filtration.*

Proof. Let C be the (non-adapted) increasing process $C_t = 1_{\{t \geq L\}}$. Since C has only one jump at time L we have $E\{X_L\} = E\{\int_0^\infty X_s dC_s\}$. Since X is a martingale it jumps only at stopping times, hence it does not jump at L, and using that A^L is predictable and hence natural we get

$$E\{X_L\} = E\{\int_0^\infty X_{s-} dC_s\} = E\{\int_0^\infty X_{s-} dA_s^L\} = E\{X_\infty A_\infty^L\}$$
$$= E\{X_\infty M_\infty^L\} = E\{[X, M^L]_\infty\} = E\{\langle X, M^L \rangle_\infty\}. \qquad (*)$$

Suppose that H is a predictable process for $\mathbb{F}^L$, and J is a predictable process for $\mathbb{F}$ which vanishes on $\{Z_- = 0\}$ and is such that $J = H$ on $(0, L]$. We are assured such a process J exists by the lemma preceding Theorem 13. Suppose first that H has the simple form $H = h1_{(t,\infty)}$ for bounded $h \in \mathcal{F}_t^L$. If j is an $\mathcal{F}_t$ random variable equal to h on $\{t < L\}$, then we can take $J = j1_{(t,\infty)}$ and we obtain $H \cdot X_\infty = h(X_L - X_t)1_{\{t<L\}}$. In this way we can define stochastic integrals for non-adapted simple processes. We have then $E\{(H \cdot X)_\infty\} = E\{(J \cdot X)_L\}$, and using our previous calculation, since $J \cdot X$ is another square integrable martingale, we get

$$E\{(H \cdot X)_\infty\} = E\{\langle J \cdot X, M^L \rangle_\infty\} = E\{\int_0^\infty J_s d\langle X, M^L \rangle_s\}.$$

Since $\langle X, M^L \rangle$ is $\mathbb{F}$ predictable, we can replace H by $\frac{H1_{(0,L]}}{Z_-}$ because it has the same predictable projection on the support of $d\langle X, M^L \rangle$. This yields

$$E\{\int_0^\infty H_s dX_s\} = E\{\int_0^L \frac{H_s}{Z_{s-}} d\langle X, M^L \rangle_s\}. \qquad (**)$$

Last if we take the bounded $\mathbb{F}^L$ predictable process H to be a stochastic interval $[0, T \wedge L]$, where T is an $\mathbb{F}^L$ stopping time, we obtain $E\{X_{T\wedge L} - \int_0^{T\wedge L} \frac{1}{Z_{s-}} d\langle X, M^L \rangle_s\} = 0$, which implies by Theorem 21 of Chap. I that $X_{T\wedge L} - \int_0^{T\wedge L} \frac{1}{Z_{s-}} d\langle X, M^L \rangle_s$ is a martingale.   □

We do not need the assumption that X is a square integrable martingale, which we made for convenience. In fact the conclusion of the theorem holds even if X is only assumed to be a local martingale. We get our main result as a corollary to Theorem 15.

Corollary. Let X be a semimartingale for the $\mathbb{F}$ filtration. Then $(X_{t\wedge L})_{t\geq 0}$ is a semimartingale for the filtration $\mathbb{F}^L$.

Proof. If X is a semimartingale then it has a decomposition $X = M + D$. The local martingale term M can be decomposed into $X = V + N$, where V and N are both local martingales, but V has paths of bounded variation on compact time sets, and N has bounded jumps. (This is the Fundamental Theorem of Local Martingales, Theorem 29 of Chap. III.) Clearly V and D remain finite variation processes in the expanded filtration $\mathbb{F}^L$, and since M has bounded jumps it is locally bounded, hence locally square integrable, and since every $\mathbb{F}$ stopping time remains a stopping time for the $\mathbb{F}^L$ filtration, the corollary follows from Theorem 15.   □

We need to add a restriction on the random variable L in order to study the evolution of semimartingales in the expanded filtration after the time L.

Definition. A random variable L is called **honest** if for every $t \leq \infty$ there exists an $\mathcal{F}_t$ measurable random variable L_t such that $L = L_t$ on $\{L \leq t\}$.

Note that in particular if L is honest then it is $\mathcal{F}_\infty$ measurable. Also, any stopping time is honest, since then we can take $L_t = L \wedge t$ which is of course $\mathcal{F}_t$ measurable by the stopping time property.

Example. Let X be a bounded càdlàg adapted processes, and let $X_t^{+*} = \sup_{s \leq t} X_s^+$ and $X^{+*} = \sup_s X_s^+$. Then $L = \inf\{s : X_s^{+*} = X^{+*}\}$, is honest because on the set $\{L \leq t\}$ one has $L = \inf\{s : X_s = X_s^{+*}\}$, which is $\mathcal{F}_t$ measurable.

Theorem 16. *L is an honest time if and only if there exists an optional set $\Lambda \subset [0, \infty] \times \Omega$ such that $L(\omega) = \sup\{t \leq \infty : (t, \omega) \in \Lambda\}$.*

This is often described verbally by saying "L is honest if it is the end of an optional set."

Proof. The end of an optional set is always an honest random variable. Indeed, on $\{L \leq t\}$, the random variable L coincides with the end of the set $\Lambda \cap ([0, t] \times \Omega)$, which is $\mathcal{F}_t$ measurable.

For the converse we suppose L is honest. Let $(L_t)_{t \geq 0}$ be an $\mathbb{F}$ adapted process such that $L = L_t$ on $\{L \leq t\}$. Since we can replace L_t with $L_t \wedge t$ we can assume without loss of generality that $L_t \leq t$. There is also no loss of generality to assume that L_t is increasing with t, since we can further replace L_t with $\sup_{s \leq t} L_s$. Last, it is also no loss, now that it is increasing, to take it right continuous. We thus have that the process $(L_t)_{t \geq 0}$ is optional for the filtration $\mathbb{F}$. Last, L is now the end of the optional set $\{(t, \omega) : L_t(\omega) = t\}$.   □

When L is honest we can give a simple and elegant description of a filtration $\mathbb{G}$ defined in the next theorem, and which is slightly larger than is $\mathbb{F}^L$.

Theorem 17. *Let L be an honest time. Define*

$$\mathcal{G}_t = \{\Gamma : \Gamma = (A \cap \{L > t\}) \cup (B \cap \{L \leq t\}) \text{ for some } A, B \in \mathcal{F}_t\}$$

Then $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ constitutes a filtration satisfying the usual hypotheses. Moreover L is a $\mathbb{G}$ stopping time. A process U is predictable for $\mathbb{G}$ if and only if it has a representation of the form

$$U = H 1_{[0, L]} + K 1_{(L, \infty]}$$

where H and K are $\mathbb{F}$ predictable processes.

