10

Application to Optimal Stopping

10.1 The Time-Homogeneous Case

Problem 5 in the introduction is a special case of a problem of the following type:

Problem 10.1.1 (The optimal stopping problem)
Let X_t be an Itô diffusion on $\mathbf{R}^n$ and let g (*the reward function*) be a given function on $\mathbf{R}^n$, satisfying

a) $g(\xi) \geq 0$ for all $\xi \in \mathbf{R}^n$ (10.1.1)
b) g is continuous.

Find a stopping time $\tau^* = \tau^*(x, \omega)$ (called an *optimal stopping time*) for $\{X_t\}$ such that

$$E^x[g(X_{\tau^*})] = \sup_{\tau} E^x[g(X_\tau)] \qquad \text{for all } x \in \mathbf{R}^n , \tag{10.1.2}$$

the sup being taken over all stopping times τ for $\{X_t\}$. We also want to find the corresponding optimal expected reward

$$g^*(x) = E^x[g(X_{\tau^*})] . \tag{10.1.3}$$

Here $g(X_\tau)$ is to be interpreted as 0 at the points $\omega \in \Omega$ where $\tau(\omega) = \infty$ and as usual E^x denotes the expectation with respect to the probability law Q^x of the process X_t; $t \geq 0$ starting at $X_0 = x \in \mathbf{R}^n$.

We may regard X_t as the state of a game at time t, each ω corresponds to one sample of the game. For each time t we have the option of stopping the game, thereby obtaining the reward $g(X_t)$, or continue the game in the hope that stopping it at a later time will give a bigger reward. The problem is of course that we do not know what state the game is in at future times, only the probability distribution of the "future". Mathematically, this means that the possible "stopping" times we consider really are stopping times in the sense of Definition 7.2.1: The decision whether $\tau \le t$ or not should only depend on the behaviour of the Brownian motion B_r (driving the process X) up to time t, or perhaps only on the behaviour of X_r up to time t. So, among all possible stopping times τ we are asking for the optimal one, τ^*, which gives the best result "in the long run", i.e. the biggest expected reward in the sense of (10.1.2).

In the following we will outline how a solution to this problem can be obtained using the material from the preceding chapter. Later in this chapter we shall see that our discussion of problem (10.1.2)–(10.1.3) also covers the apparently more general problems

$$g^*(s,x) = \sup_\tau E^{(s,x)}[g(s+\tau, X_\tau)] = E^{(s,x)}[g(s+\tau^*, X_{\tau^*})] \qquad (10.1.4)$$

and

$$\begin{aligned} G^*(s,x) &= \sup_\tau E^{(s,x)}\Big[\int_0^\tau f(s+t,X_t)dt + g(s+\tau, X_\tau)\Big] \\ &= E^{(s,x)}\Big[\int_0^{\tau^*} f(s+t,X_t)dt + g(s+\tau^*, X_{\tau^*})\Big] \end{aligned} \qquad (10.1.5)$$

where f is a given *profit rate (or reward rate) function* (satisfying certain conditions).

We shall also discuss possible extensions of problem (10.1.2)–(10.1.3) to cases where g is not necessarily continuous or where g may assume negative values.

A basic concept in the solution of (10.1.2)–(10.1.3) is the following:

Definition 10.1.2 *A measurable function* $f: \mathbf{R}^n \to [0,\infty]$ *is called* supermeanvalued *(w.r.t.* X_t*) if*

$$f(x) \ge E^x[f(X_\tau)] \qquad (10.1.6)$$

for all stopping times τ *and all* $x \in \mathbf{R}^n$.

If, in addition, f *is also* lower semicontinuous*, then* f *is called* l.s.c. superharmonic *or just* superharmonic *(w.r.t.* X_t*).*

Note that if $f: \mathbf{R}^n \to [0,\infty]$ is lower semicontinuous then by the Fatou lemma

$$f(x) \le E^x[\underline{\lim}_{k\to\infty} f(X_{\tau_k})] \le \underline{\lim}_{k\to\infty} E^x[f(X_{\tau_k})] , \qquad (10.1.7)$$

for any sequence $\{\tau_k\}$ of stopping times such that $\tau_k \to 0$ a.s. P. Combining this with (10.1.6) we see that if f is (l.s.c.) superharmonic, then

$$f(x) = \lim_{k\to\infty} E^x[f(X_{\tau_k})] \qquad \text{for all } x , \qquad (10.1.8)$$

for all such sequences τ_k.

Remarks. 1) In the literature (see e.g. Dynkin (1965 II)) one often finds a weaker concept of X_t-superharmonicity, defined by the supermeanvalued property (10.1.6) plus the stochastic continuity requirement (10.1.8). This weaker concept corresponds to the X_t-harmonicity defined in Chapter 9.

2) If $f \in C^2(\mathbf{R}^n)$ it follows from Dynkin's formula that f is superharmonic w.r.t. X_t if and only if

$$\mathcal{A}f \leq 0$$

where $\mathcal{A}$ is the characteristic operator of X_t. This is often a useful criterion (See e.g. Example 10.2.1).

3) If $X_t = B_t$ is Brownian motion in $\mathbf{R}^n$ then the superharmonic functions for X_t coincide with the (nonnegative) superharmonic functions in classical potential theory. See Doob (1984) or Port and Stone (1979).

We state some useful properties of superharmonic and supermeanvalued functions.

Lemma 10.1.3 a) *If f is superharmonic (supermeanvalued) and $\alpha > 0$, then αf is superharmonic (supermeanvalued).*

b) *If f_1, f_2 are superharmonic (supermeanvalued), then $f_1 + f_2$ is superharmonic (supermeanvalued).*

c) *If $\{f_j\}_{j \in J}$ is a family of supermeanvalued functions, then $f(x) := \inf_{j \in J}\{f_j(x)\}$ is supermeanvalued if it is measurable (J is any set).*

d) *If $f_1, f_2, \cdots$ are superharmonic (supermeanvalued) functions and $f_k \uparrow f$ pointwise, then f is superharmonic (supermeanvalued).*

e) *If f is supermeanvalued and $\sigma \leq \tau$ are stopping times, then $E^x[f(X_\sigma)] \geq E^x[f(X_\tau)]$.*

f) *If f is supermeanvalued and H is a Borel set, then $\widetilde{f}(x) := E^x[f(X_{\tau_H})]$ is supermeanvalued.*

Proof of Lemma 10.1.3.

a) and b) are straightforward.

c) Suppose f_j is supermeanvalued for all $j \in J$. Then

$$f_j(x) \geq E^x[f_j(X_\tau)] \geq E^x[f(X_\tau)] \qquad \text{for all } j \ .$$

So $f(x) = \inf f_j(x) \geq E^x[f(X_\tau)]$, as required.

d) Suppose f_j is supermeanvalued, $f_j \uparrow f$. Then

$$\begin{aligned} f(x) &\geq f_j(x) \geq E^x[f_j(X_\tau)] \qquad \text{for all } j, \text{ so} \\ f(x) &\geq \lim_{j\to\infty} E^x[f_j(X_\tau)] = E^x[f(X_\tau)] \ , \end{aligned}$$

by monotone convergence. Hence f is supermeanvalued. If each f_j is also lower semicontinuous then if $y_k \to x$ as $k \to \infty$ we have

$$f_j(x) \leq \varliminf_{k\to\infty} f_j(y_k) \leq \varliminf_{k\to\infty} f(y_k) \qquad \text{for each } j \ .$$

Hence, by letting $j \to \infty$,

$$f(x) \le \varliminf_{k\to\infty} f(y_k) \ .$$

e) If f is supermeanvalued we have by the Markov property when $t > s$

$$E^x[f(X_t)|\mathcal{F}_s] = E^{X_s}[f(X_{t-s})] \le f(X_s) \ , \tag{10.1.9}$$

i.e. the process

$$\zeta_t = f(X_t)$$

is a *supermartingale* w.r.t. the σ-algebras $\mathcal{F}_t$ generated by $\{B_r; r \le t\}$. (Appendix C). Therefore, by Doob's optional sampling theorem (see Gihman and Skorohod (1975, Theorem 6 p. 11)) we have

$$E^x[f(X_\sigma)] \ge E^x[f(X_\tau)]$$

for all stopping times σ, τ with $\sigma \le \tau$ a.s. Q^x.

f) Suppose f is supermeanvalued. By the strong Markov property (7.2.2) and formula (7.2.6) we have, for any stopping time α,

$$\begin{aligned} E^x[\widetilde{f}(X_\alpha)] &= E^x[E^{X_\alpha}[f(X_{\tau_H})]] = E^x[E^x[\theta_\alpha f(X_{\tau_H})|\mathcal{F}_\alpha]] \\ &= E^x[\theta_\alpha f(X_{\tau_H})] = E^x[f(X_{\tau_H^\alpha})] \end{aligned} \tag{10.1.10}$$

where $\tau_H^\alpha = \inf\{t > \alpha; X_t \notin H\}$. Since $\tau_H^\alpha \ge \tau_H$ we have by e)

$$E^x[\widetilde{f}(X_\alpha)] \le E^x[f(X_{\tau_H})] = \widetilde{f}(x) \ ,$$

so $\widetilde{f}$ is supermeanvalued. □

The following concepts are fundamental:

Definition 10.1.4 *Let h be a real measurable function on $\mathbf{R}^n$. If f is a superharmonic (supermeanvalued) function and $f \ge h$ we say that f is a* superharmonic (supermeanvalued) majorant *of h (w.r.t. X_t). The function*

$$\overline{h}(x) = \inf_f f(x); \quad x \in \mathbf{R}^n \ , \tag{10.1.11}$$

the inf *being taken over all supermeanvalued majorants f of h, is called the* least supermeanvalued majorant *of h.*

Similarly, suppose there exists a function $\widehat{h}$ such that

(i) $\widehat{h}$ is a superharmonic majorant of h and
(ii) if f is any other superharmonic majorant of h then $\widehat{h} \le f$.

Then $\widehat{h}$ is called *the least superharmonic majorant* of h.

