1
Vector Algebra

1.1 Vectors and scalars

This book is concerned with the mathematical description of physical quantities. These physical quantities include vectors and scalars, which are defined below.

1.1.1 Definition of a vector and a scalar

A *vector* is a physical quantity which has both magnitude and direction. There are many examples of such quantities, including velocity, force and electric field. A *scalar* is a physical quantity which has magnitude only. Examples of scalars include mass, temperature and pressure.

In this book, vectors will be written in bold italic type (for example, $\boldsymbol{u}$ is a vector) while scalar quantities will be written in plain italic type (for example, a is a scalar). There are two other commonly used ways of denoting vectors which are more convenient when writing by hand: an arrow over the symbol ($\vec{u}$) or a line under the symbol ($\underline{u}$).

Vectors can be represented diagrammatically by a line with an arrow at the end, as shown in Figure 1.1. The length of the line shows the magnitude of the vector and the arrow indicates its direction. If the vector has magnitude one, it is said to be a *unit vector*. Two vectors are said to be equal if they have the same magnitude and the same direction.

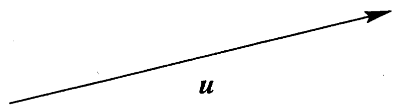


Fig. 1.1. Representation of a vector.

Example 1.1

Classify the following quantities according to whether they are vectors or scalars: energy, electric charge, electric current.

Energy and electric charge are scalars since there is no direction associated with them. Electric current is a vector because it flows in a particular direction.

1.1.2 Addition of vectors

Two vector quantities can be added together by the 'triangle rule' as shown in Figure 1.2. The vector $\boldsymbol{a} + \boldsymbol{b}$ is obtained by drawing the vector $\boldsymbol{a}$ and then drawing the vector $\boldsymbol{b}$ starting from the arrow at the end of $\boldsymbol{a}$.

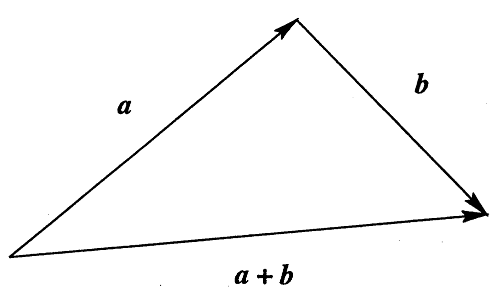


Fig. 1.2. Addition of vectors.

The vector $-\boldsymbol{a}$ is defined as the vector with magnitude equal to that of $\boldsymbol{a}$ but pointing in the opposite direction.

By adding $\boldsymbol{a}$ and $-\boldsymbol{a}$ we obtain the *zero vector*, $\mathbf{0}$. This has magnitude zero and so does not have a direction; nevertheless it is sensible to regard $\mathbf{0}$ as a vector.

1.1.3 Components of a vector

Vectors are often written using a Cartesian coordinate system with axes x, y, z. Such a system is usually assumed to be *right-handed*, which means that a screw rotated from the x-axis to the y-axis would move in the direction of the z-axis. Alternatively, if the thumb of the right hand points in the x direction and the first finger in the y direction, then the second finger points in the z direction.

Suppose that a vector $\boldsymbol{a}$ is drawn in a Cartesian coordinate system and extends from the point $(x_1,\ y_1,\ z_1)$ to the point $(x_2,\ y_2,\ z_2)$, as shown in Figure 1.3. Then the *components* of the vector are defined to be the three numbers $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$ and $a_3 = z_2 - z_1$. The vector can then be written in the form $\boldsymbol{a} = (a_1,\ a_2,\ a_3)$.

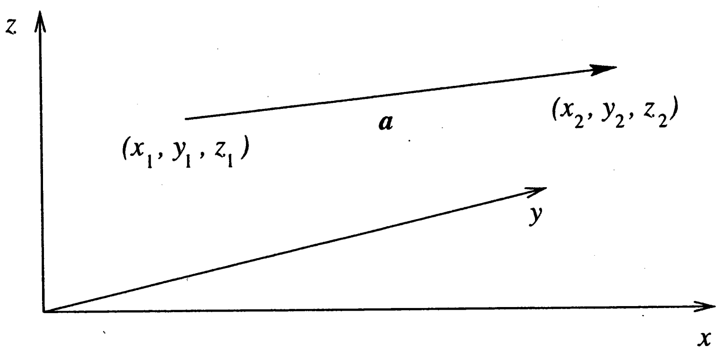


Fig. 1.3. The components of the vector $\boldsymbol{a}$ are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

By introducing three unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, which point along the coordinate axes x, y and z respectively, the vector can also be written in the form $\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3$. Using this form, the sum of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{a}+\boldsymbol{b} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3 = (a_1+b_1)\boldsymbol{e}_1 + (a_2 + b_2)\boldsymbol{e}_2 + (a_3+b_3)\boldsymbol{e}_3$. It follows that vectors can be added simply by adding their components, so that the vector equation $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ is equivalent to the three equations $c_1 = a_1 + b_1$, $c_2 = a_2 + b_2$, $c_3 = a_3 + b_3$.

The magnitude of the vector is written $|\boldsymbol{a}|$. It can be deduced from Pythagoras's theorem that the magnitude of the vector can be written in terms of its components as $|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

The position of a point in space (x, y, z) defines a vector which points from the origin of the coordinate system to the point (x, y, z). This vector is called the ***position vector*** of the point, and is usually denoted by the symbol $\boldsymbol{r}$, with components given by $\boldsymbol{r} = (x, y, z)$.

