# The Geometry of Physics

## An Introduction

*Second Edition*

**Theodore Frankel**

*University of California, San Diego*

CAMBRIDGE UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011–4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org



First published 2004

Printed in the United States of America

*Typeface* Times Roman 10.5/13 pt. *System* LaTeX $2_\varepsilon$ [TB]

*A catalog record for this book is available from the British Library.*

*Library of Congress Cataloging in Publication data*

Frankel, Theodore, 1929–
The geometry of physics : an introduction / Theodore Frankel.– 2nd ed.
p. cm.
Includes bibliographical references and index.
ISBN 0-521-53927-7 (pbk.)
1. Geometry, Differential. 2. Mathematical physics. I. Title.
QC20.7 D52F73 2003
530.15′ 636–dc21 2003044030

ISBN 0 521 83330 2 hardback
ISBN 0 521 53927 7 paperback

# Contents


*Preface to the Second Edition* — *page* xix
*Preface to the Revised Printing* — xxi
*Preface to the First Edition* — xxiii

**I Manifolds, Tensors, and Exterior Forms**

**1 Manifolds and Vector Fields** **3**
- **1.1.** Submanifolds of Euclidean Space 3
  - **1.1a.** Submanifolds of $\mathbb{R}^N$ 4
  - **1.1b.** The Geometry of Jacobian Matrices: The "Differential" 7
  - **1.1c.** The Main Theorem on Submanifolds of $\mathbb{R}^N$ 8
  - **1.1d.** A Nontrivial Example: The Configuration Space of a Rigid Body 9
- **1.2.** Manifolds 11
  - **1.2a.** Some Notions from Point Set Topology 11
  - **1.2b.** The Idea of a Manifold 13
  - **1.2c.** A Rigorous Definition of a Manifold 19
  - **1.2d.** Complex Manifolds: The Riemann Sphere 21
- **1.3.** Tangent Vectors and Mappings 22
  - **1.3a.** Tangent or "Contravariant" Vectors 23
  - **1.3b.** Vectors as Differential Operators 24
  - **1.3c.** The Tangent Space to $M^n$ at a Point 25
  - **1.3d.** Mappings and Submanifolds of Manifolds 26
  - **1.3e.** Change of Coordinates 29
- **1.4.** Vector Fields and Flows 30
  - **1.4a.** Vector Fields and Flows on $\mathbb{R}^n$ 30
  - **1.4b.** Vector Fields on Manifolds 33
  - **1.4c.** Straightening Flows 34

**2 Tensors and Exterior Forms** **37**
**2.1.** Covectors and Riemannian Metrics 37
**2.1a.** Linear Functionals and the Dual Space 37
**2.1b.** The Differential of a Function 40
**2.1c.** Scalar Products in Linear Algebra 42
**2.1d.** Riemannian Manifolds and the Gradient Vector 45
**2.1e.** Curves of Steepest Ascent 46
**2.2.** The Tangent Bundle 48
**2.2a.** The Tangent Bundle 48
**2.2b.** The Unit Tangent Bundle 50
**2.3.** The Cotangent Bundle and Phase Space 52
**2.3a.** The Cotangent Bundle 52
**2.3b.** The Pull-Back of a Covector 52
**2.3c.** The Phase Space in Mechanics 54
**2.3d.** The Poincaré 1-Form 56
**2.4.** Tensors 58
**2.4a.** Covariant Tensors 58
**2.4b.** Contravariant Tensors 59
**2.4c.** Mixed Tensors 60
**2.4d.** Transformation Properties of Tensors 62
**2.4e.** Tensor Fields on Manifolds 63
**2.5.** The Grassmann or Exterior Algebra 66
**2.5a.** The Tensor Product of Covariant Tensors 66
**2.5b.** The Grassmann or Exterior Algebra 66
**2.5c.** The Geometric Meaning of Forms in $\mathbb{R}^n$ 70
**2.5d.** Special Cases of the Exterior Product 70
**2.5e.** Computations and Vector Analysis 71
**2.6.** Exterior Differentiation 73
**2.6a.** The Exterior Differential 73
**2.6b.** Examples in $\mathbb{R}^3$ 75
**2.6c.** A Coordinate Expression for $d$ 76
**2.7.** Pull-Backs 77
**2.7a.** The Pull-Back of a Covariant Tensor 77
**2.7b.** The Pull-Back in Elasticity 80
**2.8.** Orientation and Pseudoforms 82
**2.8a.** Orientation of a Vector Space 82
**2.8b.** Orientation of a Manifold 83
**2.8c.** Orientability and 2-Sided Hypersurfaces 84
**2.8d.** Projective Spaces 85
**2.8e.** Pseudoforms and the Volume Form 85
**2.8f.** The Volume Form in a Riemannian Manifold 87
**2.9.** Interior Products and Vector Analysis 89
**2.9a.** Interior Products and Contractions 89
**2.9b.** Interior Product in $\mathbb{R}^3$ 90
**2.9c.** Vector Analysis in $\mathbb{R}^3$ 92