Note that by Lemma 10.1.3 c) the function $\overline{h}$ is supermeanvalued if it is measurable. Moreover, if $\overline{h}$ is lower semicontinuous, then $\widehat{h}$ exists and $\widehat{h} = \overline{h}$. Later we will prove that if g is nonnegative (or lower bounded) and lower semicontinuous, then $\widehat{g}$ exists and $\widehat{g} = \overline{g}$ (Theorem 10.1.7).

Let $g \ge 0$ and let f be a supermeanvalued majorant of g. Then if τ is a stopping time

$$f(x) \ge E^x[f(X_\tau)] \ge E^x[g(X_\tau)] \ .$$

So

$$f(x) \ge \sup_\tau E^x[g(X_\tau)] = g^*(x) \ .$$

Therefore we always have, if $\widehat{g}$ exists,

$$\widehat{g}(x) \ge g^*(x) \qquad \text{for all } x \in \mathbf{R}^n \ . \tag{10.1.12}$$

What is not so easy to see is that the converse inequality also holds, i.e. that in fact

$$\widehat{g} = g^* \ . \tag{10.1.13}$$

We will prove this after we have established a useful iterative procedure for calculating $\widehat{g}$. Before we give such a procedure let us introduce a concept which is related to superharmonic functions:

Definition 10.1.5 *A lower semicontinuous function* $f\colon \mathbf{R}^n \to [0,\infty]$ *is called* excessive *(w.r.t.* X_t*) if*

$$f(x) \ge E^x[f(X_s)] \qquad \textit{for all } s \ge 0,\ x \in \mathbf{R}^n \ . \tag{10.1.14}$$

It is clear that a superharmonic function must be excessive. What is not so obvious, is that the converse also holds:

Theorem 10.1.6 *Let* $f\colon \mathbf{R}^n \to [0,\infty]$*. Then* f *is excessive w.r.t.* X_t *if and only if* f *is superharmonic w.r.t.* X_t*.*

Proof in a special case. Let L be the differential operator associated to X (given by the right hand side of (7.3.3)), so that L coincides with the generator A of X on C_0^2. We only prove the theorem in the special case when $f \in C^2(\mathbf{R}^n)$ and Lf is bounded: Then by Dynkin's formula we have

$$E^x[f(X_t)] = f(x) + E^x\left[\int_0^t Lf(X_r)dr\right] \qquad \text{for all } t \ge 0 \ ,$$

so if f is excessive then $Lf \le 0$. Therefore, if τ is a stopping time we get

$$E^x[f(X_{t\wedge\tau})] \le f(x) \qquad \text{for all } t \ge 0 \ .$$

Letting $t \to \infty$ we see that f is superharmonic. □

A proof in the general case can be found in Dynkin (1965 II, p. 5).

The first iterative procedure for the least superharmonic majorant $\widehat{g}$ of g is the following:

Theorem 10.1.7 (Construction of the least superharmonic majorant) *Let $g = g_0$ be a nonnegative, lower semicontinuous function on $\mathbf{R}^n$ and define inductively*

$$g_n(x) = \sup_{t \in S_n} E^x[g_{n-1}(X_t)] \ , \tag{10.1.15}$$

where $S_n = \{k \cdot 2^{-n}; 0 \leq k \leq 4^n\}$, $n = 1, 2, \ldots$. Then $g_n \uparrow \widehat{g}$ and $\widehat{g}$ is the least superharmonic *majorant of g. Moreover, $\widehat{g} = \overline{g}$.*

Proof. Note that $\{g_n\}$ is increasing. Define $\check{g}(x) = \lim_{n \to \infty} g_n(x)$. Then

$$\check{g}(x) \geq g_n(x) \geq E^x[g_{n-1}(X_t)] \qquad \text{for all } n \text{ and all } t \in S_n \ .$$

Hence

$$\check{g}(x) \geq \lim_{n \to \infty} E^x[g_{n-1}(X_t)] = E^x[\check{g}(X_t)] \tag{10.1.16}$$

for all $t \in S = \bigcup_{n=1}^{\infty} S_n$.

Since $\check{g}$ is an increasing limit of lower semicontinuous functions (Lemma 8.1.4) $\check{g}$ is lower semicontinuous. Fix $t \in \mathbf{R}$ and choose $t_k \in S$ such that $t_k \to t$. Then by (10.1.16), the Fatou lemma and lower semicontinuity

$$\check{g}(x) \geq \varliminf_{k \to \infty} E^x[\check{g}(X_{t_k})] \geq E^x[\varliminf_{k \to \infty} \check{g}(X_{t_k})] \geq E^x[\check{g}(X_t)] \ .$$

So $\check{g}$ is an excessive function. Therefore $\check{g}$ is superharmonic by Theorem 10.1.6 and hence $\check{g}$ is a superharmonic majorant of g. On the other hand, if f is any supermeanvalued majorant of g, then clearly by induction

$$f(x) \geq g_n(x) \qquad \text{for all } n$$

and so $f(x) \geq \check{g}(x)$. This proves that $\check{g}$ is the least supermeanvalued majorant $\overline{g}$ of g. So $\check{g} = \widehat{g}$. □

It is a consequence of Theorem 10.1.7 that we may replace the finite sets S_n by the whole interval $[0, \infty]$:

Corollary 10.1.8 *Define $h_0 = g$ and inductively*

$$h_n(x) = \sup_{t \geq 0} E^x[h_{n-1}(X_t)] \ ; \qquad n = 1, 2, \ldots$$

Then $h_n \uparrow \widehat{g}$.

Proof. Let $h = \lim h_n$. Then clearly $h \geq \check{g} = \widehat{g}$. On the other hand, since $\widehat{g}$ is excessive we have

$$\widehat{g}(x) \geq \sup_{t \geq 0} E^x[\widehat{g}(X_t)].$$

So by induction

$$\widehat{g} \geq h_n \qquad \text{for all } n .$$

Thus $\widehat{g} = h$ and the proof is complete.

We are now ready for our first main result on the optimal stopping problem. The following result is basically due to Dynkin (1963) (and, in a martingale context, Snell (1952)):

Theorem 10.1.9 (Existence theorem for optimal stopping)
Let g^* denote the optimal reward and $\widehat{g}$ the least superharmonic majorant of a continuous reward function $g \geq 0$.

a) *Then*

$$g^*(x) = \widehat{g}(x) . \tag{10.1.17}$$

b) *For $\epsilon > 0$ let*

$$D_\epsilon = \{x; g(x) < \widehat{g}(x) - \epsilon\} . \tag{10.1.18}$$

Suppose g is bounded. *Then stopping at the first time τ_ϵ of exit from D_ϵ is close to being optimal, in the sense that*

$$|g^*(x) - E^x[g(X_{\tau_\epsilon})]| \leq 2\epsilon \tag{10.1.19}$$

for all x.

c) *For arbitrary continuous $g \geq 0$ let*

$$D = \{x; g(x) < g^*(x)\} \quad \text{(the continuation region)} . \tag{10.1.20}$$

For $N = 1, 2, \ldots$ define $g_N = g \wedge N$, $D_N = \{x; g_N(x) < \widehat{g_N}(x)\}$ and $\sigma_N = \tau_{D_N}$. Then $D_N \subset D_{N+1}$, $D_N \subset D \cap g^{-1}([0, N))$, $D = \bigcup_N D_N$. If $\sigma_N < \infty$ a.s. Q^x for all N then

$$g^*(x) = \lim_{N \to \infty} E^x[g(X_{\sigma_N})] . \tag{10.1.21}$$

d) *In particular, if $\tau_D < \infty$ a.s. Q^x and the family $\{g(X_{\sigma_N})\}_N$ is uniformly integrable w.r.t. Q^x (Appendix C), then*

$$g^*(x) = E^x[g(X_{\tau_D})]$$

and $\tau^* = \tau_D$ is an optimal stopping time.

Proof. First assume that g is bounded and define

$$\widetilde{g}_\epsilon(x) = E^x[\widehat{g}(X_{\tau_\epsilon})] \qquad \text{for } \epsilon > 0 \ . \tag{10.1.22}$$

Then $\widetilde{g}_\epsilon$ is supermeanvalued by Lemma 10.1.3 f). We claim that

$$g(x) \le \widetilde{g}_\epsilon(x) + \epsilon \qquad \text{for all } \ x \ . \tag{10.1.23}$$

To see this suppose

$$\beta := \sup_x \{g(x) - \widetilde{g}_\epsilon(x)\} > \epsilon \ . \tag{10.1.24}$$

Then for all $\eta > 0$ we can find x_0 such that

$$g(x_0) - \widetilde{g}_\epsilon(x_0) \ge \beta - \eta \ . \tag{10.1.25}$$

On the other hand, since $\widetilde{g}_\epsilon + \beta$ is a supermeanvalued majorant of g, we have

$$\widehat{g}(x_0) \le \widetilde{g}_\epsilon(x_0) + \beta \ . \tag{10.1.26}$$

Combining (10.1.25) and (10.1.26) we get

$$\widehat{g}(x_0) \le g(x_0) + \eta \ . \tag{10.1.27}$$

Consider the two possible cases:

Case 1: $\tau_\epsilon > 0$ a.s. Q^{x_0}. Then by (10.1.27) and the definition of D_ϵ

$$g(x_0) + \eta \ge \widehat{g}(x_0) \ge E^{x_0}[\widehat{g}(X_{t\wedge\tau_\epsilon})] \ge E^{x_0}[(g(X_t) + \epsilon)\mathcal{X}_{\{t<\tau_\epsilon\}}] \quad \text{for all } t > 0 \ .$$

Hence by the Fatou lemma and lower semicontinuity of g

$$\begin{aligned} g(x_0) + \eta \ &\ge \ \varliminf_{t\to 0} E^{x_0}[(g(X_t) + \epsilon)\mathcal{X}_{\{t<\tau_\epsilon\}}] \\ &\ge \ E^{x_0}[\varliminf_{t\to 0}(g(X_t) + \epsilon)\mathcal{X}_{\{t<\tau_\epsilon\}}] \ge g(x_0) + \epsilon \ . \end{aligned}$$

This is a contradiction if $\eta < \epsilon$.