Example 1.2

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are defined by $\boldsymbol{a} = (1, 1, 1)$, $\boldsymbol{b} = (1, 2, 2)$. Find the magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$, and find the vectors $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$.

The magnitude of the vector $\boldsymbol{a}$ is $|\boldsymbol{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. The magnitude of $\boldsymbol{b}$ is $|\boldsymbol{b}| = \sqrt{1^2 + 2^2 + 2^2} = 3$. The vector $\boldsymbol{a} + \boldsymbol{b}$ is $(1, 1, 1) + (1, 2, 2) = (2, 3, 3)$ and $\boldsymbol{a} - \boldsymbol{b} = (0, -1, -1)$.

1.2 Dot product

The *dot product* or *scalar product* of two vectors is a scalar quantity. It is written $\boldsymbol{a} \cdot \boldsymbol{b}$ and is defined as the product of the magnitudes of the two vectors and the cosine of the angle between them:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta. \tag{1.1}$$

A number of properties of the dot product follow from this definition:

- The dot product is commutative, i.e. $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$.
- If the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular (orthogonal) then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- Conversely, if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ then either the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular or one of the vectors is the zero vector.
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Since the quantity $|\boldsymbol{b}| \cos\theta$ represents the component of the vector $\boldsymbol{b}$ in the direction of the vector $\boldsymbol{a}$, the scalar $\boldsymbol{a} \cdot \boldsymbol{b}$ can be thought of as the magnitude of $\boldsymbol{a}$ multiplied by the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ (see Figure 1.4).
- The dot product is distributive over addition, i.e. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$. This follows geometrically from the fact that the component of $\boldsymbol{b} + \boldsymbol{c}$ in the direction of $\boldsymbol{a}$ is the same as the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ plus the component of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$ (see Figure 1.5).

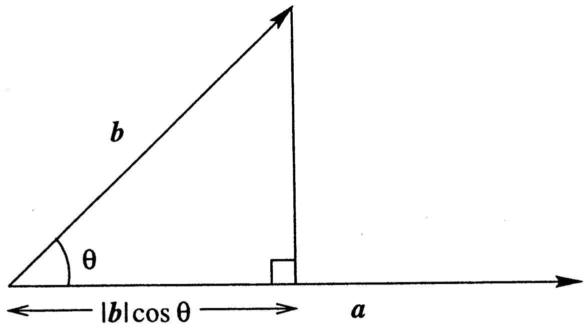


Fig. 1.4. The component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is $|\boldsymbol{b}| \cos\theta$.

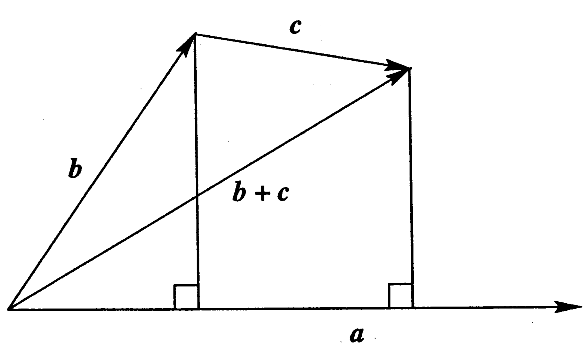


Fig. 1.5. Geometrical demonstration that the dot product is distributive over addition.

A formula for the dot product $\boldsymbol{a} \cdot \boldsymbol{b}$ in terms of the components of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be derived from the above properties. Considering first the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, it follows from the fact that these vectors have magnitude 1 and are orthogonal to each other that

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = 1,\ \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = 1,\ \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1,\ \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0,\ \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 = 0,\ \boldsymbol{e}_3 \cdot \boldsymbol{e}_1 = 0.$$

The dot product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is therefore

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= (a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3) \cdot (b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3) \\ &= a_1b_1\boldsymbol{e}_1 \cdot \boldsymbol{e}_1 + a_2b_2\boldsymbol{e}_2 \cdot \boldsymbol{e}_2 + a_3b_3\boldsymbol{e}_3 \cdot \boldsymbol{e}_3 \\ &= a_1b_1 + a_2b_2 + a_3b_3. \end{aligned} \tag{1.2}$$

Example 1.3

Find the dot product of the vectors $(1, 1, 2)$ and $(2, 3, 2)$.

$(1, 1, 2) \cdot (2, 3, 2) = 1 \times 2 + 1 \times 3 + 2 \times 2 = 9$.

Example 1.4

For what value of c are the vectors $(c, 1, 1)$ and $(-1, 2, 0)$ perpendicular?

They are perpendicular when their dot product is zero. The dot product is $-c + 2 + 0$ so the vectors are perpendicular if $c = 2$.

Example 1.5

Show that a triangle inscribed in a circle is right-angled if one of the sides of the triangle is a diameter of the circle.

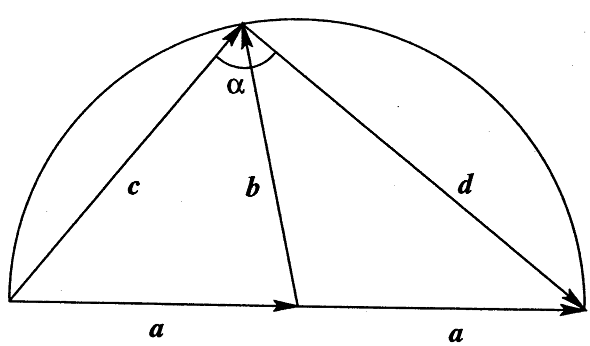


Fig. 1.6. Geometrical construction to show that α is a right angle.

