1
The Laplace Transform

1.1 Introduction

As a discipline, mathematics encompasses a vast range of subjects. In pure mathematics an important concept is the idea of an axiomatic system whereby axioms are proposed and theorems are proved by invoking these axioms logically. These activities are often of little interest to the applied mathematician to whom the pure mathematics of algebraic structures will seem like tinkering with axioms for hours in order to prove the obvious. To the engineer, this kind of pure mathematics is even more of an anathema. The value of knowing about such structures lies in the ability to generalise the "obvious" to other areas. These generalisations are notoriously unpredictable and are often very surprising. Indeed, many say that there is no such thing as non-applicable mathematics, just mathematics whose application has yet to be found.

The Laplace Transform expresses the conflict between pure and applied mathematics splendidly. There is a temptation to begin a book such as this on linear algebra outlining the theorems and properties of normed spaces. This would indeed provide a sound basis for future results. However most applied mathematicians and all engineers would probably turn off. On the other hand, engineering texts present the Laplace Transform as a toolkit of results with little attention being paid to the underlying mathematical structure, regions of validity or restrictions. What has been decided here is to give a brief introduction to the underlying pure mathematical structures, enough it is hoped for the pure mathematician to appreciate what kind of creature the Laplace Transform is, whilst emphasising applications and giving plenty of examples. The point of view from which this book is written is therefore definitely that of the applied mathematician. However, pure mathematical asides, some of which can be quite

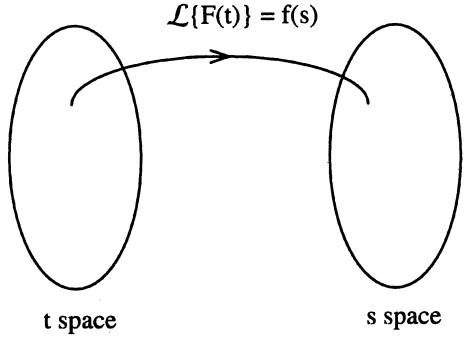

Figure 1.1: The Laplace Transform as a mapping

extensive, will occur. It remains the view of this author that Laplace Transforms only come alive when they are used to solve real problems. Those who strongly disagree with this will find pure mathematics textbooks on integral transforms much more to their liking.

The main area of pure mathematics needed to understand the fundamental properties of Laplace Transforms is analysis and, to a lesser extent the normed vector space. Analysis, in particular integration, is needed from the start as it governs the existence conditions for the Laplace Transform itself; however as is soon apparent, calculations involving Laplace Transforms can take place without explicit knowledge of analysis. Normed vector spaces and associated linear algebra put the Laplace Transform on a firm theoretical footing, but can be left until a little later in a book aimed at second year undergraduate mathematics students.

1.2 The Laplace Transform

The definition of the Laplace Transform could hardly be more straightforward. Given a suitable function $F(t)$ the Laplace Transform, written $f(s)$ is defined by

$$f(s) = \int_0^\infty F(t)e^{-st}dt.$$

This bald statement may satisfy most engineers, but not mathematicians. The question of what constitutes a "suitable function" will now be addressed. The integral on the right has infinite range and hence is what is called an improper integral. This too needs careful handling. The notation $\mathcal{L}\{F(t)\}$ is used to denote the Laplace Transform of the function $F(t)$.

Another way of looking at the Laplace Transform is as a mapping from points in the t domain to points in the s domain. Pictorially, Figure 1.1 indicates this mapping process. The time domain t will contain all those functions $F(t)$ whose Laplace Transform exists, whereas the frequency domain s contains all the images $\mathcal{L}\{F(t)\}$. Another aspect of Laplace Transforms that needs mentioning at this stage is that the variable s often has to take complex values. This means that $f(s)$ is a function of a complex variable, which in turn places restrictions on the (real) function $F(t)$ given that the improper integral must converge. Much of the analysis involved in dealing with the image of the function $F(t)$ in the s plane is therefore complex analysis which may be quite new to some readers.