**2.10.** Dictionary 94

**3 Integration of Differential Forms** **95**

**3.1.** Integration over a Parameterized Subset 95

**3.1a.** Integration of a $p$-Form in $\mathbb{R}^p$ 95

**3.1b.** Integration over Parameterized Subsets 96

**3.1c.** Line Integrals 97

**3.1d.** Surface Integrals 99

**3.1e.** Independence of Parameterization 101

**3.1f.** Integrals and Pull-Backs 102

**3.1g.** Concluding Remarks 102

**3.2.** Integration over Manifolds with Boundary 104

**3.2a.** Manifolds with Boundary 105

**3.2b.** Partitions of Unity 106

**3.2c.** Integration over a Compact Oriented Submanifold 108

**3.2d.** Partitions and Riemannian Metrics 109

**3.3.** Stokes's Theorem 110

**3.3a.** Orienting the Boundary 110

**3.3b.** Stokes's Theorem 111

**3.4.** Integration of Pseudoforms 114

**3.4a.** Integrating Pseudo-$n$-Forms on an $n$-Manifold 115

**3.4b.** Submanifolds with Transverse Orientation 115

**3.4c.** Integration over a Submanifold with Transverse Orientation 116

**3.4d.** Stokes's Theorem for Pseudoforms 117

**3.5.** Maxwell's Equations 118

**3.5a.** Charge and Current in Classical Electromagnetism 118

**3.5b.** The Electric and Magnetic Fields 119

**3.5c.** Maxwell's Equations 120

**3.5d.** Forms and Pseudoforms 122

**4 The Lie Derivative** **125**

**4.1.** The Lie Derivative of a Vector Field 125

**4.1a.** The Lie Bracket 125

**4.1b.** Jacobi's Variational Equation 127

**4.1c.** The Flow Generated by $[X, Y]$ 129

**4.2.** The Lie Derivative of a Form 132

**4.2a.** Lie Derivatives of Forms 132

**4.2b.** Formulas Involving the Lie Derivative 134

**4.2c.** Vector Analysis Again 136

**4.3.** Differentiation of Integrals 138

**4.3a.** The Autonomous (Time-Independent) Case 138

**4.3b.** Time-Dependent Fields 140

**4.3c.** Differentiating Integrals 142

**4.4.** A Problem Set on Hamiltonian Mechanics 145

**4.4a.** Time-Independent Hamiltonians 147

**4.4b.** Time-Dependent Hamiltonians and Hamilton's Principle 151
**4.4c.** Poisson Brackets 154

**5 The Poincaré Lemma and Potentials** **155**
**5.1.** A More General Stokes's Theorem 155
**5.2.** Closed Forms and Exact Forms 156
**5.3.** Complex Analysis 158
**5.4.** The Converse to the Poincaré Lemma 160
**5.5.** Finding Potentials 162

**6 Holonomic and Nonholonomic Constraints** **165**
**6.1.** The Frobenius Integrability Condition 165
**6.1a.** Planes in $\mathbb{R}^3$ 165
**6.1b.** Distributions and Vector Fields 167
**6.1c.** Distributions and 1-Forms 167
**6.1d.** The Frobenius Theorem 169
**6.2.** Integrability and Constraints 172
**6.2a.** Foliations and Maximal Leaves 172
**6.2b.** Systems of Mayer–Lie 174
**6.2c.** Holonomic and Nonholonomic Constraints 175
**6.3.** Heuristic Thermodynamics via Caratheodory 178
**6.3a.** Introduction 178
**6.3b.** The First Law of Thermodynamics 179
**6.3c.** Some Elementary Changes of State 180
**6.3d.** The Second Law of Thermodynamics 181
**6.3e.** Entropy 183
**6.3f.** Increasing Entropy 185
**6.3g.** Chow's Theorem on Accessibility 187