Proof. Let $s < t$ and take $H \in \mathcal{G}_s$, of the form $(A \cap \{L > s\}) \cup (B \cap \{L \leq s\})$ with $A, B \in \mathcal{F}_s$. We will show that $H \in \mathcal{G}_t$, which shows that the collection $\mathbb{G}$ is filtering to the right.[3] Since L is an honest time, there must exist $D \in \mathcal{F}_t$ such that $\{L \leq s\} = D \cap \{L \leq t\}$. Therefore

$$H \cap \{L \leq t\} = [(A \cap D^c) \cup (B \cap D)] \cap \{L \leq t\},$$

---

[3] That is, if $s < t$, then $\mathcal{G}_s \subset \mathcal{G}_t$.

with $[(A \cap D^c) \cup (B \cap D)] \in \mathcal{F}_t$. The fact that each $\mathcal{G}_t$ is a σ-algebra, and also that $\mathbb{G}$ is right continuous, we leave to the reader. Note that $\{L \leq t\} \in \mathcal{G}_t$ which implies that L is a $\mathbb{G}$ stopping time, as we observed at the start of the proof of Theorem 16. For the last part of the theorem, let $U = (U_t)_{t \geq 0}$ be a càdlàg process adapted to the $\mathbb{G}$ filtration. For s rational let H_t^0 and K_t^0 be $\mathcal{F}_t$ measurable random variables such that

$$U_t = H_t^0 \ \text{ on } \ \{L > t\} \ \text{ and } \ U_t = K_t^0 \ \text{ on } \ \{L \leq t\}.$$

For $t > 0$ real set $H_t = \liminf_{s \nearrow t} H_s^0; K_t = \liminf_{s \nearrow t} K_s^0$ as s increases to t through the rationals. The processes H and J formed this way can be seen to be $\mathbb{F}$ predictable processes , and we have the desired representation:

$$U = H 1_{[0,L]} + K 1_{(L,\infty]}.$$

$\square$

Theorem 18. *Let X be a square integrable martingale for the $\mathbb{F}$ filtration. Then X is a semimartingale for the $\mathbb{G}$ filtration if L is an honest time. Moreover X has a $\mathbb{G}$ decomposition*

$$X_t = \{X_t - \int_0^{t \wedge L} \frac{1}{Z_{s-}} d\langle X, M^L \rangle_s + 1_{\{t \geq L\}} \int_L^t \frac{1}{1 - Z_{s-}} d\langle X, M^L \rangle_s\}$$
$$+ \{\int_0^{t \wedge L} \frac{1}{Z_{s-}} d\langle X, M^L \rangle_s - 1_{\{t \geq L\}} \int_L^t \frac{1}{1 - Z_{s-}} d\langle X, M^L \rangle_s\}.$$

Before beginning the proof of the theorem, we establish a lemma we will need in the proof. It is a small extension of the local behavior of the stochastic integral established in Chap. IV.

Lemma (Local behavior of the stochastic integral at random times). Let X and Y be two semimartingales and let $H \in L(X)$ and $J \in L(Y)$. Let U and V be two positive random variables with $U \leq V$. (U and V are not assumed to be stopping times.) Define

$$\Lambda = \{\omega : H_t(\omega) = J_t(\omega) \ \text{ and } \ X_t(\omega) - X_U(\omega) = Y_t(\omega) - Y_U(\omega),$$
$$\text{for all } \ t \in [U(\omega), V(\omega))\}.$$

Let $W_t = H \cdot X_t$ and $Z_t = J \cdot Y_t$. Then a.s. on Λ, $W_t(\omega) - W_U(\omega) = Z_t(\omega) - Z_U(\omega)$ for all $t \in [U(\omega), V(\omega))$.

Proof. We know by the Corollary to Theorem 26 of Chap. IV that the conclusion of the lemma is true when U and V are stopping times. Let (u, v) be rationals with $0 < u < v$, and let

$$\Lambda_{uv} = \{\omega : U(\omega) < u < v < V(\omega)\} \cap \Lambda.$$

Then also $W_t - W_u = Z_t - Z_u$, all $t \in [u, v)$, a.s. on Λ_{uv}, since u, v are fixed times and hence *a fortiori* stopping times. Next let

$$A = \left\{ \omega : \bigcap_{\substack{u,v \in \mathbb{Q}_+ \\ u < v}} \{U(\omega) < u < v < V(\omega), \text{ and } W_t(\omega) - W_u(\omega) = Z_t(\omega) - Z_u(\omega) \right.$$

$$\left. \text{for all } t \in [u, v)\} \right\}.$$

The intersection in the definition of A is countable, so null sets do not accumulate. Finally let u decrease to $U(\omega)$ and then v increase to $V(\omega)$, which gives the result.   □

Proof of Theorem 18. We first observe that without loss of generality we can assume $X_0 = 0$. Let H be a bounded $\mathbb{F}$ predictable process. We *define* stochastic integrals at the random time L by

$$\int_0^L H_s dX_s = (H \cdot X)_L, \quad \int_L^\infty H_s dX_s = (H \cdot X)_\infty - (H \cdot X)_L$$

where $H \cdot X = (H \cdot X_t)_{t \geq 0}$ is the $\mathbb{F}$ stochastic integral process. That is, we use the usual definition of the stochastic integral, sampling it at the random time L. When H is a simple predictable process, these are reasonable definitions. Moreover we know from the lemma that if H and J are both bounded $\mathbb{F}$ predictable processes and if also $H = J$ on $(L, \infty]$, then $\int_L^\infty H_s dX_s = \int_L^\infty J_s dX_s$ a.s., so the definition is well-defined.

Since X is an $\mathbb{F}$ martingale in L^2 with $X_0 = 0$, we have $E\{\int_0^\infty H_s dX_s\} = 0$. Applying the equalities $(*)$ on page 380 to $H \cdot X$ we have

$$E\{\int_0^L H_s dX_s\} = E\{\int_0^\infty H_s d\langle X, M^L\rangle_s\} = E\{\int_0^L \frac{H_s}{Z_{s-}} d\langle X, M^L\rangle_s\} \quad (\dagger)$$

where the second equality uses $(**)$ on page 381. Recalling $E\{\int_0^\infty H_s dX_s\} = 0$ and combining this with the above gives (where we have changed the name of the process H to K for clarity slightly later in the proof)

$$E\{\int_L^\infty K_s dX_s\} = -E\{\int_0^\infty K_s d\langle X, M^L\rangle_s\}$$