Introduce two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as shown in Figure 1.6. Since these two vectors are both along radii of the circle they are of equal magnitude. The two sides $\boldsymbol{c}$ and $\boldsymbol{d}$ of the triangle are then given by $\boldsymbol{c} = \boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{d} = \boldsymbol{a}-\boldsymbol{b}$. The dot product of these two vectors is $\boldsymbol{c}\cdot\boldsymbol{d} = (\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) = |\boldsymbol{a}|^2 - \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{a}-|\boldsymbol{b}|^2 = 0$. Since the dot product is zero the vectors are perpendicular, so the angle α is a right angle. This is just one of many geometrical results that can be obtained using vector methods.

1.2.1 Applications of the dot product

Work done against a force

Suppose that a constant force $\boldsymbol{F}$ acts on a body and that the body is moved a distance $\boldsymbol{d}$. Then the work done against the force is given by the magnitude of the force times the distance moved in the direction opposite to the force; this is simply $-\boldsymbol{F}\cdot\boldsymbol{d}$ (Figure 1.7).

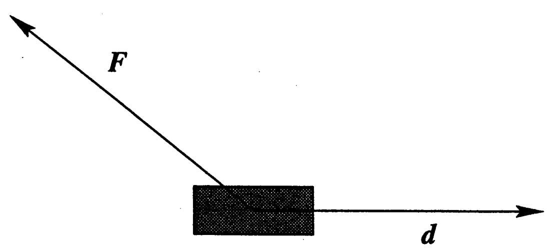


Fig. 1.7. The work done against a force $\boldsymbol{F}$ when an object is moved a distance $\boldsymbol{d}$ is $-\boldsymbol{F}\cdot\boldsymbol{d}$.

Equation of a plane

Consider a two-dimensional plane in three-dimensional space (Figure 1.8). Let $\boldsymbol{r}$ be the position vector of any point in the plane, and let $\boldsymbol{a}$ be a vector perpendicular to the plane. The condition for a point with position vector $\boldsymbol{r}$ to lie in the plane is that the component of $\boldsymbol{r}$ in the direction of $\boldsymbol{a}$ is equal to the perpendicular distance p from the origin to the plane. The general form of the equation of a plane is therefore

$$\boldsymbol{r}\cdot\boldsymbol{a} = \text{constant}.$$

An alternative way to write this is in terms of components. Writing $\boldsymbol{r} = (x, y, z)$ and $\boldsymbol{a} = (a_1, a_2, a_3)$, the equation of a plane becomes

$$a_1x + a_2y + a_3z = \text{constant}. \tag{1.3}$$

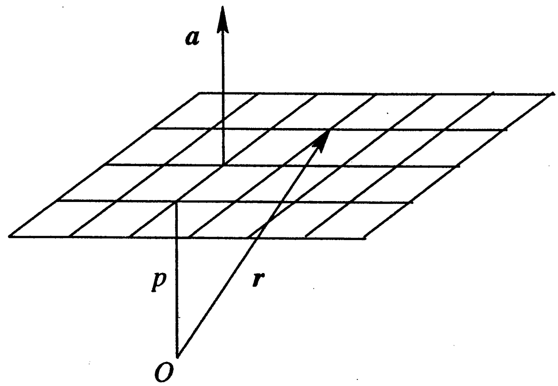


Fig. 1.8. The equation of a plane is $\boldsymbol{r} \cdot \boldsymbol{a} = \text{constant}$.

EXERCISES

1.1 Classify the following quantities according to whether they are vectors or scalars: density, magnetic field strength, power, momentum, angular momentum, acceleration.

1.2 If $\boldsymbol{a} = (2, 0, 3)$ and $\boldsymbol{b} = (1, 0, -1)$, find $|\boldsymbol{a}|$, $|\boldsymbol{b}|$, $\boldsymbol{a} + \boldsymbol{b}$, $\boldsymbol{a} - \boldsymbol{b}$ and $\boldsymbol{a} \cdot \boldsymbol{b}$. What is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$?

1.3 If $\boldsymbol{u} = (1, 2, 2)$ and $\boldsymbol{v} = (-6, 2, 3)$, find the component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$ and the component of $\boldsymbol{v}$ in the direction of $\boldsymbol{u}$.

1.4 Find the equation of the plane that is perpendicular to the vector $(1, 1, -1)$ and passes through the point $x = 1$, $y = 2$, $z = 1$.

1.5 Use vector methods to show that the diagonals of a rhombus are perpendicular.

1.6 What is the angle between any two diagonals of a cube?

1.7 Use vectors to show that for any triangle, the three lines drawn from each vertex to the midpoint of the opposite side all pass through the same point.

1.3 Cross product

The *cross product* or *vector product* of two vectors is a vector quantity, written $\boldsymbol{a} \times \boldsymbol{b}$. Since it is a vector, its definition must specify both its magnitude and direction. The magnitude of $\boldsymbol{a}\times\boldsymbol{b}$ is $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$, where θ is the angle between the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The direction of $\boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ in a right-handed sense, i.e. a right-handed screw rotated from $\boldsymbol{a}$ towards $\boldsymbol{b}$ moves in the direction of $\boldsymbol{a}\times\boldsymbol{b}$ (Figure 1.9). We may therefore write $\boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\,\boldsymbol{u}$, where $\boldsymbol{u}$ is a unit vector perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$ in a right-handed sense.