Case 2: $\tau_\epsilon = 0$ a.s. Q^{x_0}. Then $\widetilde{g}_\epsilon(x_0) = \widehat{g}(x_0)$, so $g(x_0) \le \widetilde{g}_\epsilon(x_0)$, contradicting (10.1.25) for $\eta < \beta$.

Therefore (10.1.24) leads to a contradiction. Thus (10.1.23) is proved and we conclude that $\widetilde{g}_\epsilon + \epsilon$ is a supermeanvalued majorant of g. Therefore

$$\widehat{g} \le \widetilde{g}_\epsilon + \epsilon = E[\widehat{g}(X_{\tau_\epsilon})] + \epsilon \le E[(g + \epsilon)(X_{\tau_\epsilon})] + \epsilon \le g^* + 2\epsilon \tag{10.1.28}$$

and since ϵ was arbitrary we have by (10.1.12)

$$\widehat{g} = g^* \ .$$

If g is not bounded, let

$$g_N = \min(N, g) , \qquad N = 1, 2, \ldots$$

and as before let $\widehat{g_N}$ be the least superharmonic majorant of g_N. Then

$$g^* \geq g_N^* = \widehat{g_N} \uparrow h \qquad \text{as } N \to \infty \text{, where } h \geq \widehat{g}$$

since h is a superharmonic majorant of g. Thus $h = \widehat{g} = g^*$ and this proves (10.1.17) for general g. From (10.1.28) and (10.1.17) we obtain (10.1.19).

Finally, to obtain c) and d) let us again first assume that g is bounded. Then, since

$$\tau_\epsilon \uparrow \tau_D \qquad \text{as } \epsilon \downarrow 0$$

and $\tau_D < \infty$ a.s we have

$$E^x[g(X_{\tau_\epsilon})] \to E^x[g(X_{\tau_D})] \qquad \text{as } \epsilon \downarrow 0 , \tag{10.1.29}$$

and hence by (10.1.28) and (10.1.17)

$$g^*(x) = E^x[g(X_{\tau_D})] \qquad \text{if } g \text{ is bounded} . \tag{10.1.30}$$

Finally, if g is not bounded define

$$h = \lim_{N\to\infty} \widehat{g_N} .$$

Then h is superharmonic by Lemma 10.1.3 d) and since $\widehat{g_N} \leq \widehat{g}$ for all N we have $h \leq \widehat{g}$. On the other hand $g_N \leq \widehat{g_N} \leq h$ for all N and therefore $g \leq h$. Since $\widehat{g}$ is the least superharmonic majorant of g we conclude that

$$h = \widehat{g} . \tag{10.1.31}$$

Hence by (10.1.30), (10.1.31) we obtain (10.1.21):

$$g^*(x) = \lim_{N\to\infty} \widehat{g_N}(x) = \lim_{N\to\infty} E^x[g_N(X_{\sigma_N})] \leq \lim_{N\to\infty} E^x[g(X_{\sigma_N})] \leq g^*(x) .$$

Note that $\widehat{g_N} \leq N$ everywhere, so if $g_N(x) < \widehat{g_N}(x)$ then $g_N(x) < N$ and therefore $g(x) = g_N(x) < \widehat{g_N}(x) \leq \widehat{g}(x)$ and $g_{N+1}(x) = g_N(x) < \widehat{g_N}(x) \leq \widehat{g_{N+1}}(x)$. Hence $D_N \subset D \cap \{x; g(x) < N\}$ and $D_N \subset D_{N+1}$ for all N. So by (10.1.31) we conclude that D is the increasing union of the sets D_N; $N = 1, 2, \ldots$ Therefore

$$\tau_D = \lim_{N\to\infty} \sigma_N .$$

So by (10.1.21) and uniform integrability we have

$$\begin{aligned} \widehat{g}(x) &= \lim_{N\to\infty} \widehat{g_N}(x) = \lim_{N\to\infty} E^x[g_N(X_{\sigma_N})] \\ &= E^x[\lim_{N\to\infty} g_N(X_{\sigma_N})] = E^x[g(X_{\tau_D})] , \end{aligned}$$

and the proof of Theorem 10.1.9 is complete. □

Remarks.

1) Note that the sets D, D_ϵ and D_N are open, since $\widehat{g} = g^*$ is lower semicontinuous and g is continuous.
2) By inspecting the proof of a) we see that (10.1.17) holds under the weaker assumption that $g \geq 0$ is lower semicontinuous.

The following consequence of Theorem 10.1.9 is often useful:

Corollary 10.1.10 *Suppose there exists a Borel set H such that*

$$\widetilde{g}_H(x) := E^x[g(X_{\tau_H})]$$

is a supermeanvalued majorant of g. Then

$$g^*(x) = \widetilde{g}_H(x) , \quad \textit{so } \tau^* = \tau_H \textit{ is optimal} .$$

Proof. If $\widetilde{g}_H$ is a supermeanvalued majorant of g then clearly

$$\overline{g}(x) \leq \widetilde{g}_H(x) .$$

On the other hand we of course have

$$\widetilde{g}_H(x) \leq \sup_\tau E^x[g(X_\tau)] = g^*(x) ,$$

so $g^* = \widetilde{g}_H$ by Theorem 10.1.7 and Theorem 10.1.9 a). □

Corollary 10.1.11 *Let*

$$D = \{x; g(x) < \widehat{g}(x)\}$$

and put

$$\widetilde{g}(x) = \widetilde{g}_D(x) = E^x[g(X_{\tau_D})] .$$

If $\widetilde{g} \geq g$ then $\widetilde{g} = g^*$.

Proof. Since $X_{\tau_D} \notin D$ we have $g(X_{\tau_D}) \geq \widehat{g}(X_{\tau_D})$ and therefore $g(X_{\tau_D}) = \widehat{g}(X_{\tau_D})$, a.s. Q^x. So $\widetilde{g}(x) = E^x[\widehat{g}(X_{\tau_D})]$ is supermeanvalued since $\widehat{g}$ is, and the result follows from Corollary 10.1.10. □

Theorem 10.1.9 gives a sufficient condition for the existence of an optimal stopping time τ^*. Unfortunately, τ^* need not exist in general. For example, if

$$X_t = t \quad \text{for } t \geq 0 \ \text{(deterministic)}$$

and

$$g(\xi) = \frac{\xi^2}{1+\xi^2} ; \qquad \xi \in \mathbf{R}$$

then $g^*(x) = 1$, but there is no stopping time τ such that

$$E^x[g(X_\tau)] = 1 .$$

However, we can prove that if an optimal stopping time τ^* exists, then the stopping time given in Theorem 10.1.9 is optimal:

Theorem 10.1.12 (Uniqueness theorem for optimal stopping) *Define as before*

$$D = \{x; g(x) < g^*(x)\} \subset \mathbf{R}^n \ .$$

Suppose there exists an optimal stopping time $\tau^* = \tau^*(x,\omega)$ *for the problem (10.1.2) for all* x. *Then*

$$\tau^* \geq \tau_D \qquad \text{for all } x \in D \tag{10.1.32}$$

and

$$g^*(x) = E^x[g(X_{\tau_D})] \qquad \text{for all } x \in \mathbf{R}^n \ . \tag{10.1.33}$$

Hence τ_D *is an optimal stopping time for the problem (10.1.2).*

Proof. Choose $x \in D$. Let τ be an $\mathcal{F}_t$-stopping time and assume $Q^x[\tau < \tau_D] > 0$. Since $g(X_\tau) < g^*(X_\tau)$ if $\tau < \tau_D$ and $g \leq g^*$ always, we have

$$\begin{aligned} E^x[g(X_\tau)] &= \int\limits_{\tau<\tau_D} g(X_\tau)dQ^x + \int\limits_{\tau\geq\tau_D} g(X_\tau)dQ^x \\ &< \int\limits_{\tau<\tau_D} g^*(X_\tau)dQ^x + \int\limits_{\tau\geq\tau_D} g^*(X_\tau)dQ^x = E^x[g^*(X_\tau)] \leq g^*(x) \ , \end{aligned}$$

since g^* is superharmonic. This proves (10.1.32).

To obtain (10.1.33) we first choose $x \in D$. Since $\widehat{g}$ is superharmonic we have by (10.1.32) and Lemma 10.1.3 e)

$$\begin{aligned} g^*(x) &= E^x[g(X_{\tau}*)] \leq E^x[\widehat{g}(X_{\tau}*)] \leq E^x[\widehat{g}(X_{\tau_D})] \\ &= E^x[g(X_{\tau_D})] \leq g^*(x) \ , \qquad \text{which proves (10.1.33) for } x \in D \ . \end{aligned}$$

Next, choose $x \in \partial D$ to be an *irregular* boundary point of D. Then $\tau_D > 0$ a.s. Q^x. Let $\{\alpha_k\}$ be a sequence of stopping times such that $0 < \alpha_k < \tau_D$ and $\alpha_k \to 0$ a.s. Q^x, as $k \to \infty$. Then $X_{\alpha_k} \in D$ so by (10.1.32), (7.2.6) and the strong Markov property (7.2.2)

$$E^x[g(X_{\tau_D})] = E^x[\theta_{\alpha_k} g(X_{\tau_D})] = E^x[E^{X_{\alpha_k}}[g(X_{\tau_D})]] = E^x[g^*(X_{\alpha_k})] \quad \text{for all } k \ .$$

Hence by lower semicontinuity and the Fatou lemma

$$g^*(x) \leq E^x[\varliminf_{k\to\infty} g^*(X_{\alpha_k})] \leq \varliminf_{k\to\infty} E^x[g^*(X_{\alpha_k})] = E^x[g(X_{\tau_D})] \ .$$

Finally, if $x \in \partial D$ is a *regular* boundary point of D or if $x \notin \overline{D}$ we have $\tau_D = 0$ a.s. Q^x and hence $g^*(x) = E^x[g(X_{\tau_D})]$. □

Remark. The following observation is sometimes useful:

Let $\mathcal{A}$ be the characteristic operator of X. Assume $g \in C^2(\mathbf{R}^n)$. Define

$$U = \{x; \mathcal{A}g(x) > 0\} \ . \tag{10.1.34}$$

Then, with D as before, (10.1.20),

$$U \subset D\ . \tag{10.1.35}$$

Consequently, from (10.1.32) we conclude that it is *never optimal to stop the process before it exits from* U. But there may be cases when $U \neq D$, so that it is optimal to proceed beyond U before stopping. (This is in fact the typical situation.) See e.g. Example 10.2.2.