We next replace K with $K1_{\{Z_- < 1\}}$ to obtain

$$E\{\int_L^\infty K_s dX_s\} = -E\{\int_0^\infty K_s 1_{\{Z_{s-} < 1\}} d\langle X, M^L\rangle_s\}$$

$$= -E\{\int_L^\infty \frac{K_s}{1 - Z_{s-}} d\langle X, M^L\rangle_s\} \quad (\ddagger)$$

because the predictable projection of $\frac{1}{1-Z_{s-}} 1_{(L,\infty)}(s)$ is $1_{\{Z_{s-} < 1\}}$. Finally we combine equalities $(\dagger)$ and $(\ddagger)$, and define $U = H1_{[0,L]} + K1_{(L,\infty]}$ to get

$$E\{\int_0^L H_s dX_s + \int_L^\infty K_s dX_s\}$$
$$= E\{\int_0^\infty U_s d\{\int_0^{t\wedge L} \frac{1}{Z_{s-}} d\langle X, M^L\rangle_s - 1_{\{t\geq L\}} \int_L^t \frac{1}{1-Z_{s-}} d\langle X, M^L\rangle_s\}$$

Up to this point we have not used the hypothesis that L is an honest time. Now we do. Let us take U to be a simple $\mathbb{G}$ predictable process, and using Theorem 17 we have

$$U = H1_{[0,L]} + K1_{(L,\infty]}$$

where H and K are $\mathbb{F}$ predictable processes (not necessarily simple predictable, however). If we further take U bounded by 1 and take the supremum over all such simple predictable processes of the elementary $\mathbb{G}$ stochastic integrals $\int_0^\infty U_s dX_s$, we get that the variation of X, $\text{Var}(X)$ as defined in Chap. III, is less than or equal the expected total variation of

$$d\alpha_t = d\{\int_0^{t\wedge L} \frac{1}{Z_{s-}} d\langle X, M^L\rangle_s + 1_{\{t\geq L\}} \int_L^t \frac{1}{1-Z_{s-}} d\langle X, M^L\rangle_s\}.$$

Thus the process X is a $\mathbb{G}$ quasimartingale, and hence also a semimartingale. We also note that by the preceding we have that $E\{X_T - \alpha_T\} = 0$ for all $\mathbb{G}$ stopping times T, and thus it is a martingale for the $\mathbb{G}$ filtration. □

The next theorem is really a corollary of Theorem 18.

Theorem 19. *Let X be a semimartingale for the $\mathbb{F}$ filtration. Then X is a semimartingale for the $\mathbb{G}$ filtration if L is an honest time.*

Proof. Let $X = M + A$ be a decomposition of X in the $\mathbb{F}$ filtration, where M is a local martingale and A is a finite variation process. Since A remains a finite variation process in the larger $\mathbb{G}$ filtration, we need only concern ourselves with M. By the Fundamental Theorem of Local Martingales (Theorem 29 on page 126), we know that we can write $M = N + V$ where N is a local martingale with bounded jumps, and V is a martingale with paths of finite variation on compacts. Again, V remains a semimartingale in the $\mathbb{G}$ filtration since it is of finite variation, and further N can be locally stopped with $\mathbb{F}$ stopping times T_n tending to ∞ a.s. to be a square integrable martingale for each T_n. Then by Theorem 18 each X^{T_n} is also a $\mathbb{G}$ semimartingale. Since X is locally a $\mathbb{G}$ semimartingale, it is an actual $\mathbb{G}$ semimartingale. □

4 Time Reversal

In this section we apply the results on initial expansions of filtration to some elementary issues regarding time reversal of semimartingales, stochastic integrals, and stochastic differential equations.

Definition. Let $Y = (Y_t)_{0 \leq t \leq 1}$ be a càdlàg process. The **time reversal** of Y on $[0, 1]$ is defined to be

$$\tilde{Y}^t = \begin{cases} 0, & \text{if } t = 0, \\ Y_{(1-t)-} - Y_{1-}, & \text{if } 0 < t < 1, \\ Y_0 - Y_{1-}, & \text{if } t = 1. \end{cases}$$

Note that time as a superscript denotes reversal. Let $\mathbb{F}$ denote the forward filtration, and correspondingly let $\tilde{\mathbb{F}} = (\mathcal{F}^t)_{0 \leq t \leq 1}$ denote a backward filtration. As an example, if B is a standard Brownian motion, $\mathcal{F}_t^0 = \sigma\{B_s; s \leq t\}$, and $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{N}$, where $\mathcal{N}$ are the P-null sets of $\mathcal{F}_1$, while $\mathcal{F}^t = \mathcal{F}_0^t \vee \mathcal{N}$ is the analogous backward filtration for $\tilde{B}$. Our first two theorems are obvious, and we omit the proofs.

Theorem 20. *Let B be a standard Brownian motion on $[0, 1]$. Then the time reversal $\tilde{B}$ of B is also a Brownian motion with respect to its natural filtration.*

Theorem 21. *Let Z be a Lévy process on $[0, 1]$. Then $\tilde{Z}$ is also a Lévy process, with the same law as $-Z$, with respect to its natural filtration.*

In what follows let us assume that we are given a forward filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq 1}$ and a backward filtration $\tilde{\mathbb{G}} = (\mathcal{G}^t)_{0 \leq t \leq 1}$. We further assume both filtrations satisfy the usual hypotheses.

Definition. A càdlàg process Y is called an $(\mathbb{F}, \tilde{\mathbb{G}})$ **reversible semimartingale** if Y is an $\mathbb{F}$ semimartingale on $[0, 1]$ and $\tilde{Y}$ is a $\tilde{\mathbb{G}}$ semimartingale on $[0, 1)$.

Note the small lack of symmetry: for the time reversed process we do not include the final time 1. This is due to the occurrence of singularities at the terminal time 1, and including it as a requirement would exclude interesting examples.

Let $\tau_t = (t_0, \ldots, t_k)$ denote a partition of $[0, t]$ with $t_0 = 0$, $t_k = t$, $0 \leq t \leq 1$, and let

$$S_{\tau_t}(H, Y) = H_0 Y_0 + \sum_i (H_{t_{i+1}} - H_{t_i})(Y_{t_{i+1}} - Y_{t_i}).$$

Definition. Let H, Y be two càdlàg processes. The **quadratic covariation** of H and Y, denoted $[H, Y]$, is defined to be

$$\lim_{n \to \infty} S_{\tau_t^n}(H, Y) = [H, Y]_t$$

when this limit exists in probability as $\text{mesh}(\tau_t) \to 0$, and is such that $[H, Y]$ is càdlàg, adapted, and of finite variation a.s.