## II Geometry and Topology

**7 $\mathbb{R}^3$ and Minkowski Space** **191**
**7.1.** Curvature and Special Relativity 191
**7.1a.** Curvature of a Space Curve in $\mathbb{R}^3$ 191
**7.1b.** Minkowski Space and Special Relativity 192
**7.1c.** Hamiltonian Formulation 196
**7.2.** Electromagnetism in Minkowski Space 196
**7.2a.** Minkowski's Electromagnetic Field Tensor 196
**7.2b.** Maxwell's Equations 198

**8 The Geometry of Surfaces in $\mathbb{R}^3$** **201**
**8.1.** The First and Second Fundamental Forms 201
**8.1a.** The First Fundamental Form, or Metric Tensor 201
**8.1b.** The Second Fundamental Form 203
**8.2.** Gaussian and Mean Curvatures 205
**8.2a.** Symmetry and Self-Adjointness 205

**8.2b.** Principal Normal Curvatures 206
**8.2c.** Gauss and Mean Curvatures: The Gauss Normal Map 207
**8.3.** The Brouwer Degree of a Map: A Problem Set 210
**8.3a.** The Brouwer Degree 210
**8.3b.** Complex Analytic (Holomorphic) Maps 214
**8.3c.** The Gauss Normal Map Revisited: The Gauss–Bonnet Theorem 215
**8.3d.** The Kronecker Index of a Vector Field 215
**8.3e.** The Gauss Looping Integral 218
**8.4.** Area, Mean Curvature, and Soap Bubbles 221
**8.4a.** The First Variation of Area 221
**8.4b.** Soap Bubbles and Minimal Surfaces 226
**8.5.** Gauss's *Theorema Egregium* 228
**8.5a.** The Equations of Gauss and Codazzi 228
**8.5b.** The *Theorema Egregium* 230
**8.6.** Geodesics 232
**8.6a.** The First Variation of Arc Length 232
**8.6b.** The Intrinsic Derivative and the Geodesic Equation 234
**8.7.** The Parallel Displacement of Levi-Civita 236

**9 Covariant Differentiation and Curvature** **241**
**9.1.** Covariant Differentiation 241
**9.1a.** Covariant Derivative 241
**9.1b.** Curvature of an Affine Connection 244
**9.1c.** Torsion and Symmetry 245
**9.2.** The Riemannian Connection 246
**9.3.** Cartan's Exterior Covariant Differential 247
**9.3a.** Vector-Valued Forms 247
**9.3b.** The Covariant Differential of a Vector Field 248
**9.3c.** Cartan's Structural Equations 249
**9.3d.** The Exterior Covariant Differential of a Vector-Valued Form 250
**9.3e.** The Curvature 2-Forms 251
**9.4.** Change of Basis and Gauge Transformations 253
**9.4a.** Symmetric Connections Only 253
**9.4b.** Change of Frame 253
**9.5.** The Curvature Forms in a Riemannian Manifold 255
**9.5a.** The Riemannian Connection 255
**9.5b.** Riemannian Surfaces $M^2$ 257
**9.5c.** An Example 257
**9.6.** Parallel Displacement and Curvature on a Surface 259
**9.7.** Riemann's Theorem and the Horizontal Distribution 263
**9.7a.** Flat Metrics 263
**9.7b.** The Horizontal Distribution of an Affine Connection 263
**9.7c.** Riemann's Theorem 266

