Er redet nur von Regelflächen,
ich werd' mich an dem Flegel rächen!

Dedicated to the memory of Karl Stein (1913 - 2000)

Preface

The simplest surfaces, aside from planes, are the traces of a line moving in ambient space or, more precisely, the unions of one-parameter families of lines. The fact that these lines can be produced using a ruler explains their name, "ruled surfaces." The mechanical production of ruled surfaces is relatively easy, and they can be visualized by means of wire models. These surfaces are not only of practical use, but also provide artistic inspiration.

Mathematically, ruled surfaces are the subject of several branches of geometry, especially differential geometry and algebraic geometry. In classical geometry, we know that surfaces of vanishing Gaussian curvature have a ruling that is even developable. Analytically, developable means that the tangent plane is the same for all points of the ruling line, which is equivalent to saying that the surface can be covered by pieces of paper. A classical result from algebraic geometry states that rulings are very rare for complex algebraic surfaces in three-space: Quadrics have two rulings, smooth cubics contain precisely twenty-seven lines, and in general, a surface of degree at least four contains no line at all. There are exceptions, such as cones or tangent surfaces of curves. It is also well-known that these two kinds of surfaces are the only developable ruled algebraic surfaces in projective three-space.

The natural generalization of a ruled surface is a ruled variety, i.e., a variety of arbitrary dimension that is "swept out" by a moving linear subspace of ambient space. It should be noted that a ruling is not an intrinsic but an extrinsic property of a variety, which only makes sense relative to an ambient affine or projective space. In this book, we consider ruled varieties mainly from the point of view of complex projective algebraic geometry, where the strongest tools are available. Some local techniques could be generalized to complex analytic varieties, but in the real analytic or even differentiable case there is little hope for generalization: the reason being that rulings, and especially developable rulings, have the tendency to produce severe singularities.

As in the classical case of surfaces, there is a strong relationship between the subject of this book, ruled varieties, and differential geometry. For our purpose, however, the Hermitian Fubini-Study metric and the related concepts of curvature are not necessary. In order to detect developable rulings, it suffices to consider a bilinear second fundamental form that is the differential of the Gauß map. This method does not give curvature as a number, but rather measures the degree of vanishing of curvature; this point of view has been used in a fundamental paper of GRIFFITHS and HARRIS [GH_2]. The purposes of this book are to make parts of this paper more accessible, to give detailed and more elementary proofs, and to report on recent progress in this area. This text can also serve as an introduction to "bilinear" complex differential geometry, a useful method in algebraic geometry.

In Chapter 0 we recall some classical facts about developable surfaces in real affine and complex projective three-space. The basic results are *a)* the linearity of the fibers of the Gauß map and *b)* the classification of developable surfaces, locally in the real differentiable case and globally in the complex projective case. We present very elementary proofs, which can be adapted to more general situations.

Families of linear spaces correspond to subsets of Grassmannians. Chapter 1 contains all the facts we need about these varieties and more; for instance, there is a meticulous and yet simple proof of the Plücker relations, which is not easy to find in the existing literature. Varieties that are unions of one-parameter families of linear spaces correspond to curves in the Grassmannian. Following [GH_2], we derive a normal form representation of such curves, which then implies the classification of developable varieties of Gauß rank one in Chapter 2.

Chapter 2 is the core of the book. We first introduce the different concepts of rulings of a variety and methods to construct them. Then we concentrate on the additional condition of developability; this class of ruled varieties is accessible to the methods of differential geometry, especially the Gauß map and the associated second fundamental form. The basic theorem of this chapter is the linearity of the fibers of the Gauß map, for which we give proofs from various points of view. This result shows the Gauß ruling to be the maximal developable ruling and the Gauß rank to be an invariant of the variety.

The central problem concerning developable varieties is their classification, and there are two cases where it can be solved:

- Arbitrary dimension and Gauß rank one (2.2.8).
- Codimension one and Gauß rank two (2.5.6).

The key to the second case is duality or, more precisely, the dual variety of a given variety, which is a subvariety of the dual projective space.

Chapter 3 is devoted to a very important class of ruled varieties: tangent and secant varieties. In general, the canonical ruling of the tangent variety is not developable, but it contains a developable subruling by lines. An amazing result is the formation of a developable ruling with larger fiber dimension if the tangent variety has a smaller than expected dimension. The dimension of the tangent variety, like nearly all other invariants in this book, can be computed from the second fundamental form. This is not the case, however, for the dimension of the secant variety, where third order invariants become important. Until now, all attempts to find the right third or higher order invariant for the problem of this dimension have failed, even for smooth varieties. Thus the book closes with one more open problem.

The only prerequisites for this textbook are the basic concepts of complex affine and projective algebraic geometry, the results of which are outlined in Chapter 1. The relatively elementary methods that we use throughout are based on linear algebra with algebraic or holomorphic parameters.

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