Theorem 22. *Let Y be an $(\mathbb{F}, \tilde{\mathbb{G}})$ reversible semimartingale, and let H be a càdlàg process such that $H_t \in \mathcal{F}_t$, and $H_t \in \mathcal{G}^{1-t}, 0 \leq t \leq 1$. Suppose the*

quadratic variation $[H, Y]$ exists. Then the two processes $[H, Y]$ and $X_t = \int_0^t H_{s-} dY_s$ are $(\mathbb{F}, \tilde{\mathbb{G}})$ reversible semimartingales. Moreover

$$\tilde{X}_t + \widetilde{[H, Y]}_t = \int_0^t H_{1-s} d\tilde{Y}^s.$$

Proof. Since $H_t \in \mathcal{G}^{1-t}, 0 \le t \le 1$, we have that $H_{1-s} \in \mathcal{G}^{1-(1-s)} = \mathcal{G}^s$, and thus it is adapted and left continuous, so the stochastic integral $\int_0^t H_{1-s} d\tilde{Y}^s$ is well-defined. We choose and fix t, $0 < t < 1$. Let τ be a partition of $[1 - t, 1]$, $\tau = (s_1, \ldots, s_n)$, chosen such that $\Delta H_{s_i} = 0$ and $\Delta Y_{s_i} = 0$ a.s., $i = 2, \ldots, n - 1$. Note that one can always choose the partition points $\{s_i\}$ in this manner because

$$E\{\int_0^1 1_{\{|\Delta H_s| > 0\}} 1_{\{|\Delta Y_s| > 0\}} ds\} = 0,$$

since for each ω, $s \mapsto Y_s(\omega)$ has only countably many jumps, and hence $\int_0^1 1_{\{|\Delta Y_s| > 0\}} ds = 0$ a.s. But then $\int_0^1 P(|\Delta Y_s| > 0) ds = 0$, which in turn implies that $P(|\Delta Y_s| > 0) = 0$ for almost all s in $[0, 1]$. We now define three new processes:

$$A^\tau = H_{(1-t)-} \Delta Y_{1-t} + \sum_{i=1}^{n-2} H_{s_i}(Y_{s_{i+1}} - Y_{s_i}) + H_{s_{n-1}}(Y_{1-} - Y_{s_{n-1}})$$

$$B^\tau = -\sum_{i=1}^{n-1} H_{s_{i+1}-}(Y_{s_{i+1}-} - Y_{s_i-})$$

$$C^\tau = H_{1-t} \Delta Y_{1-t} + \sum_{i=1}^{n-2} \{(H_{s_{i+1}} - H_{s_i})(Y_{s_{i+1}} - Y_{s_i})\}$$
$$+ (H_{1-} - H_{s_{n-1}})(Y_{1-} - Y_{s_{n-1}}).$$

Let τ_n be a sequence of partitions of $[0, 1]$ with $\lim_{n \to \infty} \text{mesh}(\tau_n) = 0$. Then

$$\lim_{n \to \infty} C^{\tau_n} = [H, Y]_{1-} - [H, Y]_{1-t} + \Delta H_{1-t} \Delta Y_{1-t}$$
$$= [H, Y]_{1-} - [H, Y]_{(1-t)-}$$
$$= -\widetilde{[H, Y]}_t$$

Since Y is $(\mathbb{F}, \tilde{\mathbb{G}})$ reversible by hypothesis, we know that C^τ is $\tilde{\mathbb{G}}$ adapted. Hence $\widetilde{[H, Y]}$ is $\tilde{\mathbb{G}}$ adapted and moreover since it has paths of finite variation by hypothesis, it is a semimartingale.

Since H is càdlàg we can approximate the stochastic integral with partial sums, and thus

$$\lim_{n \to \infty} A^{\tau_n} = \int_{[1-t, 1)} H_{s-} dY_s = X_{1-} - X_{(1-t)-} = -\tilde{X}^t.$$

Since $Y_{s_{i+1}-} - Y_{s_i-} = -(\tilde{Y}^{1-s_i} - \tilde{Y}^{1-s_{i+1}})$, we have

$$\lim_{n\to\infty} B^{\tau_n} = \int_0^t H_{1-s} d\tilde{Y}^s.$$

Combining A^{τ_n}, B^{τ_n}, and C^{τ_n}, we get

$$A^{\tau_n} + B^{\tau_n} + C^{\tau_n}$$

$$= H_{1-t}\Delta Y_{1-t} + \sum_{i=1}^{n-2} H_{s_{i+1}}(Y_{s_{i+1}} - Y_{s_i}) + H_{1-}(Y_{1-} - Y_{s_{n-1}})$$

$$- \sum_{i=1}^{n-1} H_{s_{i+1}}(Y_{s_{i+1}-} - Y_{s_i-})$$

$$= H_{1-t}\Delta Y_{1-t} + \sum_{i=1}^{n-2} H_{s_{i+1}}(\Delta Y_{s_{i+1}} - \Delta Y_{s_i}) - \Delta H_1(Y_{1-} - Y_{s_{n-1}}) - H_1 \Delta Y_{s_{n-1}}.$$

Since we chose our partitions τ_n with the property that $\Delta Y_{s_i} = 0$ a.s. for each partition point s_i, the above simplifies to

$$A^{\tau_n} + B^{\tau_n} + C^{\tau_n} = (H_{1-t} - H_{s_i})\Delta Y_{1-t} + \Delta H_1(Y_{s_{n-1}} - Y_{1-}),$$

which tends to 0 since s_1 decreases to $1-t$, and s_{n-1} increases to 1. Therefore

$$0 = \lim_{n\to\infty} A^{\tau_n} + \lim_{n\to\infty} B^{\tau_n} + \lim_{n\to\infty} C^{\tau_n} = -\tilde{X}^t + \int_0^t H_{1-s} d\tilde{Y}^s - \widetilde{[H,Y]}_t.$$

This establishes the desired formula, and since we have already seen that $\widetilde{[H,Y]}_t$ is a semimartingale, we have that X is also a reversible semimartingale as a consequence of the formula.   □

We remark that in the case where $\tilde{Y}$ is a $\tilde{\mathbb{G}}$ semimartingale on $[0,1]$ (and not only on $[0,1)$), the same proof shows that $\tilde{X}$ is a $\tilde{\mathbb{G}}$ semimartingale on $[0,1]$.

Example 1 (reversibility for Lévy process integrals). Let Z be a Lévy process on $[0,1]$, and let $\tilde{\mathbb{H}} = (\tilde{\mathcal{H}}_t)_{0\le t\le 1}$ be the filtration generated by $\tilde{Z}$ and expanded by the addition of $\tilde{Z}^1 = Z_0 - Z_{1-}$. Then $\tilde{Z}$ is an $\tilde{\mathbb{H}}$ semimartingale on $[0,1]$ by Theorem 21. Let $H_t = f(Z_t)$ with f such that the quadratic covariation $[H,Z]$ exists. Clearly $H_t = f(Z_t) \in \mathcal{F}_t$. Since

$$\mathcal{H}^{1-t} \supseteq \sigma\{Z_{(1-u)-} - Z_{(1-v)-};\ 0 \le u, v \le 1\} \vee \sigma\{Z_1\},$$

we have that $Z_t = Z_{1-(1-t)} \in \mathcal{H}^{1-t}$, since $Z_t = Z_{t-}$ a.s. Therefore $H_t \in \mathcal{H}^{1-t}$. Using Theorem 22 we conclude that $X_t = \int_0^t f(Z_{s-}) dZ_s$ is an $(\mathbb{F}, \tilde{\mathbb{H}})$ reversible semimartingale.