To prove (10.1.35) choose $x \in U$ and let τ_0 be the first exit time from a bounded open set $W \ni x$, $W \subset U$. Then by Dynkin's formula, for $u > 0$

$$E^x[g(X_{\tau_0 \wedge u})] = g(x) + E^x\left[\int_0^{\tau_0 \wedge u} \mathcal{A}g(X_s)ds\right] > g(x)$$

so $g(x) < g^*(x)$ and therefore $x \in D$.

Example 10.1.13 Let $X_t = B_t$ be a Brownian motion in $\mathbf{R}^2$. Using that B_t is recurrent in $\mathbf{R}^2$ (Example 7.4.2) one can show that the only (nonnegative) superharmonic functions in $\mathbf{R}^2$ are the constants (Exercise 10.2).

Therefore

$$g^*(x) = \|g\|_\infty := \sup\{g(y); y \in \mathbf{R}^2\} \qquad \text{for all } x\ .$$

So if g is unbounded then $g^* = \infty$ and no optimal stopping time exists. Assume therefore that g is bounded. The continuation region is

$$D = \{x; g(x) < \|g\|_\infty\}\ ,$$

so if ∂D is a *polar set* i.e. cap $(\partial D) = 0$, where cap denotes the *logarithmic capacity* (see Port and Stone (1979)), then $\tau_D = \infty$ a.s. and no optimal stopping exists. On the other hand, if $\text{cap}(\partial D) > 0$ then $\tau_D < \infty$ a.s. and

$$E^x[g(B_{\tau_D})] = \|g\|_\infty = g^*(x)\ ,$$

so $\tau^* = \tau_D$ is optimal.

Example 10.1.14 The situation is different in $\mathbf{R}^n$ for $n \geq 3$.

a) To illustrate this let $X_t = B_t$ be Brownian motion in $\mathbf{R}^3$ and let the reward function be

$$g(\xi) = \begin{cases} |\xi|^{-1} & \text{for } |\xi| \geq 1 \\ 1 & \text{for } |\xi| < 1 \end{cases}; \qquad \xi \in \mathbf{R}^3\ .$$

Then g is superharmonic (in the classical sense) in $\mathbf{R}^3$, so $g^* = g$ everywhere and the best policy is to stop immediately, no matter where the starting point is.

b) Let us change g to

$$h(x) = \begin{cases} |x|^{-\alpha} & \text{for } |x| \geq 1 \\ 1 & \text{for } |x| < 1 \end{cases}$$

for some $\alpha > 1$. Let $H = \{x; |x| > 1\}$ and define

$$\widetilde{h}(x) = E^x[h(B_{\tau_H})] = P^x[\tau_H < \infty] \ .$$

Then by Example 7.4.2

$$\widetilde{h}(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ |x|^{-1} & \text{if } |x| > 1 \ , \end{cases}$$

i.e. $\widetilde{h} = g$ (defined in a)), which is a superharmonic majorant of h. Therefore by Corollary 10.1.10

$$h^* = \widetilde{h} = g \ ,$$

$H = D$ and $\tau^* = \tau_H$ is an optimal stopping time.

Reward Functions Assuming Negative Values

The results we have obtained so far regarding the problem (10.1.2)–(10.1.3) are based on the assumptions (10.1.1). To some extent these assumptions can be relaxed, although neither can be removed completely. For example, we have noted that Theorem 10.1.9 a) still holds if $g \geq 0$ is only assumed to be lower semicontinuous.

The nonnegativity assumption on g can also be relaxed. First of all, note that if g is bounded below, say $g \geq -M$ where $M > 0$ is a constant, then we can put

$$g_1 = g + M \geq 0$$

and apply the theory to g_1. Since

$$E^x[g(X_\tau)] = E^x[g_1(X_\tau)] - M \qquad \text{if } \tau < \infty \text{ a.s.} \ ,$$

we have $g^*(x) = g_1^*(x) - M$, so the problem can be reduced to the optimal stopping problem for the nonnegative function g_1. (See Exercise 10.4.)

If g is not bounded below, then problem (10.1.2)–(10.1.3) is not well-defined unless

$$E^x[g^-(X_\tau)] < \infty \qquad \text{for all } \tau \tag{10.1.36}$$

where

$$g^-(x) = -\min(g(x), 0) \ .$$

If we assume that g satisfies the stronger condition that

$$\text{the family } \{g^-(X_\tau); \tau \text{ stopping time}\} \text{ is uniformly integrable} \tag{10.1.37}$$

then basically all the theory from the nonnegative case carries over. We refer to the reader to Shiryaev (1978) for more information. See also Theorem 10.4.1.

10.2 The Time-Inhomogeneous Case

Let us now consider the case when the reward function g depends on both time and space, i.e.

$$g = g(t,x)\colon \mathbf{R}\times\mathbf{R}^n \to [0,\infty)\ , \qquad g \text{ is continuous}\ . \tag{10.2.1}$$

Then the problem is to find $g_0(x)$ and τ^* such that

$$g_0(x) = \sup_\tau E^x[g(\tau, X_\tau)] = E^x[g(\tau^*, X_{\tau^*})]\ . \tag{10.2.2}$$

To reduce this case to the original case (10.1.2)–(10.1.3) we proceed as follows:

Suppose the Itô diffusion $X_t = X_t^x$ has the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t\ ; \qquad t\geq 0\ ,\ \ X_0 = x$$

where $b\colon\mathbf{R}^n \to \mathbf{R}^n$ and $\sigma\colon\mathbf{R}^n \to \mathbf{R}^{n\times m}$ are given functions satisfying the conditions of Theorem 5.2.1 and B_t is m-dimensional Brownian motion. Define the Itô diffusion $Y_t = Y_t^{(s,x)}$ in $\mathbf{R}^{n+1}$ by

$$Y_t = \begin{bmatrix} s+t \\ X_t^x \end{bmatrix}; \qquad t\geq 0\ . \tag{10.2.3}$$

Then

$$dY_t = \begin{bmatrix} 1 \\ b(X_t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sigma(X_t) \end{bmatrix} dB_t = \widehat{b}(Y_t)dt + \widehat{\sigma}(Y_t)dB_t \tag{10.2.4}$$

where

$$\widehat{b}(\eta) = \widehat{b}(t,\xi) = \begin{bmatrix} 1 \\ b(\xi) \end{bmatrix} \in \mathbf{R}^{n+1}\ , \quad \widehat{\sigma}(\eta) = \widehat{\sigma}(t,\xi) = \begin{bmatrix} 0\cdots 0 \\ \text{- - - -} \\ \sigma(\xi) \end{bmatrix} \in \mathbf{R}^{(n+1)\times m}\ ,$$

with $\eta = (t,\xi) \in \mathbf{R}\times\mathbf{R}^n$.

So Y_t is an Itô diffusion starting at $y = (s,x)$. Let $R^y = R^{(s,x)}$ denote the probability law of $\{Y_t\}$ and let $E^y = E^{(s,x)}$ denote the expectation w.r.t. R^y. In terms of Y_t the problem (10.2.2) can be written

$$g_0(x) = g^*(0,x) = \sup_\tau E^{(0,x)}[g(Y_\tau)] = E^{(0,x)}[g(Y_{\tau^*})] \tag{10.2.5}$$

which is a special case of the problem

$$g^*(s,x) = \sup_\tau E^{(s,x)}[g(Y_\tau)] = E^{(s,x)}[g(Y_{\tau^*})]\ , \tag{10.2.6}$$

which is of the form (10.1.2)–(10.1.3) with X_t replaced by Y_t.

Note that the characteristic operator $\widehat{\mathcal{A}}$ of Y_t is given by

$$\widehat{\mathcal{A}}\phi(s,x) = \frac{\partial\phi}{\partial s}(s,x) + \mathcal{A}\phi(s,x)\ ; \qquad \phi\in C^2(\mathbf{R}\times\mathbf{R}^n) \tag{10.2.7}$$

where $\mathcal{A}$ is the characteristic operator of X_t (working on the x-variables).

Example 10.2.1 Let $X_t = B_t$ be 1-dimensional Brownian motion and let the reward function be

$$g(t,\xi) = e^{-\alpha t+\beta\xi}\ ; \qquad \xi \in \mathbf{R}$$

where $\alpha, \beta \geq 0$ are constants. The characteristic operator $\widehat{\mathcal{A}}$ of $Y_t^{s,x} = \begin{bmatrix} s+t \\ B_t^x \end{bmatrix}$ is given by

$$\widehat{\mathcal{A}}f(s,x) = \frac{\partial f}{\partial s} + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}\ ; \qquad f \in C^2\ .$$

Thus

$$\mathcal{A}g = (-\alpha + \tfrac{1}{2}\beta^2)g\ ,$$

so if $\beta^2 \leq 2\alpha$ then $g^* = g$ and the best policy is to stop immediately. If $\beta^2 > 2\alpha$ we have

$$U := \{(s,x); \widehat{\mathcal{A}}g(s,x) > 0\} = \mathbf{R}^2$$

and therefore by (10.1.35) $D = \mathbf{R}^2$ and hence τ^* does not exist. If $\beta^2 > 2\alpha$ we can use Theorem 10.1.7 to prove that $g^* = \infty$:

$$\begin{aligned}\sup_{t\in S_n} E^{(s,x)}[g(Y_t)] &= \sup_{t\in S_n} E[e^{-\alpha(s+t)+\beta B_t^x}] \\ &= \sup_{t\in S_n}[e^{-\alpha(s+t)}\cdot e^{\beta x+\frac{1}{2}\beta^2 t}] \qquad \text{(see the remark following (5.1.6))} \\ &= \sup_{t\in S_n} g(s,x)\cdot e^{(-\alpha+\frac{1}{2}\beta^2)t} = g(s,x)\cdot\exp((-\alpha+\tfrac{1}{2}\beta^2)2^n)\ ,\end{aligned}$$

so $g_n(s,x) \to \infty$ as $n \to \infty$.