**10 Geodesics** **269**
**10.1.** Geodesics and Jacobi Fields 269
**10.1a.** Vector Fields Along a Surface in $M^n$ 269
**10.1b.** Geodesics 271
**10.1c.** Jacobi Fields 272
**10.1d.** Energy 274
**10.2.** Variational Principles in Mechanics 275
**10.2a.** Hamilton's Principle in the Tangent Bundle 275
**10.2b.** Hamilton's Principle in Phase Space 277
**10.2c.** Jacobi's Principle of "Least" Action 278
**10.2d.** Closed Geodesics and Periodic Motions 281
**10.3.** Geodesics, Spiders, and the Universe 284
**10.3a.** Gaussian Coordinates 284
**10.3b.** Normal Coordinates on a Surface 287
**10.3c.** Spiders and the Universe 288

**11 Relativity, Tensors, and Curvature** **291**
**11.1.** Heuristics of Einstein's Theory 291
**11.1a.** The Metric Potentials 291
**11.1b.** Einstein's Field Equations 293
**11.1c.** Remarks on Static Metrics 296
**11.2.** Tensor Analysis 298
**11.2a.** Covariant Differentiation of Tensors 298
**11.2b.** Riemannian Connections and the Bianchi Identities 299
**11.2c.** Second Covariant Derivatives: The Ricci Identities 301
**11.3.** Hilbert's Action Principle 303
**11.3a.** Geodesics in a Pseudo-Riemannian Manifold 303
**11.3b.** Normal Coordinates, the Divergence and Laplacian 303
**11.3c.** Hilbert's Variational Approach to General Relativity 305
**11.4.** The Second Fundamental Form in the Riemannian Case 309
**11.4a.** The Induced Connection and the Second Fundamental Form 309
**11.4b.** The Equations of Gauss and Codazzi 311
**11.4c.** The Interpretation of the Sectional Curvature 313
**11.4d.** Fixed Points of Isometries 314
**11.5.** The Geometry of Einstein's Equations 315
**11.5a.** The Einstein Tensor in a (Pseudo-)Riemannian Space–Time 315
**11.5b.** The Relativistic Meaning of Gauss's Equation 316
**11.5c.** The Second Fundamental Form of a Spatial Slice 318
**11.5d.** The Codazzi Equations 319
**11.5e.** Some Remarks on the Schwarzschild Solution 320

**12 Curvature and Topology: Synge's Theorem** **323**
**12.1.** Synge's Formula for Second Variation 324
**12.1a.** The Second Variation of Arc Length 324
**12.1b.** Jacobi Fields 326

**12.2.** Curvature and Simple Connectivity 329
**12.2a.** Synge's Theorem 329
**12.2b.** Orientability Revisited 331

**13 Betti Numbers and De Rham's Theorem 333**
**13.1.** Singular Chains and Their Boundaries 333
**13.1a.** Singular Chains 333
**13.1b.** Some 2-Dimensional Examples 338
**13.2.** The Singular Homology Groups 342
**13.2a.** Coefficient Fields 342
**13.2b.** Finite Simplicial Complexes 343
**13.2c.** Cycles, Boundaries, Homology, and Betti Numbers 344
**13.3.** Homology Groups of Familiar Manifolds 347
**13.3a.** Some Computational Tools 347
**13.3b.** Familiar Examples 350
**13.4.** De Rham's Theorem 355
**13.4a.** The Statement of De Rham's Theorem 355
**13.4b.** Two Examples 357

**14 Harmonic Forms 361**
**14.1.** The Hodge Operators 361
**14.1a.** The $*$ Operator 361
**14.1b.** The Codifferential Operator $\delta = d^*$ 364
**14.1c.** Maxwell's Equations in Curved Space–Time $M^4$ 366
**14.1d.** The Hilbert Lagrangian 367
**14.2.** Harmonic Forms 368
**14.2a.** The Laplace Operator on Forms 368
**14.2b.** The Laplacian of a 1-Form 369
**14.2c.** Harmonic Forms on Closed Manifolds 370
**14.2d.** Harmonic Forms and De Rham's Theorem 372
**14.2e.** Bochner's Theorem 374
**14.3.** Boundary Values, Relative Homology, and Morse Theory 375
**14.3a.** Tangential and Normal Differential Forms 376
**14.3b.** Hodge's Theorem for Tangential Forms 377
**14.3c.** Relative Homology Groups 379
**14.3d.** Hodge's Theorem for Normal Forms 381
**14.3e.** Morse's Theory of Critical Points 382

