Preface

The body of this text is by now written it remains to find some words to explain to you what to expect from this book.

This could be a first attempt of characterizing the content:

MSC 2000: 05-01, 05C10, 05C62, 52-01, 52C10, 52C30, 52C42.

In words: The questions posed and partially answered in this book are from the intersection of graph theory and discrete geometry. There is some graph theory with a geometric flavor and there is combinatorial geometry of the plane. Though, the investigations always start in the geometry of the plane it is sometimes appropriate to go to higher dimensions to get a more global understanding of the structures under investigation. This happens e.g. in Chapter 7 when the study of triangulations of a point configuration leads to the definition of secondary polytopes.

I like to think of the book as a collection which makes up a kind of bouquet. A bouquet of problems, ideas and results, each of a special character and beauty combined in the intention that they supplement each other to form an interesting and appealing whole.

The main mathematical part of the text contains few citations and sparse references to related material. These additional informations are supplemented in the last section 'Notes and References' of each chapter. On average the bibliography of a chapter contains 30 items. Still, this is far from being a thorough collection of the related literature. The intention is to provide meaningful pointers to the literature so that these sections can serve as entry points for further studies. I have supplemented the text by many figures they should make the material more attractive and help getting a sensual impression of the objects. In some cases I have confined the presentation to results which fall behind todays state of the art. The intention is to emphasize the main ideas and stop before the technical complexity starts taking over. This strategy should make the mathematics accessibility to a rather broad audience including students of computer science, students of mathematics, instructors and researchers.

The idea is that the book can serve different purposes. It may be used as textbook for a course or as a collection of material for a seminar. It should be helpful to people who want to learn one or more specific themes, they can read chapters as stand alone surveys.

Topics

Chapter 1. We introduce basic notions of graph theory and learn what geometric and topological graphs are. Planar graphs and some big theorems about them are reviewed. The main results of this chapter are bounds for some extremal problems for geometric graphs.

Chapter 2. We show that a 3-connected planar graph with f faces admits a convex drawing on the $(f-1)\times(f-1)$ grid. The result is based on Schnyder woods, a special cover of the edges of a 3-connected planar graph with three trees. Schnyder woods bring along connections to geodesic embeddings of planar graphs and to the order dimension of planar graphs and 3-polytopes.

Chapter 3. This is about non-planar graphs. How many crossing pairs of edges do we need in any drawing of a given graph in the plane? The Crossing Lemma provides a bound and has beautiful applications to deep extremal problems. We explain some of them.

Chapter 4. Let $\mathcal{P}$ be a configuration of n points in the plane. A k-set of $\mathcal{P}$ is a subset $\mathcal{S}$ of k points of $\mathcal{P}$ which can be separated from the complement $\bar{\mathcal{S}} = \mathcal{P} \setminus \mathcal{S}$ by a line. The notorious k-set problem of discrete geometry asks for asymptotic bounds of this number as a function of n. We present some bounds, prove Welzl's generalization of the Lovász Lemma to higher dimensions and close with the surprisingly related problem of bounding the rectilinear crossing number of complete graphs from below.

Chapter 5. This chapter contains selected results from the extremal theory for configurations of points and arrangements of lines. The main results are bounds for the number of ordinary lines of a point configuration and for the number of triangles of an arrangement.

Chapter 6. Compared to arrangements of lines, arrangements of pseudolines have the advantage that they can be encoded nicely by combinatorial data. We introduce several combinatorial representations and prove relations between them. For each representation we give an applications which makes use of specific properties. The encoding by triangle orientations has a natural generalization which leads to higher Bruhat orders.

Chapter 7. In this chapter we study triangulations of a point configuration. The flip operation allows to move between different triangulations. The Delaunay triangulation is investigated as a special element in the graph of triangulations. This graph is shown to be related to the skeleton graph of the secondary polytope. In the special case of a point configuration in convex position they coincide. To bound the diameter of the graph of triangulations in this case we have to go through hyperbolic geometry.

Chapter 8. Rigidity allows a different view to geometric graphs. We introduce rigidity theory and prove three characterizations of minimal generically rigid graphs in the plane. Pseudotriangulations are shown to be the planar minimal generically rigid graphs in the plane. The set of pseudotriangulations with vertices embedded in a fixed point configuration $\mathcal{P}$ has a nice structure. There is a notion of flip that allows to move between different pseudotriangulations. The flip-graph is a connected graph and it turns out that it is the skeleton graph of a polytope. The beautiful theory finds a surprising application in the Carpenter's Rule Problem.

The selection of topics is clearly governed by my personal taste. This is a drawback, because your taste is likely to differ from mine at least in some details. The advantage is that though there are several books on related topics each of the books is clearly distinguishable by its style and content. In the spirit of 'Customers who bought this book also bought' I advertise the following four books:

- J. Matoušek
  Lectures on Discrete Geometry
  Graduate Texts in Math. 212
  Springer-Verlag, 2002.

- J. Pach and P. K. Agarwal
  Combinatorial Geometry
  John Wiley & Sons, 1995.

- G. M. Ziegler
  Lectures on Polytopes
  Graduate Texts in Math. 152
  Springer-Verlag, 1994.

- H. Edelsbrunner
  Geometry and Topology for Mesh Generation,
  Cambridge University Press, 2001.

Feedback

There will be something wrong. You may find errors of different nature. I may inadvertently not have given proper credit for some contribution. You may know of new work or have additional comments. In all these cases: Please let me know.

- `felsner@math.tu-berlin.de`

You can find an erratum and a collection of comments and pointers related to the book at the web-location:

- `http://www.math.tu-berlin.de/~felsner/gga-book.html`

Acknowledgments

There are *sine qua non* conditions for the existence of this book:

- The work of a vast number of mathematicians who laid out the ground and produced what I am retelling.
- Friends and family of mine, in particular Diana and my parents, without their belief I would not have been able to finish this task.

I'm truly grateful to these people.

In Berlin we have a wonderful environment for discrete mathematics. While working at this book I had the privilege of being member of teams of discrete mathematicians at Freie Universität and at Technische Unversität. I want to thank the nice people in these groups for providing me with such a friendly and supportive 'ambiente'.

I have been involved in the European graduate college 'Combinatorics, Geometry and Computing' and in the European network COMBSTRU and acknowledge support from their side.

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Stefan Felsner