Hence no optimal stopping exists in this case.

Example 10.2.2 (When is the right time to sell the stocks? (Part 1)) We now return to a specified version of Problem 5 in the introduction:

Suppose the price X_t at time t of a person's assets (e.g. a house, stocks, oil ...) varies according to a stochastic differential equation of the form

$$dX_t = rX_t dt + \alpha X_t dB_t, X_0 = x > 0\ ,$$

where B_t is 1-dimensional Brownian motion and r, α are known constants. (The problem of estimating α and r from a series of observations can be approached using the quadratic variation $\langle X, X\rangle_t$ of the process $\{X_t\}$ (Exercise 4.7) and filtering theory (Example 6.2.11), respectively. Suppose that connected to the sale of the assets there is a fixed fee/tax or transaction cost $a > 0$. Then if the person decides to sell at time t the discounted net of the sale is

$$e^{-\rho t}(X_t - a)\ ,$$

where $\rho > 0$ is given discounting factor. The problem is to find a stopping time τ that maximizes

$$E^{(s,x)}[e^{-\rho\tau}(X_\tau - a)] = E^{(s,x)}[g(\tau, X_\tau)]\ ,$$

where

$$g(t, \xi) = e^{-\rho t}(\xi - a) .$$

The characteristic operator $\widehat{\mathcal{A}}$ of the process $Y_t = (s + t, X_t)$ is given by

$$\widehat{\mathcal{A}}f(s, x) = \frac{\partial f}{\partial s} + rx\frac{\partial f}{\partial x} + \tfrac{1}{2}\alpha^2 x^2 \frac{\partial^2 f}{\partial x^2} ; \qquad f \in C^2(\mathbf{R}^2) .$$

Hence $\widehat{\mathcal{A}}g(s, x) = -\rho e^{-\rho s}(x - a) + rxe^{-\rho s} = e^{-\rho s}((r - \rho)x + \rho a)$. So

$$U := \{(s, x); \widehat{\mathcal{A}}g(s, x) > 0\} = \begin{cases} \mathbf{R} \times \mathbf{R}_+ & \text{if } r \geq \rho \\ \{(s, x); x < \frac{a\rho}{\rho - r}\} & \text{if } r < \rho . \end{cases}$$

So if $r \geq \rho$ we have $U = D = \mathbf{R} \times \mathbf{R}_+$ so τ^* does not exist. If $r > \rho$ then $g^* = \infty$ while if $r = \rho$ then

$$g^*(s, x) = xe^{-\rho s} .$$

(The proofs of these statements are left as Exercise 10.5.)

It remains to examine the case $r < \rho$. (If we regard ρ as the sum of interest rate, inflation and tax etc., this is not an unreasonable assumption in applications.) First we establish that the region D must be invariant w.r.t. t, in the sense that

$$D + (t_0, 0) = D \qquad \text{for all } t_0 . \tag{10.2.8}$$

To prove (10.2.8) consider

$$\begin{aligned} D + (t_0, 0) &= \{(t + t_0, x); (t, x) \in D\} = \{(s, x); (s - t_0, x) \in D\} \\ &= \{(s, x); g(s - t_0, x) < g^*(s - t_0, x)\} = \{(s, x); e^{\rho t_0} g(s, x) < e^{\rho t_0} g^*(s, x)\} \\ &= \{(s, x); g(s, x) < g^*(s, x)\} = D , \end{aligned}$$

where we have used that

$$\begin{aligned} g^*(s - t_0, x) &= \sup_\tau E^{(s - t_0, x)}[e^{-\rho\tau}(X_\tau - a)] = \sup_\tau E[e^{-\rho(\tau + (s - t_0))}(X_\tau^x - a)] \\ &= e^{\rho t_0} \sup_\tau E[e^{-\rho(\tau + s)}(X_\tau^x - a)] = e^{\rho t_0} g^*(s, x) . \end{aligned}$$

Therefore the connected component of D that contains U must have the form

$$D(x_0) = \{(t, x); 0 < x < x_0\} \qquad \text{for some } x_0 \geq \tfrac{a\rho}{\rho - r} .$$

Note that D cannot have any other components, for if V is a component of D disjoint from U then $\widehat{\mathcal{A}}g < 0$ in V and so, if $y \in V$,

$$E^y[g(Y_\tau)] = g(y) + E^y\Big[\int_0^\tau \widehat{\mathcal{A}}g(Y_t)dt\Big] < g(y)$$

for all exit times τ bounded by the exit time from an x-bounded strip in V. From this we conclude by Theorem 10.1.9 c) that $g^*(y) = g(y)$, which implies $V = \emptyset$.

Put $\tau(x_0) = \tau_{D(x_0)}$ and let us compute

$$\widetilde{g}(s,x) = \widetilde{g}_{x_0}(s,x) = E^{(s,x)}[g(Y_{\tau(x_0)})] \;. \tag{10.2.9}$$

From Chapter 9 we know that $f = \widetilde{g}$ is the (bounded) solution of the boundary value problem

$$\left.\begin{aligned} \frac{\partial f}{\partial s} + rx\frac{\partial f}{\partial x} + \tfrac{1}{2}\alpha^2 x^2 \frac{\partial^2 f}{\partial x^2} &= 0 \qquad \text{for } 0 < x < x_0 \\ f(s, x_0) &= e^{-\rho s}(x_0 - a) \;. \end{aligned}\right\} \tag{10.2.10}$$

(Note that $\mathbf{R} \times \{0\}$ does not contain any regular boundary points of D w.r.t. $Y_t = (s+t, X_t)$.)

If we try a solution of (10.2.10) of the form

$$f(s,x) = e^{-\rho s}\phi(x)$$

we get the following 1-dimensional problem

$$\left.\begin{aligned} -\rho\phi + rx\phi'(x) + \tfrac{1}{2}\alpha^2 x^2 \phi''(x) &= 0 \quad \text{for } 0 < x < x_0 \\ \phi(x_0) &= x_0 - a \;. \end{aligned}\right\} \tag{10.2.11}$$

The general solution ϕ of (10.2.11) is

$$\phi(x) = C_1 x^{\gamma_1} + C_2 x^{\gamma_2} \;,$$

where C_1, C_2 are arbitrary constants and

$$\gamma_i = \alpha^{-2}\Big[\tfrac{1}{2}\alpha^2 - r \pm \sqrt{(r - \tfrac{1}{2}\alpha^2)^2 + 2\rho\alpha^2}\;\Big] \qquad (i = 1, 2)\,,\; \gamma_2 < 0 < \gamma_1 \;.$$

Since $\phi(x)$ is bounded as $x \to 0$ we must have $C_2 = 0$ and the boundary requirement $\phi(x_0) = x_0 - a$ gives $C_1 = x_0^{-\gamma_1}(x_0 - a)$. We conclude that the bounded solution f of (10.2.10) is

$$\widetilde{g}_{x_0}(s,x) = f(s,x) = e^{-\rho s}(x_0 - a)\left(\frac{x}{x_0}\right)^{\gamma_1} \;. \tag{10.2.12}$$

If we fix (s,x) then the value of x_0 which maximizes $\widetilde{g}_{x_0}(s,x)$ is easily seen to be given by

$$x_0 = x_{\max} = \frac{a\gamma_1}{\gamma_1 - 1} \tag{10.2.13}$$

(note that $\gamma_1 > 1$ if and only if $r < \rho$).

Thus we have arrived at the candidate $\widetilde{g}_{x_{\max}}(s,x)$ for $g^*(s,x) = \sup_\tau E^{(s,x)}[e^{-\rho\tau}(X_\tau - a)]$. To verify that we indeed have $\widetilde{g}_{x_{\max}} = g^*$ it would suffice to prove that $\widetilde{g}_{x_{\max}}$ is a supermeanvalued majorant of g (see Corollary 10.1.10). This can be done, but we do not give the details here, since this problem can be solved more easily by Theorem 10.4.1 (see Example 10.4.2).

The conclusion is therefore that one should sell the assets the first time the price of them reaches the value $x_{\max} = \frac{a\gamma_1}{\gamma_1 - 1}$. The expected discounted profit obtained from this strategy is

$$g^*(s,x) = \widetilde{g}_{x_{\max}}(s,x) = e^{-\rho s}\left(\frac{\gamma_1 - 1}{a}\right)^{\gamma_1 - 1}\left(\frac{x}{\gamma_1}\right)^{\gamma_1} .$$

Remark. The reader is invited to check that the value $x_0 = x_{\max}$ is the only value of x_0 which makes the function

$$x \to \widetilde{g}_{x_0}(s,x) \qquad \text{(given by (10.2.9))}$$

continuously differentiable at x_0. This is not a coincidence. In fact, it illustrates a general phenomenon which is known as the *high contact* (or smooth fit) *principle*. See Samuelson (1965), McKean (1965), Bather (1970) and Shiryaev (1978). This principle is the basis of the fundamental connection between optimal stopping and *variational inequalities*. Later in this chapter we will discuss some aspects of this connection. More information can be found in Bensoussan and Lions (1978) and Friedman (1976). See also Brekke and Øksendal (1991).