As has been said earlier, engineers are quite happy to use Laplace Transforms to help solve a variety of problems without questioning the convergence of the improper integrals. This goes for some applied mathematicians too. The argument seems to be on the lines that if it gives what looks a reasonable answer, then fine! In our view, this takes the engineer's maxim "if it ain't broke, don't fix it" too far. This is primarily a mathematics textbook, therefore in this opening chapter we shall be more mathematically explicit than is customary in books on Laplace Transforms. In Chapter 4 there is some more pure mathematics when Fourier series are introduced. That is there for similar reasons. One mathematical question that ought to be asked concerns uniqueness. Given a function $F(t)$, its Laplace Transform is surely unique from the well defined nature of the improper integral. However, is it possible for two different functions to have the same Laplace Transform? To put the question a different but equivalent way, is there a function $N(t)$, not identically zero, whose Laplace Transform is zero? For this function, called a *null* function, could be added to any suitable function and the Laplace Transform would remain unchanged. Null functions do exist, but as long as we restrict ourselves to piecewise continuous functions this ceases to be a problem. Here is the definition of piecewise continuous:

Definition 1.1 *If an interval $[0, t_0]$ say can be partitioned into a finite number of subintervals $[0, t_1], [t_1, t_2], [t_2, t_3], \ldots, [t_n, t_0]$ with $0, t_1, t_2, \ldots, t_n, t_0$ an increasing sequence of times and such that a given function $f(t)$ is continuous in each of these subintervals but not necessarily at the end points themselves, then $f(t)$ is piecewise continuous in the interval $[0, t_0]$.*

Only functions that differ at a finite number of points have the same Laplace Transform. If $F_1(t) = F(t)$ except at a finite number of points where they differ by finite values then $\mathcal{L}\{F_1(t)\} = \mathcal{L}\{F(t)\}$. We mention this again in the next chapter when the inverse Laplace Transform is defined.

In this section, we shall examine the conditions for the existence of the Laplace Transform in more detail than is usual. In engineering texts, the simple definition followed by an explanation of exponential order is all that is required. Those that are satisfied with this can virtually skip the next few paragraphs and go on study the elementary properties, Section 1.3. However, some may need to know enough background in terms of the integrals, and so we devote a little space to some fundamentals. We will need to introduce improper integrals, but let us first define the Riemann integral. It is the integral we know and love, and is defined in terms of limits of sums. The strict definition runs as follows:-

Let $F(x)$ be a function which is defined and is bounded in the interval $a \leq x \leq b$ and suppose that m and M are respectively the lower and upper bounds of $F(x)$ in this interval (written $[a, b]$ see Appendix C). Take a set of points

$$x_0 = a, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_n = b$$

and write $\delta_r = x_r - x_{r-1}$. Let M_r, m_r be the bounds of $F(x)$ in the subinterval (x_{r-1}, x_r) and form the sums

$$S = \sum_{r=1}^{n} M_r \delta_r$$

$$s = \sum_{r=1}^{n} m_r \delta_r.$$

These are called respectively the upper and lower Riemann sums corresponding to the mode of subdivision. It is certainly clear that $S \geq s$. There are a variety of ways that can be used to partition the interval (a, b) and each way will have (in general) different M_r and m_r leading to different S and s. Let M be the minimum of all possible M_r and m be the maximum of all possible m_r A lower bound or supremum for the set S is therefore $M(b-a)$ and an upper bound or infimum for the set s is $m(b-a)$. These bounds are of course rough. There are *exact* bounds for S and s, call them J and I respectively. If $I = J$, $F(x)$ is said to be Riemann integrable in (a, b) and the value of the integral is I or J and is denoted by

$$I = J = \int_a^b F(x)dx.$$

For the purist it turns out that the Riemann integral is not quite general enough, and the Stieltjes integral is actually required. However, we will not use this concept which belongs securely in specialist final stage or graduate texts.