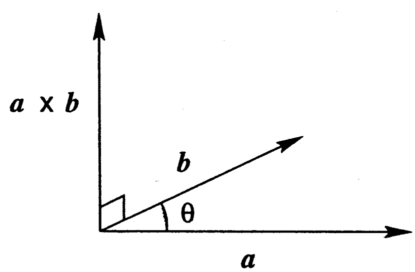


Fig. 1.9. The cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$, in a right-handed sense.

The cross product has the following properties:

- The cross product is *not* commutative. Because of the right-hand rule, $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{b} \times \boldsymbol{a}$ point in opposite directions, so $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$.
- If the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel then $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$.
- $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$.
- The magnitude of the cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is the area of the parallelogram made by the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (Figure 1.10). Similarly the area of the triangle made by $\boldsymbol{a}$ and $\boldsymbol{b}$ is $|\boldsymbol{a} \times \boldsymbol{b}|/2$.
- The cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$ only depends on the component of $\boldsymbol{b}$ perpendicular to $\boldsymbol{a}$. This is apparent from Figure 1.10 since the component of $\boldsymbol{b}$ perpendicular to $\boldsymbol{a}$ is $|\boldsymbol{b}|\sin\theta$.
- The cross product is distributive over addition, i.e. $\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{a}\times\boldsymbol{c}$. This is demonstrated geometrically in Figure 1.11, where the vector $\boldsymbol{a}$ points into the page. The vectors $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{b}+\boldsymbol{c}$ do not necessarily lie in the page, but from the previous point the cross products of these vectors with $\boldsymbol{a}$ only depend on their projections onto the page. The effect of taking the cross

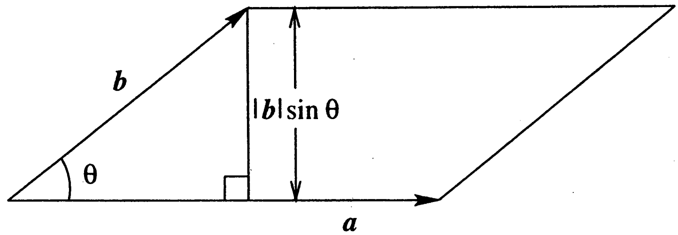


Fig. 1.10. The area of the parallelogram is the length of its base, $|\boldsymbol{a}|$, multiplied by its height, $|\boldsymbol{b}|\sin\theta$.

product with $\boldsymbol{a}$ on any vector is to project it onto the page, rotate through $\pi/2$ clockwise and then multiply by $|\boldsymbol{a}|$. Thus the triangle made by the vectors $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{b}+\boldsymbol{c}$ becomes rotated and scaled as in Figure 1.11 but remains a triangle.

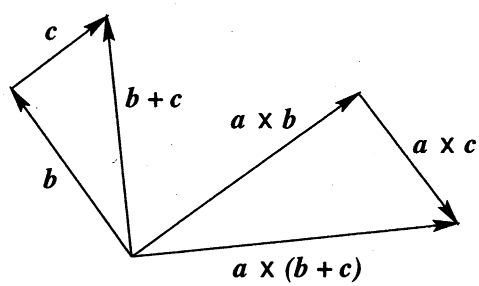


Fig. 1.11. Geometrical demonstration that the cross product is distributive over addition. The vector $\boldsymbol{a}$ points into the page.

A formula for the cross product $\boldsymbol{a}\times\boldsymbol{b}$ in terms of the components of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be derived in a similar manner to that carried out for the dot product. Consider first $\boldsymbol{e}_1 \times \boldsymbol{e}_2$. Since these two vectors have magnitude 1 and are perpendicular, $\sin\theta = 1$ and the magnitude of $\boldsymbol{e}_1 \times \boldsymbol{e}_2$ is 1. The direction of $\boldsymbol{e}_1 \times \boldsymbol{e}_2$ is perpendicular to both $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in a right-handed sense, so $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3$.

It follows that the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ obey

$$\boldsymbol{e}_1\times\boldsymbol{e}_1 = \boldsymbol{0},\ \boldsymbol{e}_2\times\boldsymbol{e}_2 = \boldsymbol{0},\ \boldsymbol{e}_3\times\boldsymbol{e}_3 = \boldsymbol{0},\ \boldsymbol{e}_1\times\boldsymbol{e}_2 = \boldsymbol{e}_3,\ \boldsymbol{e}_2\times\boldsymbol{e}_3 = \boldsymbol{e}_1,\ \boldsymbol{e}_3\times\boldsymbol{e}_1 = \boldsymbol{e}_2.$$

The cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is therefore

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= (a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3) \times (b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3) \\ &= a_1b_2\,\boldsymbol{e}_1 \times \boldsymbol{e}_2 + a_1b_3\,\boldsymbol{e}_1 \times \boldsymbol{e}_3 + a_2b_1\,\boldsymbol{e}_2 \times \boldsymbol{e}_1 \\ &\quad + a_2b_3\,\boldsymbol{e}_2 \times \boldsymbol{e}_3 + a_3b_1\,\boldsymbol{e}_3 \times \boldsymbol{e}_1 + a_3b_2\,\boldsymbol{e}_3 \times \boldsymbol{e}_2 \\ &= (a_2b_3 - a_3b_2)\boldsymbol{e}_1 + (a_3b_1 - a_1b_3)\boldsymbol{e}_2 + (a_1b_2 - a_2b_1)\boldsymbol{e}_3. \end{aligned} \tag{1.4}$$

This can also be written as the determinant of a 3×3 matrix as follows:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Example 1.6

Find the cross product of the vectors $(1,3,0)$ and $(2,-1,1)$.