Example 2 (reversibility for Brownian integrals). Let B be a standard Brownian motion on $[0, 1]$. Let L_t^x denote Brownian local time at the level x and let μ be a signed measure on $\mathbb{R}$. Let $\mathbb{F}$ be the minimal filtration of B satisfying the usual hypotheses, and let $(\tilde{\mathbb{H}}^t)_{0 \leq t \leq 1}$ be the minimal filtration generated by $\tilde{B}$, expanded by adding $\tilde{B}^1 = -B_1$, and satisfying the usual hypotheses. Finally let f be a càdlàg function on $\mathbb{R}$ of finite variation on compacts with primitive F (that is, $F'_+(x) = f(x)$), and define

$$U_t = \int_0^t f(B_s) dB_s + \int_{\mathbb{R}} L_t^x \mu(dx).$$

We will show U is an $(\mathbb{F}, \tilde{\mathbb{H}})$ reversible semimartingale.

The hypotheses on f imply that its primitive F is the difference of two convex functions, and therefore $M_t = F(B_t) - F(B_0)$ is an $\mathbb{F}$ semimartingale. Moreover $\tilde{M}^t = F(B_{1-t}) - F(B_1)$, and since we already know that B_{1-t} is an $\tilde{\mathbb{H}}$ semimartingale by Theorem 3, by the convexity of F we conclude that $\tilde{M}$ is also an $\tilde{\mathbb{H}}$ semimartingale. Therefore $\int_0^t f(B_s) dB_s$ is an $(\mathbb{F}, \tilde{\mathbb{H}})$ reversible semimartingale as soon as $\frac{1}{2} \int_{\mathbb{R}} L_t^a \eta(da)$ is one, where η is the signed measure 'second derivative' of F. Finally, what we want to show is that $A_t = \frac{1}{2} \int_{\mathbb{R}} L_t^a \mu(da)$ is an $(\mathbb{F}, \tilde{\mathbb{H}})$ reversible semimartingale, for any signed measure μ. This will imply that U is an $(\mathbb{F}, \tilde{\mathbb{H}})$ reversible semimartingale.

Since A is continuous with paths of finite variation on $[0, 1]$, all we really need to show is that $\tilde{A}^t \in \mathcal{H}^t$, each t. But $\tilde{A}^t = A_{1-t} - A_1 = \int_{\mathbb{R}} (L_{1-t}^a - L_1^a) \mu(da)$, and

$$L_{1-t}^a - L_1^a = \lim_{\varepsilon \to 0} \left(-\frac{1}{\varepsilon} \int_{1-t}^1 1_{[a, a+\varepsilon)}(B_s) ds \right)$$

$$= \lim_{\varepsilon \to 0} \left(-\frac{1}{\varepsilon} \int_{1-t}^1 1_{[a-B_1, a-B_1+\varepsilon)}(B_s - B_1) ds \right)$$

$$= \lim_{\varepsilon \to 0} \left(-\frac{1}{\varepsilon} \int_{1-t}^1 1_{[a-B_1, a-B_1+\varepsilon)}(\tilde{B}^u) du \right)$$

$$= -\Lambda_t^{a-B_1},$$

where Λ_t^x is the local time at level x of the standard Brownian motion $\tilde{B}$. Therefore $\tilde{A}^t = - \int_{\mathbb{R}} (\Lambda_t^{a-B_1}) \mu(da)$. Since Λ_t^x is $\tilde{\mathbb{H}}$ adapted, and since $B_1 \in \mathcal{H}^0$, we conclude $\tilde{A}$ is $\tilde{\mathbb{H}}$ adapted, and therefore U is an $(\mathbb{F}, \tilde{\mathbb{H}})$ reversible semimartingale.

We next consider the time reversal of stochastic differential equations, which is perhaps more interesting than the previous two examples. Let B be a standard Brownian motion and let X be the unique solution of the stochastic differential equation

$$X_t = X_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds$$

where σ and b are Lipschitz, and moreover, σ and b are chosen so that for h Borel and bounded,

$$E\{h(X_1)|\mathcal{F}_t\} = \int h(x)\pi(1-t, X_t, x)dx$$

where $\pi(1-t, u, x)$ is a deterministic function. As in the example *expansion via the end of a stochastic differential equation* treated earlier on page 375, we know (as a consequence of Theorem 10) that if $\mathbb{F}$ is the natural, completed Brownian filtration, we can expand $\mathbb{F}$ with X_1 to get $\mathbb{H}: \mathcal{H}_t = \bigcap_{u>t}\mathcal{F}_u \vee \sigma\{X_1\}$, and then all $\mathbb{F}$ semimartingales remain $\mathbb{H}$ semimartingales on $[0,1)$.

We fix (t, ω) and define $\phi: \mathbb{R} \to \mathbb{R}$ by $\phi(x) = X(t, \omega, x)$ where $X_t = X(t, \omega, x)$ is the unique solution of

$$X_t = x + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds$$

for $x \in \mathbb{R}$. Recall from Chap. V that ϕ is called the **flow of the solution of the stochastic differential equation**. Again we have seen in Chap. V that under our hypotheses on σ and b the flow ϕ is injective. For $0 < s < t < 1$ define the function $\phi_{s,t}$ to be the flow of the equation

$$X_{s,t} = x + \int_s^t \sigma(X_{s,u})dB_u + \int_s^t b(X_{s,u})du. \tag{$*$}$$

It now follows from the uniqueness of solutions that $X_t = \phi_{s,t}(X_s)$, and in particular $X_1 = \phi_{t,1}(X_t)$, and therefore $\phi_{t,1}^{-1}(X_1) = X_t$, where $\phi_{t,1}^{-1}$ is of course the inverse function of $\phi_{s,t}$. Since the solution $X_{s,t}$ of equation $(*)$ is $\mathcal{G} = \sigma\{B_v - B_u; \ s \leq u, v \leq t\}$ measurable, we have $\phi_{t,1} \in \mathcal{F}^{1-t}$, where $\mathcal{F}^t = \sigma\{\tilde{B}^s; 0 \leq s \leq t\} \vee \mathcal{N}$ where $\mathcal{N}$ are the null sets of $\mathcal{F}$. Let $\mathcal{H}^t = \bigcap_{u>t}\mathcal{F}^u \vee \sigma\{X_1\}$ and we have $\tilde{B}$ is an $\tilde{\mathbb{H}} = (\mathcal{H}^t)_{0 \leq t \leq 1}$ semimartingale, and that $X_t = \phi_{t,1}^{-1}(X_1)$ is $\mathcal{H}^{1-t}$ measurable. Therefore $X_t \in \mathcal{F}_t$ and also at the same time $X_t \in \mathcal{H}^{1-t}$. Finally note that since X is a semimartingale and σ is $\mathcal{C}^1$, the quadratic covariation $[\sigma(X), B]$ exists and is of finite variation, and we are in a position to apply Theorem 22.