The improper integral is defined in the obvious way by taking the limit:

$$\lim_{R\to\infty} \int_a^R F(x)dx = \int_0^\infty F(x)dx$$

provided $F(x)$ is continuous in the interval $a \leq x \leq R$ for every R, and the limit on the left exists.

This is enough of general theory, we now apply it to the Laplace Transform. The parameter x is defined to take the increasing values from 0 to ∞. The condition $|F(x)| \leq Me^{\alpha x}$ is termed "$F(x)$ is of exponential order" and is, speaking loosely, quite a weak condition. All polynomial functions and (of course) exponential functions of the type e^{kx} (k constant) are included as well as bounded functions. Excluded functions are those that have singularities such as $\ln(x)$ or $1/(x-1)$ and functions that have a growth rate more rapid than exponential, for example e^{x^2}. Functions that have a finite number of finite discontinuities are also included. These have a special role in the theory of Laplace Transforms (see Chapter 3) so we will not dwell on them here: suffice to say that a function such as

$$F(x) = \begin{cases} 1 & 2n < x < 2n+1 \\ 0 & 2n+1 < x < 2n+2 \end{cases} \quad \text{where } n = 0, 1, \ldots$$

is one example. However, the function

$$F(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

is excluded because although all the discontinuities are finite, there are infinitely many of them.

We shall now follow standard practice and use t (time) instead of x as the dummy variable.

1.3 Elementary Properties

The Laplace Transform has many interesting and useful properties, the most fundamental of which is linearity. It is linearity that enables us to add results together to deduce other more complicated ones and is so basic that we state it as a theorem and prove it first.

Theorem 1.2 (Linearity) *If $F_1(t)$ and $F_2(t)$ are two functions whose Laplace Transform exists, then*

$$\mathcal{L}\{aF_1(t) + bF_2(t)\} = a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\}$$

where a and b are arbitrary constants.

Proof

$$\begin{aligned} \mathcal{L}\{aF_1(t) + bF_2(t)\} &= \int_0^\infty (aF_1 + bF_2)e^{-st}dt \\ &= \int_0^\infty \left(aF_1e^{-st} + bF_2e^{-st}\right)dt \\ &= a\int_0^\infty F_1e^{-st}dt + b\int_0^\infty F_2e^{-st}dt \\ &= a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\} \end{aligned}$$

where we have assumed that

$$|F_1| \le M_1e^{\alpha_1 t} \text{ and } |F_2| \le M_2e^{\alpha_2 t}$$

so that

$$\begin{aligned} |aF_1 + bF_2| &\le |a||F_1| + |b||F_2| \\ &\le (|a|M_1 + |b|M_2)e^{\alpha_3 t} \end{aligned}$$

where $\alpha_3 = max\{\alpha_1, \alpha_2\}$. This proves the theorem.

□

In this section, we shall concentrate on those properties of the Laplace Transform that do not involve the calculus. The first of these takes the form of another theorem because of its generality.

Theorem 1.3 (First Shift Theorem) *If it is possible to choose constants M and α such that $|F(t)| \leq Me^{\alpha t}$, that is $F(t)$ is of exponential order, then*

$$\mathcal{L}\{e^{-bt}F(t)\} = f(s+b)$$

provided $b \leq \alpha$. (In practice if $F(t)$ is of exponential order then the constant α can be chosen such that this inequality holds.)

Proof The proof is straightforward and runs as follows:-

$$\begin{aligned}
\mathcal{L}\{e^{-bt}F(t)\} &= \lim_{T\to\infty}\int_0^T e^{-st}e^{-bt}F(t)dt \\
&= \int_0^\infty e^{-st}e^{-bt}F(t)dt \text{ (as the limit exists)} \\
&= \int_0^\infty e^{-(s+b)t}F(t)dt \\
&= f(s+b).
\end{aligned}$$

This establishes the theorem.

□

We shall make considerable use of this once we have established a few elementary Laplace Transforms. This we shall now proceed to do.