10.3 Optimal Stopping Problems Involving an Integral

Let

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t\ , \qquad Y_0 = y \tag{10.3.1}$$

be an Itô diffusion in $\mathbf{R}^k$. Let $g\colon \mathbf{R}^k \to [0,\infty)$ be continuous and let $f\colon \mathbf{R}^k \to [0,\infty)$ be Lipschitz continuous with at most linear growth. (These conditions can be relaxed. See (10.1.37) and Theorem 10.4.1.) Consider the optimal stopping problem: Find $\Phi(y)$ and τ^* such that

$$\Phi(y) = \sup_\tau E^y\left[\int_0^\tau f(Y_t)dt + g(Y_\tau)\right] = E^y\left[\int_0^{\tau^*} f(Y_t)dt + g(Y_{\tau^*})\right] . \tag{10.3.2}$$

This problem can be reduced to our original problem (10.1.2)–(10.1.3) by proceeding as follows: Define the Itô diffusion Z_t in $\mathbf{R}^k \times \mathbf{R} = \mathbf{R}^{k+1}$ by

$$dZ_t = \begin{bmatrix} dY_t \\ dW_t \end{bmatrix} := \begin{bmatrix} b(Y_t) \\ f(Y_t) \end{bmatrix} dt + \begin{bmatrix} \sigma(Y_t) \\ 0 \end{bmatrix} dB_t\ ; \quad Z_0 = z = (y,w)\ . \tag{10.3.3}$$

Then we see that

$$\Phi(y) = \sup_\tau E^{(y,0)}[W_\tau + g(Y_\tau)] = \sup_\tau E^{(y,0)}[\widetilde{g}(Z_\tau)]$$

with

$$\widetilde{g}(z) := \widetilde{g}(y,w) := g(y) + w \; ; \qquad z = (y,w) \in \mathbf{R}^k \times \mathbf{R} \; . \tag{10.3.4}$$

This is again a problem of the type (10.1.2)–(10.1.3) with X_t replaced by Z_t and g replaced by $\widetilde{g}$. Note that the connection between the characteristic operators $\mathcal{A}_Y$ of Y_t and $\mathcal{A}_Z$ of Z_t is given by

$$\mathcal{A}_Z\phi(z) = \mathcal{A}_Z\phi(y,w) = \mathcal{A}_Y\phi(y,w) + f(y)\frac{\partial\phi}{\partial w} \; , \quad \phi \in C^2(\mathbf{R}^{k+1}) \; . \tag{10.3.5}$$

In particular, if $\widetilde{g}(y,w) = g(y) + w \in C^2(\mathbf{R}^{k+1})$ then

$$\mathcal{A}_Z\,\widetilde{g}(y,w) = \mathcal{A}_Y g(y) + f(y) \; . \tag{10.3.6}$$

Hence, in this general case the domain U of (10.1.34) gets the form

$$U = \{y; \mathcal{A}_Y g(y) + f(y) > 0\} \; . \tag{10.3.7}$$

Example 10.3.1 Consider the optimal stopping problem

$$\Phi(x) = \sup_{\tau} E^x\left[\int_0^{\tau} \theta e^{-\rho t} X_t dt + e^{-\rho\tau} X_\tau\right] ,$$

where

$$dX_t = \alpha X_t dt + \beta X_t dB_t \; ; \qquad X_0 = x > 0$$

is geometric Brownian motion (α, β, θ constants, $\theta > 0$). We put

$$dY_t = \begin{bmatrix} dt \\ dX_t \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha X_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ \beta X_t \end{bmatrix} dB_t \; ; \qquad Y_0 = (s,x)$$

and

$$dZ_t = \begin{bmatrix} dY_t \\ dW_t \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha X_t \\ \theta e^{-\rho t} X_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ \beta X_t \\ 0 \end{bmatrix} dB_t \; ; \qquad Z_0 = (s,x,w) \; .$$

Then with

$$f(y) = f(s,x) = \theta e^{-\rho s} x \; , \qquad g(y) = e^{-\rho s} x$$

and

$$\widetilde{g}(s,x,w) = g(s,x) + w = e^{-\rho s} x + w$$

we have

$$\mathcal{A}_Z\,\widetilde{g} = \frac{\partial\widetilde{g}}{\partial s} + \alpha x \frac{\partial\widetilde{g}}{\partial x} + \tfrac{1}{2}\beta^2 x^2 \frac{\partial^2\widetilde{g}}{\partial x^2} + \theta e^{-\rho s} x \frac{\partial\widetilde{g}}{\partial w} = (-\rho + \alpha + \theta) e^{-\rho s} x \; .$$

Hence

$$U = \{(s,x,w); \mathcal{A}_Z\, \widetilde{g}(s,x,w) > 0\} = \begin{cases} \mathbf{R}^3 & \text{if } \rho < \alpha + \theta \\ \emptyset & \text{if } \rho \geq \alpha + \theta \ . \end{cases}$$

From this we conclude (see Exercise 10.6):

If $\rho \geq \alpha + \theta$ then $\tau^* = 0$

$$\text{and } \Phi(s,x,w) = \widetilde{g}(s,x,w) = e^{-\rho s}x + w \ . \tag{10.3.8}$$

If $\alpha < \rho < \alpha + \theta$ then τ^* does not exist

$$\text{and } \Phi(s,x,w) = \tfrac{\theta x}{\rho - \alpha} e^{-\rho s} + w \ . \tag{10.3.9}$$

If $\rho \leq \alpha$ then τ^* does not exist and $\Phi = \infty$. (10.3.10)

10.4 Connection with Variational Inequalities

The 'high contact principle' says, roughly, that – under certain conditions – the solution g^* of (10.1.2)–(10.1.3) is a C^1 function on $\mathbf{R}^n$ if $g \in C^2(\mathbf{R}^n)$. This is a useful information which can help us to determine g^*. Indeed, this principle is so useful that it is frequently applied in the literature also in cases where its validity has not been rigorously proved.

Fortunately it turns out to be easy to prove a *sufficiency* condition of high contact type, i.e. a kind of verification theorem for optimal stopping, which makes it easy to verify that a given candidate for g^*(that we may have found by guessing or intuition) is actually equal to g^*. The result below is a simplified variant of a result in Brekke and Øksendal (1991):

In the following we fix a domain G in $\mathbf{R}^k$ and we let

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t \ ; \qquad Y_0 = y \tag{10.4.1}$$

be an Itô diffusion in $\mathbf{R}^k$. Define

$$\tau_G = \tau_G(y,\omega) = \inf\{t > 0; Y_t(\omega) \notin V\} \ . \tag{10.4.2}$$

Let $f\colon \mathbf{R}^k \to \mathbf{R}$, $g\colon \mathbf{R}^k \to \mathbf{R}$ be continuous functions satisfying

(a) $E^y[\int_0^{\tau_G} f^-(Y_t)dt] < \infty$ for all $y \in \mathbf{R}^k$ (10.4.3)

and

(b) the family $\{g^-(Y_\tau); \tau \text{ stopping time}, \tau \leq \tau_G\}$ is uniformly integrable w.r.t. R^y (the probability law of Y_t), for all $y \in \mathbf{R}^k$. (10.4.4)

Let $\mathcal{T}$ denote the set of all stopping times $\tau \leq \tau_G$. Consider the following problem: Find $\Phi(y)$ and $\tau^* \in \mathcal{T}$ such that

$$\Phi(y) = \sup_{\tau \in \mathcal{T}} J^\tau(y) = J^{\tau^*}(y) \ , \tag{10.4.5}$$

where

$$J^\tau(y) = E^y\Big[\int_0^\tau f(Y_t)dt + g(Y_\tau)\Big] \qquad \text{for } \tau \in \mathcal{T} .$$

Note that since $J^0(y) = g(y)$ we have

$$\Phi(y) \ge g(y) \qquad \text{for all } y \in G . \tag{10.4.6}$$

We can now formulate the variational inequalities. As usual we let

$$L = L_Y = \sum_{i=1}^k b_i(y)\frac{\partial}{\partial y_i} + \tfrac{1}{2}\sum_{i,j=1}^k (\sigma\sigma^T)_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j}$$

be the partial differential operator which coincides with the generator A_Y of Y_t on $C_0^2(\mathbf{R}^k)$.

Theorem 10.4.1 (Variational inequalities for optimal stopping)
a) *Suppose we can find a function* $\phi: \overline{G} \to \mathbf{R}$ *such that*

(i) $\phi \in C^1(G) \cap C(\overline{G})$
(ii) $\phi \ge g$ *on* G $\lim_{t\to\tau_G^-} \phi(Y_t) = g(Y_{\tau_G})\mathcal{X}_{\{\tau_G<\infty\}}$ *a.s.*
Define

$$D = \{x \in G; \phi(x) > g(x)\} .$$

Suppose Y_t *spends 0 time on* ∂D *a.s., i.e.*

(iii) $E^y\Big[\int_0^{\tau_G} \mathcal{X}_{\partial D}(Y_t)dt\Big] = 0$ *for all* $y \in G$
and suppose that

(iv) ∂D *is a Lipschitz surface, i.e.* ∂D *is locally the graph of a function* $h: \mathbf{R}^{k-1} \to \mathbf{R}$ *such that there exists* $K < \infty$ *with*

$$|h(x) - h(y)| \le K|x - y| \qquad \textit{for all } x, y .$$

Moreover, suppose the following:

(v) $\phi \in C^2(G\backslash\partial D)$ *and the second order derivatives of* ϕ *are locally bounded near* ∂D
(vi) $L\phi + f \le 0$ *on* $G \setminus D$.
Then

$$\phi(y) \ge \Phi(y) \qquad \textit{for all } y \in G.$$

b) *Suppose, in addition to the above, that*

(vii) $L\phi + f = 0$ *on* D
(viii) $\tau_D := \inf\{t > 0; Y_t \notin D\} < \infty$ *a.s.* R^y *for all* $y \in G$
and

(ix) *the family* $\{\phi(Y_\tau); \tau \leq \tau_D, \tau \in \mathcal{T}\}$ *is uniformly integrable w.r.t.* R^y, *for all* $y \in G$.

Then

$$\phi(y) = \Phi(y) = \sup_{\tau \in \mathcal{T}} E^y \Big[\int_0^\tau f(Y_t)dt + g(Y_\tau) \Big] ; \qquad y \in G \tag{10.4.7}$$

and

$$\tau^* = \tau_D \tag{10.4.8}$$

is an optimal stopping time for this problem.