$(1,3,0) \times (2,-1,1) = (3-0, 0-1, -1-6) = (3,-1,-7)$.

Example 1.7

Find a unit vector which is perpendicular to both $(1,0,1)$ and $(0,1,1)$.

A perpendicular vector is $(1,0,1) \times (0,1,1) = (-1,-1,1)$. To make this a unit vector we must divide by its magnitude, which is $\sqrt{3}$, so the unit vector perpendicular to $(1,0,1)$ and $(0,1,1)$ is $(-1,-1,1)/\sqrt{3}$.

Example 1.8

What is the area of the triangle which has its vertices at the points $P = (1,1,1)$, $Q = (2,3,3)$ and $R = (4,1,2)$?

First construct two vectors that make up two sides of the triangle. The vector from P to Q is $\boldsymbol{a} = (1,2,2)$ and the vector from P to R is $\boldsymbol{b} = (3,0,1)$. The cross product of these vectors is $\boldsymbol{a} \times \boldsymbol{b} = (2,5,-6)$. The area of the triangle is then $|\boldsymbol{a} \times \boldsymbol{b}|/2 = \sqrt{65}/2 \approx 4.03$.

1.3.1 Applications of the cross product

Solid body rotation

Suppose that a solid body is rotating steadily about an axis. What is the velocity vector of a point within the body?

Consider a body rotating with angular velocity Ω (this means that in a time t the body rotates through an angle Ωt radians). Since there is a rotation axis, a vector $\boldsymbol{\Omega}$ can be defined, with magnitude $|\boldsymbol{\Omega}| = \Omega$ and directed along the rotation axis. Since this vector could point in either direction, the following form of the right-hand rule is used to define the direction of $\boldsymbol{\Omega}$: a screw rotating in the same direction as the body moves in the direction of $\boldsymbol{\Omega}$. Alternatively, if the fingers of the right hand point in the direction of the rotation, the thumb of the right hand points in the direction of $\boldsymbol{\Omega}$. This means that for a body which is rotating to the right, $\boldsymbol{\Omega}$ points upwards (Figure 1.12).

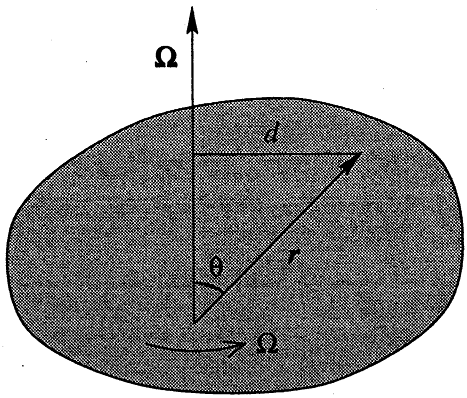


Fig. 1.12. Motion of a rotating body.

Now consider the motion of a point at a position vector $\boldsymbol{r}$, which makes an angle θ with the rotation axis. The speed at which this point moves is Ωd, where d is the perpendicular distance from the point to the rotation axis. Since $d = |\boldsymbol{r}| \sin\theta$ (Figure 1.12), the speed of motion is $v = \Omega|\boldsymbol{r}|\sin\theta$. Note that this is equal to $|\boldsymbol{\Omega} \times \boldsymbol{r}|$. Now consider the direction of the motion. In Figure 1.12, where both $\boldsymbol{\Omega}$ and $\boldsymbol{r}$ lie in the plane of the page, the direction of motion is into the page, perpendicular to both $\boldsymbol{\Omega}$ and $\boldsymbol{r}$ and so in the direction of $\boldsymbol{\Omega} \times \boldsymbol{r}$. Therefore the velocity vector of the point at $\boldsymbol{r}$ is

$$\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r}, \tag{1.5}$$

since this vector has both the correct magnitude and the correct direction.

Equation of a straight line

The equation of a straight line can be written in terms of the cross product as follows. Suppose that $\boldsymbol{a}$ is the position vector of a particular fixed point on the line, and that $\boldsymbol{u}$ is a vector pointing along the line (Figure 1.13). Then any point $\boldsymbol{r}$ on the line can be reached from the origin by travelling first along the vector $\boldsymbol{a}$ onto the line and then some multiple of the vector $\boldsymbol{u}$ along the line:

$$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{u}, \tag{1.6}$$

where λ is a parameter. This is referred to as the *parametric* form of the equation of a line.

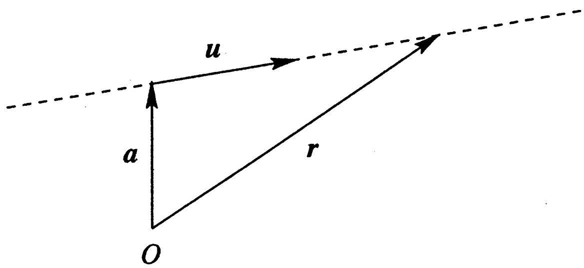


Fig. 1.13. The equation of a line is $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{u}$.