Theorem 23. *Let B be a standard Brownian motion and let X be the unique solution of the stochastic differential equation*

$$X_t = X_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds$$

for $0 \leq t \leq 1$, where σ and b are Lipschitz, and moreover, σ and b are chosen so that for h Borel and bounded,

$$E\{h(X_1)|\mathcal{F}_t\} = \int h(x)\pi(1-t, X_t, x)dx$$

where $\pi(1 - t, u, x)$ is a deterministic function. Let $\mathbb{H}$ be given by $\mathcal{H}_t = \bigcap_{u>t} \mathcal{F}_u \vee \sigma\{X_1\}$. Then B is an $(\mathbb{F}, \tilde{\mathbb{H}})$ reversible semimartingale, and $\hat{X}^t = X_{1-t}$ satisfies the following backward stochastic differential equation

$$Y_t = X_1 + \int_0^t \sigma(Y_s)d\tilde{B}^s + \int_0^t \sigma'(Y_s)\sigma(Y_s)ds + \int_0^t b(Y_s)ds,$$

for $0 \le t \le 1$. In particular, X is an $(\mathbb{F}, \tilde{\mathbb{H}})$ reversible semimartingale.

Proof. We first note that B is an $(\mathbb{F}, \tilde{\mathbb{H}})$ reversible semimartingale as we saw in the example on page 375. We have that $[\sigma(X), B] = \int_0^t \sigma'(X_s)\sigma(X_s)ds$, which is clearly of finite variation, since σ' is continuous, and thus $\sigma'(X)$ has paths bounded by a random constant on $[0, 1]$. In the discussion preceding this theorem we further established that $\sigma(X_t) \in \mathcal{H}^{1-t}$, and of course $\sigma(X_t) \in \mathcal{F}_t$. Therefore by Theorem 22 we have

$$\tilde{X}^t = X_{1-t} - X_1 = \int_0^t \sigma(X_{1-s})d\tilde{B}^s + [\widetilde{\sigma(X)}, B]_t + \int_{1-t}^1 b(X_s)ds.$$

Observe that $[\widetilde{\sigma(X)}, B]_t = \int_{1-t}^1 \sigma'(X_s)\sigma(X_s)ds$. Use the change of variable $u = 1 - s$ in the preceding integral and also in the term $\int_{1-t}^1 b(X_s)ds$ to get

$$X_{1-t} = X_1 + \int_0^t \sigma(X_{1-s})d\tilde{B}^s + \int_0^t \sigma'(X_{1-s})\sigma(X_{1-s})ds + \int_0^t b(X_{1-s})ds,$$

and the proof is complete. $\square$

Bibliographic Notes

The theory of expansion of filtrations began with the work of K. Ito [104] for initial expansions, and with the works of M. Barlow [9], M. Yor [264], and Th. Jeulin and M. Yor [121], [122] for progressive expansions. Our treatment has benefited from some private notes P. A. Meyer shared with the author, as well as the pedagogic treatment found in [46]. A comprehensive treatment, including most of the early important results, is in the book of Th. Jeulin [119]. Excellent later summaries of main results, including many not covered here, can be found in the lecture notes volume edited by Th. Jeulin and M. Yor [125] and also in the little book by M. Yor [271].

Theorem 1 is due to J. Jacod, but we first learned of it through a paper of P. A. Meyer [191], while Theorem 3 (Itô's Theorem extended to Lévy processes) was established in [112], with the help of T. Kurtz. Theorems 6 and 7 are of course due to Th. Jeulin (see [120] and page 44 of [119] for Theorem 6 and [123] for Theorem 7 (alternatively, see [125])); see also M. Yor [269]. Theorem 10, Jacod's criterion, and Theorem 11 are taken from [109]; Jacod's

criterion is the best general result on initial expansions of filtrations. The small result on filtration shrinkage is new and is due to J. Jacod and P. Protter.

Progressive expansions began with the simultaneous publication of M. Barlow [9] and M. Yor [264]. Two slightly later seminal papers are those of Th. Jeulin and M. Yor [121] and [123]. The idea of using a slightly larger filtration than the minimal one when one progressively expands, dates to M. Yor [264], and the idea to use the process $Z_t =^o 1_{\{L>t\}}$ originates with J. Azéma [5]. The lemma on local behavior of stochastic integrals at random times is due to C. Dellacherie and P. A. Meyer [48], and the proof of Theorem 18 is taken from [119] and [46]. The results on time reversal are inspired by the work of J. Jacod and P. Protter [112] and also E. Pardoux [206]. Related results can be found in J. Picard [209] and P. Sundar [242]. More recently time reversal has been used to extend Itô's formula for Brownian motion to functions which are more general than just being convex, although we did not present these results in this book. The interested reader can consult for example H. Föllmer, P. Protter and A.N. Shiryaev [80] as well as F. Russo and P. Vallois [228]. See also H. Föllmer and P. Protter [79] for the multidimensional case. The case for diffusions is treated by X. Bardina and M. Jolis [8].

For a study of Hypothesis (H) referred to just before Exercises 21 and 22, see P. Brémaud and M. Yor [27].

Exercises for Chapter VI

Exercise 1. Let $(\Omega, \mathcal{F}, \mathbb{F}, B, P)$ be a standard Brownian motion. Expand $\mathbb{F}$ by the initial addition of $\sigma\{B_1\}$. Let M be the local martingale in the formula

$$B_t = B_0 + M_t + \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds.$$

Show that the Brownian motion $M = (M_t)_{0 \le t \le 1}$ is independent of $\sigma\{B_1\}$.

Exercise 2. Show that the processes J^i defined in the proof of Itô's Theorem for Lévy processes (Theorem 3) are compound Poisson processes. Let N^i denote the Poisson process comprised of the arrival times of J^i, and let $\mathbb{G}^i$ be the natural completed filtration of N^i. Further, show the following three formulae (where we suppress the i superscripts) hold.