**III Lie Groups, Bundles, and Chern Forms**

**15 Lie Groups 391**
**15.1.** Lie Groups, Invariant Vector Fields, and Forms 391
**15.1a.** Lie Groups 391
**15.1b.** Invariant Vector Fields and Forms 395
**15.2.** One-Parameter Subgroups 398
**15.3.** The Lie Algebra of a Lie Group 402
**15.3a.** The Lie Algebra 402

**15.3b.** The Exponential Map 403
**15.3c.** Examples of Lie Algebras 404
**15.3d.** Do the 1-Parameter Subgroups Cover *G*? 405
**15.4.** Subgroups and Subalgebras 407
**15.4a.** Left Invariant Fields Generate Right Translations 407
**15.4b.** Commutators of Matrices 408
**15.4c.** Right Invariant Fields 409
**15.4d.** Subgroups and Subalgebras 410

**16 Vector Bundles in Geometry and Physics 413**
**16.1.** Vector Bundles 413
**16.1a.** Motivation by Two Examples 413
**16.1b.** Vector Bundles 415
**16.1c.** Local Trivializations 417
**16.1d.** The Normal Bundle to a Submanifold 419
**16.2.** Poincaré's Theorem and the Euler Characteristic 421
**16.2a.** Poincaré's Theorem 422
**16.2b.** The Stiefel Vector Field and Euler's Theorem 426
**16.3.** Connections in a Vector Bundle 428
**16.3a.** Connection in a Vector Bundle 428
**16.3b.** Complex Vector Spaces 431
**16.3c.** The Structure Group of a Bundle 433
**16.3d.** Complex Line Bundles 433
**16.4.** The Electromagnetic Connection 435
**16.4a.** Lagrange's Equations without Electromagnetism 435
**16.4b.** The Modified Lagrangian and Hamiltonian 436
**16.4c.** Schrödinger's Equation in an Electromagnetic Field 439
**16.4d.** Global Potentials 443
**16.4e.** The Dirac Monopole 444
**16.4f.** The Aharonov–Bohm Effect 446

**17 Fiber Bundles, Gauss–Bonnet, and Topological Quantization 451**
**17.1.** Fiber Bundles and Principal Bundles 451
**17.1a.** Fiber Bundles 451
**17.1b.** Principal Bundles and Frame Bundles 453
**17.1c.** Action of the Structure Group on a Principal Bundle 454
**17.2.** Coset Spaces 456
**17.2a.** Cosets 456
**17.2b.** Grassmann Manifolds 459
**17.3.** Chern's Proof of the Gauss–Bonnet–Poincaré Theorem 460
**17.3a.** A Connection in the Frame Bundle of a Surface 460
**17.3b.** The Gauss–Bonnet–Poincaré Theorem 462
**17.3c.** Gauss–Bonnet as an Index Theorem 465
**17.4.** Line Bundles, Topological Quantization, and Berry Phase 465
**17.4a.** A Generalization of Gauss–Bonnet 465
**17.4b.** Berry Phase 468
**17.4c.** Monopoles and the Hopf Bundle 473

**18 Connections and Associated Bundles** **475**
**18.1.** Forms with Values in a Lie Algebra 475
**18.1a.** The Maurer–Cartan Form 475
**18.1b.** $\mathfrak{g}$-Valued $p$-Forms on a Manifold 477
**18.1c.** Connections in a Principal Bundle 479
**18.2.** Associated Bundles and Connections 481
**18.2a.** Associated Bundles 481
**18.2b.** Connections in Associated Bundles 483
**18.2c.** The Associated $Ad$ Bundle 485
**18.3.** $r$-Form Sections of a Vector Bundle: Curvature 488
**18.3a.** $r$-Form Sections of $E$ 488
**18.3b.** Curvature and the $Ad$ Bundle 489