To obtain a form of (1.6) that does not involve the parameter λ, the term involving the vector $\boldsymbol{u}$ must be eliminated. This can be done by taking the cross product of (1.6) with $\boldsymbol{u}$. This gives $\boldsymbol{r} \times \boldsymbol{u} = \boldsymbol{a} \times \boldsymbol{u}$. Since the vector $\boldsymbol{a} \times \boldsymbol{u}$ is a constant, it can be relabelled $\boldsymbol{b}$, giving the second form for the equation of a straight line:

$$\boldsymbol{r} \times \boldsymbol{u} = \boldsymbol{b}. \tag{1.7}$$

Physical applications of the cross product

There are many physical quantities that are defined in terms of the cross product. These include the following:

- A particle of mass m has position vector $\boldsymbol{r}$ and is moving with velocity $\boldsymbol{v}$. Its angular momentum about the origin is $\boldsymbol{h} = m\boldsymbol{r} \times \boldsymbol{v}$.
- A particle of mass m moves with velocity $\boldsymbol{u}$ in a frame which is rotating with angular velocity $\boldsymbol{\Omega}$. Due to the rotation, the particle experiences a sideways force called the Coriolis force, $\boldsymbol{F} = 2m\boldsymbol{u} \times \boldsymbol{\Omega}$. Since the Earth is rotating, this force influences motion on the surface of the Earth. The effect deflects particles to the right in the northern hemisphere and is strongest for motions on large scales such as ocean currents and weather systems.
- A particle with electric charge q moves with velocity $\boldsymbol{v}$ in the presence of a magnetic field $\boldsymbol{B}$. This results in a force, called the Lorentz force, equal to $q\boldsymbol{v} \times \boldsymbol{B}$. This is the force which is responsible for the operation of an electric motor.

1.4 Scalar triple product

The *scalar triple product* of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is defined to be $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$. In fact the brackets here are unnecessary: $(\boldsymbol{a}\cdot\boldsymbol{b})\times\boldsymbol{c}$ is meaningless since $(\boldsymbol{a}\cdot\boldsymbol{b})$ is a scalar and so cannot be crossed with the vector $\boldsymbol{c}$. Therefore the expression $\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}$ is well defined.

The formula for the scalar triple product in terms of the components of the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ can be obtained using the formula for the cross product (1.4):

$$\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1. \qquad (1.8)$$

The scalar triple product has a number of properties, listed below. The first four follow directly from (1.8).

- The dot and the cross can be interchanged:

$$\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c} = \boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}.$$

- The vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ can be permuted cyclically:

$$\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c} = \boldsymbol{b}\cdot\boldsymbol{c}\times\boldsymbol{a} = \boldsymbol{c}\cdot\boldsymbol{a}\times\boldsymbol{b}.$$

- The scalar triple product can be written in the form of a determinant:

$$\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

- If any two of the vectors are equal, the scalar triple product is zero.
- Geometrically, the magnitude of the scalar triple product is the volume of the three-dimensional object known as a parallelepiped formed by the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ (Figure 1.14). This can be shown as follows. The area of the parallelogram forming the base is $|\boldsymbol{b}\times\boldsymbol{c}|$. The height is the vertical component of $\boldsymbol{a}$, which is the magnitude of the component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}\times\boldsymbol{c}$. This is $|\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}|/|\boldsymbol{b}\times\boldsymbol{c}|$, so the volume is the area of the base multiplied by the height, which is $|\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}|$. Similarly, the volume of the tetrahedron made by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is $|\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}|/6$.

The scalar triple product of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is often written $[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]$. This notation highlights the fact that the dot and the cross can be interchanged.

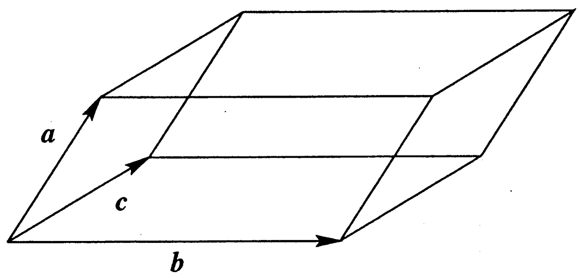


Fig. 1.14. The volume of the object formed by the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is $|\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}|$.

Example 1.9

Find the scalar triple product of the vectors (1,2,1), (0,1,1) and (2,1,0).

First find the vector $(0,1,1) \times (2,1,0) = (-1,2,-2)$. Now dot this with $(1,2,1)$, giving the answer 1.

Example 1.10

Show that if three vectors lie in a plane, then their scalar triple product is zero.

If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ lie in a plane, then the vector $\boldsymbol{b} \times \boldsymbol{c}$ is perpendicular to the plane and hence perpendicular to $\boldsymbol{a}$. Since the dot product of perpendicular vectors is always zero, it follows that $\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c} = 0$.

Example 1.11

A particle with mass m and electric charge q moves in a uniform magnetic field $\boldsymbol{B}$. Given that the force $\boldsymbol{F}$ on the particle is $\boldsymbol{F} = q\,\boldsymbol{v} \times \boldsymbol{B}$, where $\boldsymbol{v}$ is the velocity of the particle, show that the particle moves at constant speed.