Proof. By (i), (iv) and (v) we can find a sequence of functions $\phi_j \in C^2(G) \cap C(\overline{G})$, $j = 1, 2, \ldots$, such that

(a) $\phi_j \to \phi$ uniformly on compact subsets of $\overline{G}$, as $j \to \infty$
(b) $L\phi_j \to L\phi$ uniformly on compact subsets of $G \setminus \partial D$, as $j \to \infty$
(c) $\{L\phi_j\}_{j=1}^\infty$ is locally bounded on G.

(See Appendix D).

Let $\{G_R\}_{R=1}^\infty$ be a sequence of bounded open sets such that $G = \bigcup_{R=1}^\infty G_R$. Put $T_R = \min(R, \inf\{t > 0; Y_t \notin G_R\})$ and let $\tau \leq \tau_G$ be a stopping time. Let $y \in G$. Then by Dynkin's formula

$$E^y[\phi_j(Y_{\tau \wedge T_R})] = \phi_j(y) + E^y\Big[\int_0^{\tau \wedge T_R} L\phi_j(Y_t)dt \Big] \tag{10.4.9}$$

Hence by (a), (b), (c) and (iii) and the bounded a.e. convergence

$$\begin{aligned} \phi(y) &= \lim_{j \to \infty} E^y \Big[\int_0^{\tau \wedge T_R} -L\phi_j(Y_t)dt + \phi_j(Y_{\tau \wedge T_R}) \Big] \\ &= E^y \Big[\int_0^{\tau \wedge T_R} -L\phi(Y_t)dt + \phi(Y_{\tau \wedge T_R}) \Big] . \end{aligned} \tag{10.4.10}$$

Therefore, by (ii), (iii) and (vi),

$$\phi(y) \geq E^y \Big[\int_0^{\tau \wedge T_R} f(Y_t)dt + g(Y_{\tau \wedge T_R}) \Big] .$$

Hence by the Fatou lemma and (10.4.3), (10.4.4)

$$\phi(y) \geq \varliminf_{R \to \infty} E^y \Big[\int_0^{\tau \wedge T_R} f(Y_t)dt + g(Y_{\tau \wedge T_R}) \Big] \geq E^y \Big[\int_0^\tau f(Y_t)dt + g(Y_\tau) \Big] .$$

Since $\tau \le \tau_G$ was arbitrary, we conclude that

$$\phi(y) \ge \Phi(y) \qquad \text{for all } y \in G\,, \tag{10.4.11}$$

which proves a).

We proceed to prove b): If $y \notin D$ then $\phi(y) = g(y) \le \Phi(y)$ so by (10.4.11) we have

$$\phi(y) = \Phi(y) \quad \text{and} \quad \widehat{\tau} = \widehat{\tau}(y,\omega) := 0 \qquad \text{is optimal for } y \notin D\,. \tag{10.4.12}$$

Next, suppose $y \in D$. Let $\{D_k\}_{k=1}^{\infty}$ be an increasing sequence of open sets D_k such that $\overline{D}_k \subset D$, $\overline{D}_k$ is compact and $D = \bigcup\limits_{k=1}^{\infty} D_k$. Put $\tau_k = \inf\{t > 0; Y_t \notin D_k\}$, $k = 1, 2, \ldots$ By Dynkin's formula we have for $y \in D_k$,

$$\begin{aligned}\phi(y) &= \lim_{j\to\infty} \phi_j(y) = \lim_{j\to\infty} E^y\Big[\int_0^{\tau_k \wedge T_R} -L\phi_j(Y_t)dt + \phi_j(Y_{\tau_k \wedge T_R})\Big] \\ &= E^y\Big[\int_0^{\tau_k \wedge T_R} -L\phi(Y_t)dt + \phi(Y_{\tau_k \wedge T_R})\Big] = E^y\Big[\int_0^{\tau_k \wedge T_R} f(Y_t)dt + \phi(Y_{\tau_k \wedge T_R})\Big]\end{aligned}$$

So by uniform integrability and (ii), (vii), (viii) we get

$$\begin{aligned}\phi(y) &= \lim_{R,k\to\infty} E^y\Big[\int_0^{\tau_k \wedge T_R} f(Y_t)dt + \phi(Y_{\tau_k \wedge T_R})\Big] \\ &= E^y\Big[\int_0^{\tau_D} f(Y_t)dt + g(Y_{\tau_D})\Big] = J^{\tau_D}(y) \le \Phi(y)\,. \end{aligned} \tag{10.4.13}$$

Combining (10.4.11) and (10.4.13) we get

$$\phi(y) \ge \Phi(y) \ge J^{\tau_D}(y) = \phi(y)$$

so

$$\phi(y) = \Phi(y) \quad \text{and} \quad \widehat{\tau}(y,\omega) := \tau_D \quad \text{is optimal when } y \in D\,. \tag{10.4.14}$$

From (10.4.12) and (10.4.14) we conclude that

$$\phi(y) = \Phi(y) \qquad \text{for all } y \in G\,.$$

Moreover, the stopping time $\widehat{\tau}$ defined by

$$\widehat{\tau}(y,\omega) = \begin{cases} 0 & \text{for } y \notin D \\ \tau_D & \text{for } y \in D \end{cases}$$

is optimal. By Theorem 10.1.12 we conclude that τ_D is optimal also. □

Example 10.4.2 (When is the right time to sell the stocks? (Part 2))
To illustrate Theorem 10.4.1 let us apply it to reconsider Example 10.2.2:

Rather than *proving* (10.2.8) and the following properties of D, we now simply *guess/assume* that D has the form

$$D = \{(s,x); 0 < x < x_0\}$$

for some $x_0 > 0$, which is intuitively reasonable. Then we solve (10.2.11) for arbitrary x_0 and we arrive at the following candidate ϕ for g^*:

$$\phi(s,x) = \begin{cases} e^{-\rho s}(x_0 - a)(\frac{x}{x_0})^{\gamma_1} & \text{for } 0 < x < x_0 \\ e^{-\rho s}(x-a) & \text{for } x \geq x_0 \ . \end{cases}$$

The requirement that $\phi \in C^1$ (Theorem 10.4.1 (i)) gives the value (10.2.13) for x_0. It is clear that $\phi \in C^2$ outside ∂D and by construction $L\phi = 0$ on D. Moreover, conditions (iii), (iv), (viii) and (ix) clearly hold. It remains to verify that

(ii) $\phi(s,x) > g(s,x)$ for $0 < x < x_0$, i.e. $\phi(s,x) > e^{-\rho s}(x-a)$ for $0 < x < x_0$ and

(v) $L\phi(s,x) \leq 0$ for $x > x_0$, i.e. $Lg(s,x) \leq 0$ for $x > x_0$.

This is easily done by direct calculation (assuming $r < \rho$).

We conclude that $\phi = g^*$ and $\tau^* = \tau_D$ is optimal (with the value (10.2.13) for x_0).

Exercises

10.1.* In each of the optimal stopping problems below find the supremum g^* and – if it exists – an optimal stopping time τ^*. (Here B_t denotes 1-dimensional Brownian motion)

a) $g^*(x) = \sup_\tau E^x[B_\tau^2]$

b) $g^*(x) = \sup_\tau E^x[|B_\tau|^p]$, where $p > 0$.

c) $g^*(x) = \sup_\tau E^x[e^{-B_\tau^2}]$

d) $g^*(s,x) = \sup_\tau E^{(s,x)}[e^{-\rho(s+\tau)}\cosh B_\tau]$ where $\rho > 0$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

10.2.* a) Prove that the only nonnegative (B_t-) superharmonic functions in $\mathbf{R}^2$ are the constants.
(Hint: Suppose u is a nonnegative superharmonic function and that there exist $x, y \in \mathbf{R}^2$ such that

$$u(x) < u(y) \ .$$

Consider

$$E^x[u(B_\tau)] \ ,$$

where τ is the first hitting time for B_t of a small disc centered at y).

b) Prove that the only nonnegative superharmonic functions in $\mathbf{R}$ are the constants and use this to find $g^*(x)$ when

$$g(x) = \begin{cases} xe^{-x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \ . \end{cases}$$

c) Let $\gamma \in \mathbf{R}$, $n \ge 3$ and define, for $x \in \mathbf{R}^n$,

$$f_\gamma(x) = \begin{cases} |x|^\gamma & \text{for } |x| \ge 1 \\ 1 & \text{for } |x| < 1 \ . \end{cases}$$

For what values of γ is $f_\gamma(\cdot)$ ((B_t)-) harmonic for $|x| > 1$? Prove that f_γ is superharmonic in $\mathbf{R}^n$ iff $\gamma \in [2-n, 0]$.

10.3.* Find g^*, τ^* such that

$$g^*(s,x) = \sup_\tau E^{(s,x)}[e^{-\rho(s+\tau)}B_\tau^2] = E^{(s,x)}[e^{-\rho(s+\tau^*)}B_{\tau^*}^2] \ ,$$

where B_t is 1-dimensional Brownian motion, $\rho > 0$ is constant.
Hint: First assume that the continuation region has the form

$$D = \{(s,x); -x_0 < x < x_0\}$$

for some x_0 and then try to determine x_0. Then apply Theorem 10.4.1.

10.4. Let X_t be an Itô diffusion on $\mathbf{R}^n$ and $g: \mathbf{R}^n \to \mathbf{R}^+$ a continuous reward function. Define

$$g^\diamond(x) = \sup\{E^x[g(X_\tau)] \,;\ \tau \text{ stopping time, } E^x[\tau] < \infty\} \ .$$

Show that $g^\diamond = g^*$.
(Hint: If τ is a stopping time put $\tau_k = \tau \wedge k$ for $k = 1, 2, \ldots$ and consider

$$E^x[g(X_\tau) \cdot \mathcal{X}_{\tau<\infty}] \le E^x[\varliminf_{k\to\infty} g(X_{\tau_k})]) \ .$$

10.5. With g, r, ρ as in Example 10.2.2 prove that
a) if $r > \rho$ then $g^* = \infty$,
b) if $r = \rho$ then $g^*(s,x) = xe^{-\rho s}$.