**19 The Dirac Equation** **491**
**19.1.** The Groups $SO(3)$ and $SU(2)$ 491
**19.1a.** The Rotation Group $SO(3)$ of $\mathbb{R}^3$ 492
**19.1b.** $SU(2)$: The Lie Algebra $\mathfrak{su}(2)$ 493
**19.1c.** $SU(2)$ Is Topologically the 3-Sphere 495
**19.1d.** $Ad : SU(2) \to SO(3)$ in More Detail 496
**19.2.** Hamilton, Clifford, and Dirac 497
**19.2a.** Spinors and Rotations of $\mathbb{R}^3$ 497
**19.2b.** Hamilton on Composing Two Rotations 499
**19.2c.** Clifford Algebras 500
**19.2d.** The Dirac Program: The Square Root of the d'Alembertian 502
**19.3.** The Dirac Algebra 504
**19.3a.** The Lorentz Group 504
**19.3b.** The Dirac Algebra 509
**19.4.** The Dirac Operator $\not\partial$ in Minkowski Space 511
**19.4a.** Dirac Spinors 511
**19.4b.** The Dirac Operator 513
**19.5.** The Dirac Operator in Curved Space–Time 515
**19.5a.** The Spinor Bundle 515
**19.5b.** The Spin Connection in $\mathbb{S}M$ 518

**20 Yang–Mills Fields** **523**
**20.1.** Noether's Theorem for Internal Symmetries 523
**20.1a.** The Tensorial Nature of Lagrange's Equations 523
**20.1b.** Boundary Conditions 526
**20.1c.** Noether's Theorem for Internal Symmetries 527
**20.1d.** Noether's Principle 528
**20.2.** Weyl's Gauge Invariance Revisited 531
**20.2a.** The Dirac Lagrangian 531
**20.2b.** Weyl's Gauge Invariance Revisited 533
**20.2c.** The Electromagnetic Lagrangian 534
**20.2d.** Quantization of the $A$ Field: Photons 536

**20.3.** The Yang–Mills Nucleon 537
**20.3a.** The Heisenberg Nucleon 537
**20.3b.** The Yang–Mills Nucleon 538
**20.3c.** A Remark on Terminology 540
**20.4.** Compact Groups and Yang–Mills Action 541
**20.4a.** The Unitary Group Is Compact 541
**20.4b.** Averaging over a Compact Group 541
**20.4c.** Compact Matrix Groups Are Subgroups of Unitary Groups 542
**20.4d.** *Ad* Invariant Scalar Products in the Lie Algebra of a Compact Group 543
**20.4e.** The Yang–Mills Action 544
**20.5.** The Yang–Mills Equation 545
**20.5a.** The Exterior Covariant Divergence $\nabla^*$ 545
**20.5b.** The Yang–Mills Analogy with Electromagnetism 547
**20.5c.** Further Remarks on the Yang–Mills Equations 548
**20.6.** Yang–Mills Instantons 550
**20.6a.** Instantons 550
**20.6b.** Chern's Proof Revisited 553
**20.6c.** Instantons and the Vacuum 557

**21 Betti Numbers and Covering Spaces** **561**
**21.1.** Bi-invariant Forms on Compact Groups 561
**21.1a.** Bi-invariant $p$-Forms 561
**21.1b.** The Cartan $p$-Forms 562
**21.1c.** Bi-invariant Riemannian Metrics 563
**21.1d.** Harmonic Forms in the Bi-invariant Metric 564
**21.1e.** Weyl and Cartan on the Betti Numbers of $G$ 565
**21.2.** The Fundamental Group and Covering Spaces 567
**21.2a.** Poincaré's Fundamental Group $\pi_1(M)$ 567
**21.2b.** The Concept of a Covering Space 569
**21.2c.** The Universal Covering 570
**21.2d.** The Orientable Covering 573
**21.2e.** Lifting Paths 574
**21.2f.** Subgroups of $\pi_1(M)$ 575
**21.2g.** The Universal Covering Group 575
**21.3.** The Theorem of S. B. Myers: A Problem Set 576
**21.4.** The Geometry of a Lie Group 580
**21.4a.** The Connection of a Bi-invariant Metric 580
**21.4b.** The Flat Connections 581

**22 Chern Forms and Homotopy Groups** **583**
**22.1.** Chern Forms and Winding Numbers 583
**22.1a.** The Yang–Mills "Winding Number" 583
**22.1b.** Winding Number in Terms of Field Strength 585
**22.1c.** The Chern Forms for a $U(n)$ Bundle 587