(a) $E\{J_t|J_1, \mathcal{G}_t\} = 1_{\{N_1 \ge 1\}} \dfrac{N_t}{N_1} J_1.$
(b) $E\{J_t|J_1, \mathcal{G}_1\} = tJ_1.$
(c) $E\{J_t|J_1\} = tJ_1.$

Exercise 3. In Exercise 1 assume $B_0 = x$ and condition on the event $\{B_1 = y\}$. Show that B is then a **Brownian bridge** beginning at x and ending at y. (See Exercise 9 below for more results concerning filtration expansions

and Brownian bridges. Brownian bridges are also discussed in Exercise 24 of Chap. II, and on page 305.)

Exercise 4. Let $(\Omega, \mathcal{F}, \mathbb{F}, Z, P)$ be a standard Poisson process. Expand $\mathbb{F}$ by the initial addition of $\sigma\{Z_1\}$. Let M be the local martingale in the formula

$$Z_t = M_t + \int_0^{t \wedge 1} \frac{Z_1 - Z_s}{1 - s} ds.$$

Show that $M = (M_t)_{0 \le t \le 1}$ is a time changed compensated Poisson process.

Exercise 5. Let B be standard Brownian motion. Show there exists an $\mathcal{F}_1$ measurable random variable L such that if $\mathbb{G}$ is the filtration $\mathbb{F}$ expanded initially with L, then $\mathcal{G}_t = \mathcal{F}_1$ for $0 \le t \le 1$. Prove that no non-constant $\mathbb{F}$ martingale is a $\mathbb{G}$ semimartingale.

Exercises 6 through 9 are linked, with the climax being Exercise 9.

Exercise 6. Let B denote standard Brownian motion on its canonical path space on continuous functions with domain $\mathbb{R}_+$ and range $\mathbb{R}$, with $B_t(\omega) = \omega(t)$. $\mathbb{F}$ is the usual minimal filtration completed (and thus rendered right continuous *a fortiori*). Define $Y_t^U = P\{B_a \in U | \mathcal{F}_t\}$. Show that

$$Y_t^U = \int_U Y_t^u du$$

where

$$Y_t^u = g(u, t, B_t) \quad \text{and} \quad g(u, t, x) = \frac{1}{\sqrt{2\pi(a-t)}} \exp\{\frac{-(u-x)^2}{2(a-t)}\}$$

for $t < a$. Show also that $Y_t^U = 1_{\{B_a \in U\}}$ for $t \ge a$.

Exercise 7. (Continuation of Exercise 6.) Show that $Y_{a-}^u = 0$ except on the null set $\{B_a = u\}$ and infer that $(Y_t^u)_{0 \le t < a}$ is a positive martingale which is not uniformly integrable on $[0, a]$.

Exercise 8. (Continuation of Exercises 6 and 7.) Show that

$$Y_t^u = Y_0^u + \int_0^t \frac{\partial g}{\partial x}(u, s, B_s) dB_s$$

where

$$\frac{\partial g}{\partial x}(u, t, x) = \frac{u - x}{\sqrt{2\pi(a-t)^3}} \exp\{\frac{-(u-x)^2}{2(a-t)}\}$$

for $t < a$.

Exercise 9. (Continuation of Exercises 6, 7, and 8.) Show that one can define a probability P_u on $\mathcal{C}([0, a], \mathbb{R})$ such that $P_u(B_a = u) = 1$ with P_u absolutely continuous with respect to P on $\mathcal{F}_t$ for $t < a$, and singular with respect to

P on $\mathcal{F}_a$. (*Hint: Sketch of procedure.* Let $\Omega_a = \mathcal{C}([0,a),\mathbb{R})$, be the space of continuous functions mapping $[0,a)$ into $\mathbb{R}$, and define $M_t^u = Y_t^u/Y_t^0$ on $\mathcal{F}_t$ for (of course) $t < a$. Let P_u denote the (unique) probability on $\Omega_a = \mathcal{C}([0,a),\mathbb{R})$ given on $\mathcal{F}_t$ by $dP_u = M_t^u dP$, and confirm that

$$\beta_t = B_t - B_0 - \int \frac{1}{M_s^u} d[B, M^u]_s$$

is a Brownian motion under P_u. Next show that

$$A_t^u = \int \frac{1}{M_s^u} d[B, M^u]_s \quad \text{verifies} \quad dA_t^u = \frac{\frac{\partial g}{\partial x}(u, t, B_t)}{g(u, t, B_t)} = \frac{u - B_t}{a - t}.$$

Finally show that

$$B_t - \frac{a - t}{a} B_0 - \frac{t}{a} u = (a - t) \int_0^t \frac{1}{a - s} d\beta_s$$

under P_u, and thus $(a - t) \int_0^t (a - s)^{-1} d\beta_s$, which is defined only on $[0, a)$, has the same distribution under P_u as the Brownian bridge, whence

$$\lim_{t \to a} (a - t) \int_0^t \frac{1}{a - s} d\beta_s = 0$$

a.s., and deduce the desired result.)

Exercise 10 (Expansion by a natural exponential random variable). Let $\mathbb{F}$ be a filtration satisfying the usual hypotheses, and let T be a totally inaccessible stopping time, with $P(T < \infty) = 1$. Let $A = (A_t)_{t \geq 0}$ be the compensator of $1_{\{t \geq T\}}$ and let M be the martingale $M_t = 1_{\{t \geq T\}} - A_t$. Note that $A_T = A_\infty$. Let $\mathbb{G}$ be the filtration obtained by initial expansion of $\mathbb{F}$ with $\sigma\{A_\infty\}$. For a bounded, non-random, Borel measurable function f let $M_t^f = \int_0^t f(A_s) dM_s$ which becomes in $\mathbb{G}$,

$$M_t^f = f(A_T)1_{\{t \geq T\}} - \int_0^t f(A_s) dA_s = f(A_T)1_{\{t \geq T\}} - F(A_t),$$

where $F(a) = \int_0^a f(s) ds$, since A is continuous. Show that $E\{f(A_T)\} = E\{F(A_T)\}$, and since f is arbitrary deduce that the distribution of the random variable A_T is exponential of parameter 1.

Exercise 11. Let $\mathbb{F}$, T, A, and $\mathbb{G}$ be as in Exercise 10. Show that if X is an $\mathbb{F}$ martingale with $X_{T-} = X_T$ a.s., then X is also a $\mathbb{G}$ martingale.

Exercise 12. Show that every stopping time for a filtration satisfying the usual hypotheses is the end of a predictable set, and is thus an honest time.

Exercise 13. Let $B = (B_t)_{t \geq 0}$ be a standard three dimensional Brownian motion. Let $L = \sup\{t : \|B_t\| \leq 1\}$, the last exit time from the unit ball. Show that L is an honest time. (This exercise is related to Exercise 8 of Chap. 1. on page 46)

Exercise 14. Let M^L be the fundamental L martingale of a progressive expansion of the underlying filtration $\mathbb{F}$ using the non-negative random variable L.