Example 1.4 *Find the Laplace Transform of the function $F(t) = t$.*

Solution Using the definition of Laplace Transform,

$$\mathcal{L}(t) = \lim_{T\to\infty}\int_0^T te^{-st}dt.$$

Now, we have that

$$\begin{aligned}
\int_0^T te^{-st}dt &= \left[-\frac{t}{s}e^{-st}\right]_0^T - \int_0^T -\frac{1}{s}e^{-st}dt \\
&= -\frac{T}{s}e^{-sT} + \left[-\frac{1}{s^2}e^{-st}\right]_0^T \\
&= -\frac{T}{s}e^{-sT} - \frac{1}{s^2}e^{-sT} + \frac{1}{s^2}
\end{aligned}$$

this last expression tends to $\frac{1}{s^2}$ as $T \to \infty$.

Hence we have the result

$$\mathcal{L}(t) = \frac{1}{s^2}.$$

We can generalise this result to deduce the following result:
Corollary

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad n \text{ a positive integer.}$$

Proof The proof is straightforward:

$$\begin{aligned}
\mathcal{L}(t^n) &= \int_0^\infty t^n e^{-st} dt \quad \text{this time taking the limit straight away} \\
&= \left[-\frac{t^n}{s} e^{-st} \right]_0^\infty + \int_0^\infty \frac{nt^{n-1}}{s} e^{-st} dt \\
&= \frac{n}{s} \mathcal{L}(t^{n-1}).
\end{aligned}$$

If we put $n = 2$ in this recurrence relation we obtain

$$\mathcal{L}(t^2) = \frac{2}{s}\mathcal{L}(t) = \frac{2}{s^3}.$$

If we assume

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

then

$$\mathcal{L}(t^{n+1}) = \frac{n+1}{s} \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}.$$

This establishes that

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

by induction.

□

Example 1.5 *Find the Laplace Transform of $\mathcal{L}\{te^{at}\}$ and deduce the value of $\mathcal{L}\{t^n e^{at}\}$, where a is a real constant and n a positive integer.*

Solution Using the first shift theorem with $b = -a$ gives

$$\mathcal{L}\{F(t)e^{at}\} = f(s-a)$$

so with

$$F(t) = t \text{ and } f = \frac{1}{s^2}$$

we get

$$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}.$$

Using $F(t) = t^n$ the formula

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

follows.

Later, we shall generalise this formula further, extending to the case where n is not an integer.

We move on to consider the Laplace Transform of trigonometric functions. Specifically, we shall calculate $\mathcal{L}\{\sin(t)\}$ and $\mathcal{L}\{\cos(t)\}$. It is unfortunate, but the Laplace Transform of the other common trigonometric functions tan, cot, csc and sec do not exist as they all have singularities for finite t. The condition that the function $F(t)$ has to be of exponential order is not obeyed by any of these singular trigonometric functions as can be seen, for example, by noting that

$$|e^{-at}\tan(t)| \to \infty \text{ as } t \to \pi/2$$

and

$$|e^{-at}\cot(t)| \to \infty \text{ as } t \to 0$$

for all values of the constant a. Similarly neither csc nor sec are of exponential order.

In order to find the Laplace Transform of $\sin(t)$ and $\cos(t)$ it is best to determine $\mathcal{L}(e^{it})$ where $i = \sqrt{(-1)}$. The function e^{it} is complex valued, but it is both continuous and bounded for all t so its Laplace Transform certainly exists. Taking the Laplace Transform,

$$\begin{aligned}\mathcal{L}(e^{it}) &= \int_0^\infty e^{-st}e^{it}dt \\ &= \int_0^\infty e^{t(i-s)}dt \\ &= \left[\frac{e^{(i-s)t}}{i-s}\right]_0^\infty \\ &= \frac{1}{s-i} \\ &= \frac{s}{s^2+1} + i\frac{1}{s^2+1}.\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{L}(e^{it}) &= \mathcal{L}(\cos(t) + i\sin(t)) \\ &= \mathcal{L}(\cos(t)) + i\mathcal{L}(\sin(t)).\end{aligned}$$

Equating real and imaginary parts gives the two results

$$\mathcal{L}(\cos(t)) = \frac{s}{s^2+1}$$

and

$$\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}.$$

The linearity property has been used here, and will be used in future without further comment.