The equation of motion of the particle is written using Newton's second law, force equals mass times acceleration. The acceleration of the particle is the rate of change of the velocity, written $\dot{\boldsymbol{v}}$, so the equation of motion is

$$q\,\boldsymbol{v} \times \boldsymbol{B} = m\,\dot{\boldsymbol{v}}.$$

Now taking the dot product of both sides of this equation with $\boldsymbol{v}$, the scalar triple product on the left-hand side gives zero since two of the vectors are equal. Hence

$$0 = m\,\dot{\boldsymbol{v}} \cdot \boldsymbol{v} = m\frac{d}{dt}(\boldsymbol{v} \cdot \boldsymbol{v})/2 = m\frac{d}{dt}\left(|\boldsymbol{v}|^2\right)/2,$$

so the speed of the particle, $|\boldsymbol{v}|$, does not change with time.

1.5 Vector triple product

The *vector triple product* of three vectors is $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$. The brackets are important here, since $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$. Since only cross products are involved, the result is a vector. An alternative expression for $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ can be obtained by writing out the components. Since

$$\boldsymbol{b} \times \boldsymbol{c} = (b_2c_3 - b_3c_2)\boldsymbol{e}_1 + (b_3c_1 - b_1c_3)\boldsymbol{e}_2 + (b_1c_2 - b_2c_1)\boldsymbol{e}_3,$$

the first component of $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ is

$$\begin{aligned} [\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})]_1 &= a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) \\ &= b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3). \end{aligned}$$

By adding and subtracting the quantity $a_1b_1c_1$, this can be written

$$\begin{aligned} [\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})]_1 &= b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3) \\ &= b_1\boldsymbol{a} \cdot \boldsymbol{c} - c_1\boldsymbol{a} \cdot \boldsymbol{b}. \end{aligned}$$

Similar equations hold for the second and third components, so the vector triple product can be expanded as

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}. \tag{1.9}$$

From this result it also follows that

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = -\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = -(\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{a} + (\boldsymbol{c} \cdot \boldsymbol{a})\boldsymbol{b}. \tag{1.10}$$

Example 1.12

Under what conditions are $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ and $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ equal?

By comparing (1.9) with (1.10), the two are equal if $-(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} = -(\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{a}$. This can alternatively be written $\boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{c}) = \boldsymbol{0}$.

Example 1.13

Find an alternative expression for $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d})$.

Since the dot and cross can be interchanged in a scalar triple product,

$$\begin{aligned} (\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) &= \boldsymbol{a} \cdot (\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d})) \\ &= \boldsymbol{a} \cdot ((\boldsymbol{b} \cdot \boldsymbol{d})\boldsymbol{c} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{d}) \\ &= (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}). \end{aligned}$$

1.6 Scalar fields and vector fields

A scalar or vector quantity is said be a *field* if it is a function of position. An example of a scalar field is the temperature inside a room; in general the temperature has a different value at different points in space, so the temperature T is a function of position. This is indicated by writing $T(\boldsymbol{r})$, where $\boldsymbol{r}$ is the position vector of a point in space, $\boldsymbol{r} = (x, y, z)$. Other examples of scalar fields include pressure and density. An example of a vector field is the velocity of the air within a room.

In general, a scalar field T is three-dimensional, i.e. it depends on all three coordinates, $T = T(x, y, z)$. Such fields are difficult to visualise. However, if the scalar field only depends on two coordinates, $T = T(x, y)$, then it can be visualised by sketching a contour plot. To do this, the line $T(x, y) =$ constant is plotted for different values of the constant. For example, consider the scalar field $T(x, y) = x^2 + y^2$. The contour lines are the lines $x^2 + y^2 =$ constant, which are concentric circles centred at the origin, as shown in Figure 1.15(a).

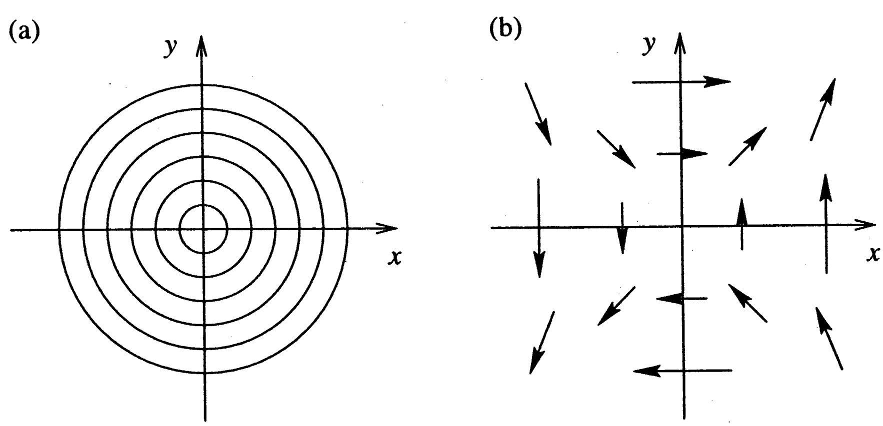


Fig. 1.15. (a) Contours of the scalar field $T(x, y) = x^2 + y^2$. (b) The vector field $\boldsymbol{u}(x, y) = (y, x)$.

Vector fields in two dimensions can also be visualised by a sketch. In this case the simplest procedure is to evaluate the vector field at a sequence of points and draw vectors indicating the magnitude and direction of the vector field at each point. An example of this procedure is the drawing of wind speeds and directions on weather maps. For example, consider the vector field $\boldsymbol{u}(x, y) =$

(y, x). At the point $(1,0)$, $\boldsymbol{u} = (0,1)$, so at this point a vector of magnitude 1 pointing in the y direction is drawn. Similarly, at $(0,1)$, $\boldsymbol{u} = (1,0)$ and at $(1,1)$, $\boldsymbol{u} = (1,1)$. By considering a few additional points, a sketch of the vector field can be built up (Figure 1.15(b)).