(a) Show that for any $\mathbb{F}$ square integrable martingale X we have $EX_L = EX_\infty M_\infty^L$.
(b) Show that M^L is the only $\mathbb{F}^L$ square integrable martingale with this property for all $\mathbb{F}$ square integrable martingales.
(c) Show that $M^L \in BMO$.

Exercise 15. Give an example of a filtration $\mathbb{F}$ and a non-negative random variable L and an $\mathbb{F}$ semimartingale X such that the post L process $Y_t = X_t - X_{t \wedge L} \equiv X_{t \vee L} - X_t$ is not a semimartingale.

Exercise 16. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space satisfying the usual hypotheses, let L be a non-negative random variable, let $\mathcal{G}_\infty = \mathcal{F}_\infty \vee \sigma\{L\}$, and let $\mathbb{G}$ be given by

$$\mathcal{G}_t = \{\Lambda \in \mathcal{G}_\infty | \exists \Lambda_t \in \mathcal{F}_t, \Lambda \cap \{t < L\} = \Lambda_t \cap \{t < L\}\}.$$

Let X be a $\mathbb{F}$ semimartingale. Give a simple proof (due to M. Yor [264]) that $X_t 1_{\{t<L\}}$ (and hence $X_{t \wedge L}$) is a semimartingale. (*Hint:* Show that it is enough to consider X a uniformly integrable martingale, hence enough to consider X a quasimartingale, and finally enough to consider X a positive supermartingale. Let $Z_t = {}^o 1_{\{L>t\}}$, the $\mathbb{F}$ optional projection of $1_{\{L>t\}}$, which is a supermartingale. Recall from Theorem 13 that $P(Z_{L-} > 0) = 1$. Show that

$$X_t^Z = \frac{X_t}{Z_t} 1_{\{t<L\}}$$

is a $\mathbb{G}$ positive supermartingale. Next show that XZ is an $\mathbb{F}$ special semimartingale, and let $XZ = M + A$ be its canonical decomposition. Observe that

$$X_t 1_{\{t<L\}} = \frac{X_t Z_t}{Z_t} 1_{\{t<L\}} = \frac{M_t + A_t}{Z_t} 1_{\{t<L\}} = M_t^Z + A_t^Z,$$

and argue as before that M^Z is a $\mathbb{G}$ semimartingale, and that A^Z is the product of the two $\mathbb{G}$ semimartingales A and $(Z_t)^{-1} 1_{\{t<L\}}$, and hence it is also a $\mathbb{G}$ semimartingale.)

Exercise 17. Let Z be a Lévy process on $[0,1]$ and let

$$X_t = X_0 + \int_0^t \sigma(X_{s-}) dZ_s + \int_0^t b(X_{s-}) ds.$$

Assume that σ and b are smooth, and also assume (which is not always true!) that the flows of X are injective. Find a stochastic differential equation solved by X_{1-t}.

Exercise 18. Let $(\Omega, \mathcal{F}, \mathbb{F}, B, P)$ be a standard Brownian space with B, a standard Brownian motion. Let $\mathbb{Z}$ denote the random zero set of B, which is a closed, perfect, nowhere dense set. Let $\{(L_n, R_n)\}_{n \geq 1}$ denote the random intervals contiguous to $\mathbb{Z}$. Without loss of generality assume that the graphs of L_n, denoted $[L_n]$, are disjoint random sets. P. A. Meyer's conjecture was that one could expand $\mathbb{F}$ to a filtration $\mathbb{G}$ in such a way that B would still be a semimartingale and each L_n would be a $\mathbb{G}$ stopping time. Show this is false (argument due to M. Barlow [10]) by taking $Y_t = |B_t|$ and showing (computed in the $\mathbb{G}$ filtration) that $\int_0^t 1_{\{Y_s > 0\}} dY_s = Y_t$, which implies that $\int_0^t 1_{\{Y_s = 0\}} dY_s = 0$, which is a contradiction, since as a consequence of Tanaka's formula $\int_0^t 1_{\{Y_s = 0\}} dY_s = L_t^0$, the Brownian local time at 0.

*__Exercise 19.__ Using the notation and hypotheses of Theorem 11, let $q(x, t, \omega)$ be the density of Q with respect to η_t as defined in the proof of the theorem. Let M be a continuous $\mathbb{F}$ local martingale. Show that there exists a predictable process k on the space $(\Omega \times E, \mathbb{F} \otimes \mathcal{E})$ such that

$$d\langle q(x, \cdot), M \rangle_t = k(x, t) q(t-, x) d\langle M, M \rangle_t.$$

__Exercise 20 (Jacod).__ [4] With the same hypotheses and notation as exercise 19 above, Show that if k satisfies $\int_0^t |k(L, s)| d\langle M, M \rangle_s < \infty$ a.s. for every $t \geq 0$, then the following process N is a $\mathbb{G}$ local martingale:

$$N_t = M_t - \int_0^t k(L, s) d\langle M, M \rangle_s.$$

In some situations it is interesting to know when a martingale remains a martingale in the enlarged filtration. Let $\mathbb{F} \subset \mathbb{G}$ be two filtrations satisfying the usual hypotheses. $\mathbb{G}$ satisfies **Hypothesis** (H) if every $\mathbb{F}$ martingale is a $\mathbb{G}$ martingale.[5] In the literature $\mathbb{G}$ is said to satisfy **Hypothesis** (H') if every $\mathbb{F}$ martingale is a $\mathbb{G}$ semimartingale. Note that it is clear that if $\mathbb{G}$ satisfies Hypothesis (H), then it satisfies Hypothesis (H').

Exercise 21. Show that if $\mathbb{G}$ satisfies Hypothesis (H'), then every $\mathbb{F}$ semimartingale is a $\mathbb{G}$ semimartingale.

*__Exercise 22.__ Let $B_t = (B_t^1, \ldots, B_t^n)$ denote standard n dimensional Brownian motion with canonical filtration $\mathbb{G}$, and with $\|B_t\|$ its Euclidean norm. Let $\mathcal{F}_t = \sigma(\|B_s\|; s \leq t) \vee \mathcal{N}$, where $\mathcal{N}$ denotes the $\mathcal{F}$ null sets. Show that $\mathbb{G}$ satisfies Hypothesis (H).

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[4] This exercise is a special case of Jacod's Decomposition Theorem. See [109], page 22, for the general result.

[5] Recently researchers in Finance have proposed an alternative name for Hypothesis H: the filtration $\mathbb{G}$ is said to be **self-sufficient** for the filtration $\mathbb{F}$ if every $\mathbb{F}$ martingale is a $\mathbb{G}$ martingale. See [4].