**22.2.** Homotopies and Extensions 591
**22.2a.** Homotopy 591
**22.2b.** Covering Homotopy 592
**22.2c.** Some Topology of $SU(n)$ 594
**22.3.** The Higher Homotopy Groups $\pi_k(M)$ 596
**22.3a.** $\pi_k(M)$ 596
**22.3b.** Homotopy Groups of Spheres 597
**22.3c.** Exact Sequences of Groups 598
**22.3d.** The Homotopy Sequence of a Bundle 600
**22.3e.** The Relation between Homotopy and Homology Groups 603
**22.4.** Some Computations of Homotopy Groups 605
**22.4a.** Lifting Spheres from $M$ into the Bundle $P$ 605
**22.4b.** $SU(n)$ Again 606
**22.4c.** The Hopf Map and Fibering 606
**22.5.** Chern Forms as Obstructions 608
**22.5a.** The Chern Forms $c_r$ for an $SU(n)$ Bundle Revisited 608
**22.5b.** $c_2$ as an "Obstruction Cocycle" 609
**22.5c.** The Meaning of the Integer $j(\Delta_4)$ 612
**22.5d.** Chern's Integral 612
**22.5e.** Concluding Remarks 615

**Appendix A. Forms in Continuum Mechanics 617**
**A.a.** The Classical Cauchy Stress Tensor and Equations of Motion 617
**A.b.** Stresses in Terms of Exterior Forms 618
**A.c.** Symmetry of Cauchy's Stress Tensor in $\mathbb{R}^n$ 620
**A.d.** The Piola–Kirchhoff Stress Tensors 622
**A.e.** Stored Energy of Deformation 623
**A.f.** Hamilton's Principle in Elasticity 626
**A.g.** Some Typical Computations Using Forms 629
**A.h.** Concluding Remarks 635

**Appendix B. Harmonic Chains and Kirchhoff's Circuit Laws 636**
**B.a.** Chain Complexes 636
**B.b.** Cochains and Cohomology 638
**B.c.** Transpose and Adjoint 639
**B.d.** Laplacians and Harmonic Cochains 641
**B.e.** Kirchhoff's Circuit Laws 643

**Appendix C. Symmetries, Quarks, and Meson Masses 648**
**C.a.** Flavored Quarks 648
**C.b.** Interactions of Quarks and Antiquarks 650
**C.c.** The Lie Algebra of $SU(3)$ 652
**C.d.** Pions, Kaons, and Etas 653
**C.e.** A Reduced Symmetry Group 656
**C.f.** Meson Masses 658

**Appendix D. Representations and Hyperelastic Bodies** **660**
**D.a.** Hyperelastic Bodies 660
**D.b.** Isotropic Bodies 661
**D.c.** Application of Schur's Lemma 662
**D.d.** Frobenius–Schur Relations 664
**D.e.** The Symmetric Traceless $3 \times 3$ Matrices Are Irreducible 666

**Appendix E. Orbits and Morse–Bott Theory in Compact Lie Groups** **670**
**E.a.** The Topology of Conjugacy Orbits 670
**E.b.** Application of Bott's Extension of Morse Theory 673

*References* 679
*Index* 683

CHAPTER 1

# Manifolds and Vector Fields

Better is the end of a thing than the beginning thereof.

Ecclesiastes 7:8

AS students we learn differential and integral calculus in the context of euclidean space $\mathbb{R}^n$, but it is necessary to apply calculus to problems involving "curved" spaces. Geodesy and cartography, for example, are devoted to the study of the most familiar curved surface of all, the surface of planet Earth. In discussing maps of the Earth, latitude and longitude serve as "coordinates," allowing us to use calculus by considering functions on the Earth's surface (temperature, height above sea level, etc.) as being functions of latitude and longitude. The familiar Mercator's projection, with its stretching of the polar regions, vividly informs us that these coordinates are badly behaved at the poles: that is, that they are not defined everywhere; they are not "global." (We shall refer to such coordinates as being "local," even though they might cover a huge portion of the surface. Precise definitions will be given in Section 1.2.) Of course we may use two sets of "polar" projections to study the Arctic and Antarctic regions. With these three maps we can study the entire surface, provided we know how to relate the Mercator to the polar maps.