Given that the restriction on the type of function one can Laplace Transform is weak, i.e. it has to be of exponential order and have at most a finite number of finite jumps, one can find the Laplace Transform of any polynomial, any combination of polynomial with sinusoidal functions and combinations of these with exponentials (provided the exponential functions grow at a rate $\leq e^{at}$ where a is a constant). We can therefore approach the problem of calculating the Laplace Transform of power series. It is possible to take the Laplace Transform of a power series term by term as long as the series uniformly converges to a piecewise continuous function. We shall investigate this further later in the text; meanwhile let us look at the Laplace Transform of functions that are not even continuous.

Functions that are not continuous occur naturally in branches of electrical and control engineering, and in the software industry. One only has to think of switches to realise how widespread discontinuous functions are throughout electronics and computing.

Example 1.6 *Find the Laplace Transform of the function represented by $F(t)$ where*

$$F(t) = \begin{cases} t & 0 \leq t < t_0 \\ 2t_0 - t & t_0 \leq t \leq 2t_0 \\ 0 & t > 2t_0. \end{cases}$$

Solution This function is of the "saw-tooth" variety that is quite common in electrical engineering. There is no question that it is of exponential order and that

$$\int_0^\infty e^{-st}F(t)dt$$

exists and is well defined. $F(t)$ is continuous but not differentiable. This is not troublesome. Carrying out the calculation is a little messy and the details can be checked using computer algebra.

$$\begin{aligned}
\mathcal{L}(F(t)) &= \int_0^\infty e^{-st}F(t)dt \\
&= \int_0^{t_0} te^{-st}dt + \int_{t_0}^{2t_0}(2t_0 - t)e^{-st}dt \\
&= \left[-\frac{t}{s}e^{-st}\right]_0^{t_0} + \int_0^{t_0}\frac{1}{s}e^{-st}dt + \left[-\frac{2t_0 - t}{s}e^{-st}\right]_{t_0}^{2t_0} - \int_{t_0}^{2t_0}\frac{1}{s}e^{-st}dt \\
&= -\frac{t_0}{s}e^{-st_0} - \frac{1}{s^2}\left[e^{-st}\right]_0^{t_0} + \frac{t_0}{s}e^{-st_0} + \frac{1}{s^2}\left[e^{-st}\right]_{t_0}^{2t_0} \\
&= \frac{1}{s^2}\left[e^{-st_0} - 1\right] + \frac{1}{s^2}\left[e^{-2st_0} - e^{-st_0}\right] \\
&= \frac{1}{s^2}\left[1 - 2e^{-st_0} + e^{-2st_0}\right] \\
&= \frac{1}{s^2}\left[1 - e^{-st_0}\right]^2
\end{aligned}$$

$$= \frac{4}{s^2} e^{-st_0} \sinh^2(\frac{1}{2} s t_0).$$

In the next chapter we shall investigate in more detail the properties of discontinuous functions such as the Heaviside unit step function. As an introduction to this, let us do the following example.

Example 1.7 *Determine the Laplace Transform of the step function $F(t)$ defined by*

$$F(t) = \begin{cases} 0 & 0 \leq t < t_0 \\ a & t \geq t_0. \end{cases}$$

Solution $F(t)$ itself is bounded, so there is no question that it is also of exponential order. The Laplace Transform of $F(t)$ is therefore

$$\begin{aligned} \mathcal{L}(F(t)) &= \int_0^\infty e^{-st} F(t) dt \\ &= \int_{t_0}^\infty a e^{-st} dt \\ &= \left[-\frac{a}{s} e^{-st} \right]_{t_0}^\infty \\ &= \frac{a}{s} e^{-st_0}. \end{aligned}$$

Here is another useful result.