10.6. Prove statements (10.3.8), (10.3.9), (10.3.10) in Example 10.3.1.

10.7. As a supplement to Exercise 10.4 it is worth noting that if g is not bounded below then the two problems

$$g^*(x) = \sup\{E^x[g(X_\tau)]\,;\ \tau \text{ stopping time}\}$$

and

$$g^\diamond(x) = \sup\{E^x[g(X_\tau)]\,;\tau \text{ stopping time, } E^x[\tau] < \infty\}$$

need not have the same solution. For example, if $g(x)=x$, $X_t=B_t\in\mathbf{R}$ prove that

$$g^*(x) = \infty \qquad \text{for all } x \in \mathbf{R}$$

while

$$g^\diamond(x) = x \qquad \text{for all } x \in \mathbf{R}\,.$$

(See Exercise 7.4.)

10.8. Give an example with g not bounded below where Theorem 10.1.9 a) fails. (Hint: See Exercise 10.7.)

10.9.* Solve the optimal stopping problem

$$\Phi(x) = \sup_\tau E^x\Big[\int_0^\tau e^{-\rho t}B_t^2 dt + e^{-\rho\tau}B_\tau^2\Big]\,.$$

10.10. Prove the following simple, but useful, observation, which can be regarded as an extension of (10.1.35):
Let $W = \{(s,x); \exists\tau \text{ with } g(s,x) < E^{(s,x)}[g(s+\tau, X_\tau)]\}$.
Then $W \subset D$.

10.11. Consider the optimal stopping problem

$$g^*(s,x) = \sup_\tau E^{(s,x)}[e^{-\rho(s+\tau)}B_\tau^+]\,,$$

where $B_t \in \mathbf{R}$ and $x^+ = \max\{x, 0\}$.

a) Use the argument for (10.2.8) and Exercise 10.10 to prove that the continuation region D has the form

$$D = \{(s,x); x < x_0\}$$

for some $x_0 > 0$.

b) Determine x_0 and find g^*.

c) Verify the *high contact principle*:

$$\frac{\partial g^*}{\partial x} = \frac{\partial g}{\partial x} \qquad \text{when } (s,x) = (s,x_0)\,,$$

where $g(t,x) = e^{-\rho t}x^+$.

10.12.* The first time the high contact principle was formulated seems to be in a paper by Samuelson (1965), who studied the optimal time for selling an asset, if the reward obtained by selling at the time t and when price is ξ is given by

$$g(t,\xi) = e^{-\rho t}(\xi - 1)^+ \ .$$

The price process is assumed to be a geometric Brownian motion X_t given by

$$dX_t = rX_t dt + \alpha X_t dB_t \ , \qquad X_0 = x > 0 \ ,$$

where $r < \rho$.
In other words, the problem is to find g^*, τ^* such that

$$g^*(s,x) = \sup_\tau E^{(s,x)}[e^{-\rho(s+\tau)}(X_\tau - 1)^+] = E^{(s,x)}[e^{-\rho(s+\tau^*)}(X_{\tau^*} - 1)^+] \ .$$

a) Use the argument for (10.2.8) and Exercise 10.10 to prove that the continuation region D has the form

$$D = \{(s,x); 0 < x < x_0\}$$

for some $x_0 > \frac{\rho}{\rho - r}$.

b) For a given $x_0 > \frac{\rho}{\rho - r}$ solve the boundary value problem

$$\begin{cases} \frac{\partial f}{\partial s} + rx\frac{\partial f}{\partial x} + \frac{1}{2}\alpha^2 x^2 \frac{\partial^2 f}{\partial x^2} = 0 & \text{for} \quad 0 < x < x_0 \\ f(s,0) = 0 & \\ f(s,x_0) = e^{-\rho s}(x_0 - 1)^+ & \end{cases}$$

by trying $f(s,x) = e^{-\rho s}\phi(x)$.

c) Determine x_0 by using the high contact principle, i.e. by using that

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} \qquad \text{when } \ x = x_0 \ .$$

d) With f, x_0 as in b), c) define

$$\gamma(s,x) = \begin{cases} f(s,x) \ ; & x < x_0 \\ e^{-\rho s}(x-1)^+ \ ; & x \geq x_0 \ . \end{cases}$$

Use Theorem 10.4.1 to verify that $\gamma = g^*$ and that $\tau^* = \tau_D$ is optimal.

10.13.* (A resource extraction problem)
Suppose the price P_t of one unit of a resource (e.g. gas, oil) at time t is varying like a geometric Brownian motion

$$dP_t = \alpha P_t dt + \beta P_t dB_t \ ; \qquad P_0 = p$$

where B_t is 1-dimensional Brownian motion and α, β are constants.

Let Q_t denote the amount of remaining resources at time t. Assume that the rate of extraction is proportional to the remaining amount, so that

$$dQ_t = -\lambda Q_t dt \; ; \qquad Q_0 = q$$

where $\lambda > 0$ is a constant.
If the running cost rate is $K > 0$ and we stop the extraction at the time $\tau = \tau(\omega)$ then the expected total discounted profit is given by

$$J^\tau(s,p,q) = E^{(s,p,q)}\Big[\int_0^\tau (\lambda P_t Q_t - K)e^{-\rho(s+t)}dt + e^{-\rho(s+\tau)}g(P_\tau, Q_\tau)\Big] \; ,$$

where $\rho > 0$ is the discounting exponent and $g(p,q)$ is a given bequest function giving the value of the remaining resource amount q when the price is p.

a) Write down the characteristic operator $\mathcal{A}$ of the diffusion process

$$dX_t = \begin{bmatrix} dt \\ dP_t \\ dQ_t \end{bmatrix} ; \qquad X_0 = (s,p,q)$$

and formulate the variational inequalities of Theorem 10.4.1 corresponding to the optimal stopping problem

$$\Phi(s,p,q) = \sup_\tau J^\tau(s,p,q) = J^{\tau^*}(s,p,q) \; .$$

b) Assume that $g(p,q) = pq$ and find the domain U corresponding to (10.1.34), (10.3.7), i.e.

$$U = \{(s,p,q); \mathcal{A}(e^{-\rho s}g(p,q)) + f(s,p,q) > 0\} \; ,$$

where

$$f(s,p,q) = e^{-\rho s}(\lambda pq - K) \; .$$

Conclude that
(i) if $\rho \geq \alpha$ then $\tau^* = 0$ and $\Phi(s,p,q) = pqe^{-\rho s}$
(ii) if $\rho < \alpha$ then $D \supset \{(s,p,q); pq > \frac{K}{\alpha - \rho}\}$.

c) As a candidate for Φ when $\rho < \alpha$ we try a function of the form

$$\phi(s,p,q) = \begin{cases} e^{-\rho s}pq \; ; & 0 < pq \leq y_0 \\ e^{-\rho s}\psi(pq) \; ; & pq > y_0 \end{cases}$$

for a suitable $\psi\colon \mathbf{R} \to \mathbf{R}$, and a suitable y_0. Use Theorem 10.4.1 to determine ψ, y_0 and to verify that with this choice of ψ, y_0 we have $\phi = \Phi$ and $\tau^* = \inf\{t > 0; P_t Q_t \leq y_0\}$, if $\rho < \alpha < \rho + \lambda$.

d) What happens if $\rho + \lambda \leq \alpha$?

10.14.* (Finding the optimal investment time (I))
Solve the optimal stopping problem

$$\Psi(s,p) = \sup_\tau E^{(s,p)}\left[\int_\tau^\infty e^{-\rho(s+t)} P_t dt - Ce^{-\rho(s+\tau)}\right] ,$$

where

$$dP_t = \alpha P_t dt + \beta P_t dB_t \; ; \qquad P_0 = p \; ,$$

B_t is 1-dimensional Brownian motion and α, β, ρ, C are constants, $0 < \alpha < \rho$ and $C > 0$. (We may interpret this as the problem of finding the optimal time τ for investment in a project. The profit rate after investment is P_t and the cost of the investment is C. Thus Ψ gives the maximal expected discounted net profit.)
Hint: Write $\int_\tau^\infty e^{-\rho(s+t)} P_t dt = e^{-\rho s}[\int_0^\infty e^{-\rho t} P_t dt - \int_0^\tau e^{-\rho t} P_t dt]$. Compute $E[\int_0^\infty e^{-\rho t} P_t dt]$ by using the solution formula for P_t (see Chapter 5) and then apply Theorem 10.4.1 to the problem

$$\Phi(s,p) = \sup_\tau E^{(s,p)}\left[-\int_0^\tau e^{-\rho(s+t)} P_t dt - Ce^{-\rho(s+\tau)}\right] .$$

10.15. Let B_t be 1-dimensional Brownian motion and let $\rho > 0$ be constant.

a) Show that the family

$$\{e^{-\rho\tau} B_\tau ; \tau \text{ stopping time}\}$$

is uniformly integrable w.r.t. P^x.

b) Solve the optimal stopping problem

$$\Phi(s,x) = \sup_\tau E^{(s,x)}[e^{-\rho(s+\tau)}(B_\tau - a)]$$

when $a > 0$ is constant. This may be regarded as a variation of Example 10.2.2/10.4.2 with the price process represented by B_t rather than X_t.

10.16. (Finding the optimal investment time (II))
Solve the optimal stopping problem

$$\Psi(s,p) = \sup_\tau E^{(s,p)}\left[\int_\tau^\infty e^{-\rho(s+t)} P_t dt - Ce^{-\rho(s+\tau)}\right]$$

where

$$dP_t = \mu\, dt + \sigma\, dB_t \; ; \qquad P_0 = p$$

with $\mu, \sigma \neq 0$ constants. (Compare with Exercise 10.14.)

10.17. a) Let

$$dX_t = \mu\, dt + \sigma\, dB_t\ ; \qquad X_0 = x \in \mathbf{R}$$

where μ and σ are constants. Prove that if $\rho > 0$ is constant then

$$E^x\Big[\int_0^\infty e^{-\rho t}|X_t|dt\Big] < \infty \qquad \text{for all } \ x\ .$$

b) Solve the optimal stopping problem

$$\Phi(s,x) = \sup_{\tau \geq 0} E^{s,x}\Big[\int_0^\tau e^{-\rho(s+t)}(X_t - a)dt\Big],$$

where $a \geq 0$ is a constant.