Summary of Chapter 1

- A *vector* is a physical quantity with magnitude and direction.
- A *scalar* is a physical quantity with magnitude only.
- In Cartesian coordinates a vector can be written in terms of its *components* as either $\boldsymbol{a} = (a_1, a_2, a_3)$ or $\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3$, where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are unit vectors along the x-, y- and z-axes respectively.
- The *magnitude* of the vector $\boldsymbol{a}$ is $|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
- The *dot product* or *scalar product* of $\boldsymbol{a}$ and $\boldsymbol{b}$ is a scalar,

  $$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

  This can also be thought of as $|\boldsymbol{a}|$ multiplied by the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$. Applications of the dot product include the work done when moving an object acted on by a force and the equation of a plane.
- The *cross product* or *vector product* of $\boldsymbol{a}$ and $\boldsymbol{b}$ is a vector, $\boldsymbol{a} \times \boldsymbol{b}$, with magnitude $|\boldsymbol{a}||\boldsymbol{b}| \sin\theta$, perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$ in a right-handed sense. In component form,

  $$\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\boldsymbol{e}_1 + (a_3 b_1 - a_1 b_3)\boldsymbol{e}_2 + (a_1 b_2 - a_2 b_1)\boldsymbol{e}_3.$$

  The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ is $|\boldsymbol{a}|$ multiplied by the component of $\boldsymbol{b}$ perpendicular to $\boldsymbol{a}$, which is the area of the parallelogram made by $\boldsymbol{a}$ and $\boldsymbol{b}$. Applications of the cross product include the equation of a straight line and the rotation of a rigid body.
- The *scalar triple product* is $\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{b} \cdot \boldsymbol{c} \times \boldsymbol{a}$.
- The *vector triple product* is $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$.
- A scalar or vector quantity is a *field* if it is a function of position.

EXERCISES

1.8 Find the equation of the straight line which passes through the points $(1,1,1)$ and $(2,3,5)$, (a) in parametric form; (b) in cross product form.

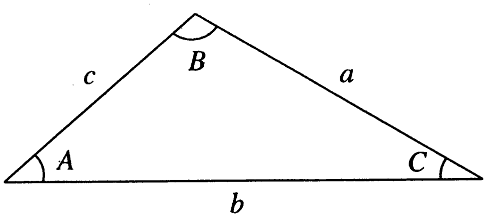
1.9 Using vector methods, prove the sine rule,

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c} \tag{1.11}$$

and the cosine rule,

$$c^2=a^2+b^2-2ab\cos C \tag{1.12}$$

for the triangle with angles A, B, C and sides a, b, c in the figure below.



1.10 (a) Show that the set of vectors and the operation of vector addition form a group. (The set of objects $a, b, c, \ldots$ and the operation $\star$ form a group if the following four conditions are satisfied: (i) for any two elements a and b, $a \star b$ is in the set; (ii) $(a \star b) \star c = a \star (b \star c)$; (iii) there is an element I obeying $a \star I = I \star a = a$; (iv) each element a has an inverse a^{-1} such that $a \star a^{-1} = a^{-1} \star a = I$.)
(b) Do the set of vectors and the dot product form a group?
(c) Do the set of vectors and the cross product form a group?

1.11 Simplify the following expressions:
(a) $|\boldsymbol{a}\times\boldsymbol{b}|^2+(\boldsymbol{a}\cdot\boldsymbol{b})^2$;
(b) $\boldsymbol{a}\times(\boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{b}))$;
(c) $(\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{b}-\boldsymbol{c})\times(\boldsymbol{c}-\boldsymbol{a})$;
(d) $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{b}\times\boldsymbol{c})\times(\boldsymbol{c}\times\boldsymbol{a})$.

1.12 The vector $\boldsymbol{x}$ obeys the two equations $\boldsymbol{x}\cdot\boldsymbol{a}=1$ and $\boldsymbol{x}\times\boldsymbol{a}=\boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors. Solve these equations to find an expression for $\boldsymbol{x}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. Give a geometrical interpretation of this question.

1.13 Find the equation of the line on which the two planes $\boldsymbol{r} \cdot \boldsymbol{a} = 1$ and $\boldsymbol{r} \cdot \boldsymbol{b} = 1$ meet.

1.14 (a) Express the vector $\boldsymbol{a} \times \boldsymbol{b}$ in the form $\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$, assuming that the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are not coplanar.

(b) Hence find an expression for $(\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c})^2$ that does not involve any cross products.

(c) Hence find the volume of a tetrahedron made from four equilateral triangles with sides of length 1.

1.15 A particle of mass m at position $\boldsymbol{r}$ and moving with velocity $\boldsymbol{v}$ is subject to a force $\boldsymbol{F}$ directed towards the origin, $\boldsymbol{F} = -f(r)\boldsymbol{r}$. Show that the angular momentum vector $\boldsymbol{h} = m\boldsymbol{r} \times \boldsymbol{v}$ is constant.

1.16 Sketch the scalar field $T(x, y) = x^2 - y$.

1.17 Sketch the vector field $\boldsymbol{u}(x, y) = (x + y, -x)$.