Theorem 1.8 *If $\mathcal{L}(F(t)) = f(s)$ then $\mathcal{L}(tF(t)) = -\frac{d}{ds} f(s)$ and in general $\mathcal{L}(t^n F(t)) = (-1)^n \frac{d^n}{ds^n} f(s)$.*

Proof Let us start with the definition of Laplace Transform

$$\mathcal{L}(F(t)) = \int_0^\infty e^{-st} F(t) dt$$

and differentiate this with respect to s to give

$$\begin{aligned} \frac{df}{ds} &= \frac{d}{ds} \int_0^\infty e^{-st} F(t) dt \\ &= \int_0^\infty -t e^{-st} F(t) dt \end{aligned}$$

assuming absolute convergence to justify interchanging differentiation and (improper) integration. Hence

$$\mathcal{L}(tF(t)) = -\frac{d}{ds} f(s).$$

One can now see how to progress by induction. Assume the result holds for n, so that

$$\mathcal{L}(t^n F(t)) = (-1)^n \frac{d^n}{ds^n} f(s)$$

and differentiate both sides with respect to s (assuming all appropriate convergence properties) to give

$$\int_0^\infty -t^{n+1} e^{-st} F(t) dt = (-1)^n \frac{d^{n+1}}{ds^{n+1}} f(s)$$

or

$$\int_0^\infty t^{n+1} e^{-st} F(t) dt = (-1)^{n+1} \frac{d^{n+1}}{ds^{n+1}} f(s).$$

So

$$\mathcal{L}(t^{n+1} F(t)) = (-1)^{n+1} \frac{d^{n+1}}{ds^{n+1}} f(s)$$

which establishes the result by induction.

□

Example 1.9 *Determine the Laplace Transform of the function* $t\sin(t)$.

Solution To evaluate this Laplace Transform we use Theorem 1.8 with $f(t) = \sin(t)$. This gives

$$\mathcal{L}\{t\sin(t)\} = -\frac{d}{ds}\left\{\frac{1}{1+s^2}\right\} = \frac{2s}{(1+s^2)^2}$$

which is the required result.

1.4 Exercises

1. For each of the following functions, determine which has a Laplace Transform. If it exists, find it; if it does not, say briefly why.

   (a) $\ln(t)$, (b) e^{3t}, (c) e^{t^2}, (d) $e^{1/t}$, (e) $1/t$,

   (f) $f(t) = \begin{cases} 1 \text{ if } t \text{ is even} \\ 0 \text{ if } t \text{ is odd.} \end{cases}$

2. Determine from first principles the Laplace Transform of the following functions:-

   (a) e^{kt}, (b) t^2, (c) $\cosh(t)$.

3. Find the Laplace Transforms of the following functions:-

(a) t^2e^{-3t}, (b) $4t + 6e^{4t}$, (c) $e^{-4t}\sin(5t)$.

4. Find the Laplace Transform of the function $F(t)$, where $F(t)$ is given by

$$F(t) = \begin{cases} t & 0 \leq t < 1 \\ 2-t & 1 \leq t < 2 \\ 0 & \text{otherwise.} \end{cases}$$

5. Use the property of Theorem 1.8 to determine the following Laplace Transforms

(a) te^{2t}, (b) $t\cos(t)$, (c) $t^2\cos(t)$.

6. Find the Laplace Transforms of the following functions:-

(a) $\sin(\omega t + \phi)$, (b) $e^{5t}\cosh(6t)$.

7. If $G(at + b) = F(t)$ determine the Laplace Transform of G in terms of $\mathcal{L}\{F\} = \bar{f}(s)$ and a finite integral.

8. Prove the following change of scale result:-

$$\mathcal{L}\{F(at)\} = \frac{1}{a}f\left(\frac{s}{a}\right).$$

Hence evaluate the Laplace Transforms of the two functions

(a) $t\cos(6t)$, (b) $t^2\cos(7t)$.