We shall soon define a "manifold" to be a space that, like the surface of the Earth, can be covered by a family of local coordinate systems. *A manifold will turn out to be the most general space in which one can use differential and integral calculus with roughly the same facility as in euclidean space*. It should be recalled, though, that calculus in $\mathbb{R}^3$ demands special care when curvilinear coordinates are required.

The most familiar manifold is $N$-dimensional euclidean space $\mathbb{R}^N$, that is, the space of ordered $N$ tuples $(x^1, \ldots, x^N)$ of real numbers. Before discussing manifolds in general we shall talk about the more familiar (and less abstract) concept of a submanifold of $\mathbb{R}^N$, generalizing the notions of curve and surface in $\mathbb{R}^3$.

## 1.1. Submanifolds of Euclidean Space

What is the configuration space of a rigid body fixed at one point of $\mathbb{R}^n$?

### 1.1a. Submanifolds of $\mathbb{R}^N$

Euclidean space, $\mathbb{R}^N$, is endowed with a global coordinate system $(x^1, \ldots, x^N)$ and is the most important example of a manifold.

In our familiar $\mathbb{R}^3$, with coordinates $(x, y, z)$, a locus $z = F(x, y)$ describes a (2-dimensional) surface, whereas a locus of the form $y = G(x)$, $z = H(x)$, describes a (1-dimensional) curve. We shall need to consider higher-dimensional versions of these important notions.

A subset $M = M^n \subset \mathbb{R}^{n+r}$ is said to be an $n$-dimensional **submanifold** of $\mathbb{R}^{n+r}$, if *locally* $M$ can be described by giving $r$ of the coordinates differentiably in terms of the $n$ remaining ones. This means that given $p \in M$, a neighborhood of $p$ on $M$ can be described in *some* coordinate system $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^r)$ of $\mathbb{R}^{n+r}$ by $r$ differentiable functions

$$y^\alpha = f^\alpha(x^1, \ldots, x^n), \qquad \alpha = 1, \ldots r$$

We abbreviate this by $y = f(x)$, or even $y = y(x)$. We say that $x^1, \ldots, x^n$ are **local (curvilinear) coordinates** for $M$ near $p$.

**Examples:**

**(i)** $y^1 = f(x^1, \ldots, x^n)$ describes an $n$-dimensional submanifold of $\mathbb{R}^{n+1}$.

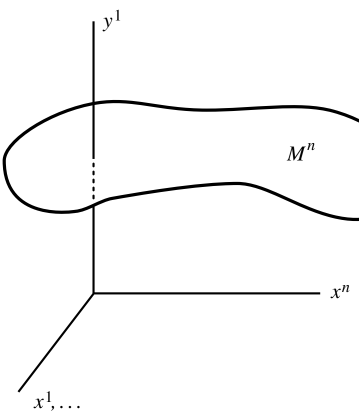


**Figure 1.1**

In Figure 1.1 we have drawn a portion of the submanifold $M$. This $M$ is the **graph** of a function $f : \mathbb{R}^n \to \mathbb{R}$, that is, $M = \{(\mathbf{x}, y) \in \mathbb{R}^{n+1} \mid y = f(\mathbf{x})\}$. When $n = 1$, $M$ is a curve; while if $n = 2$, it is a surface.

**(ii)** The *unit sphere* $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. Points in the northern hemisphere can be described by $z = F(x, y) = (1 - x^2 - y^2)^{1/2}$ and this function is differentiable everywhere except at the equator $x^2 + y^2 = 1$. Thus $x$ and $y$ are local coordinates for the northern hemisphere except at the equator. For points on the equator one can solve for $x$ or $y$ in terms of the others. If we have solved for $x$ then $y$ and $z$ are the two local coordinates. For points in the southern hemisphere one can use the negative square root for $z$. The unit sphere in $\mathbb{R}^3$ is a 2-dimensional submanifold of $\mathbb{R}^3$. We note that we have *not* been able to describe the *entire* sphere by expressing one of the coordinates, say $z$, in terms of the two remaining ones, $z = F(x, y)$. We settle for local coordinates.

More generally, given $r$ functions $F^\alpha(x_1, \ldots, x_n, y_1, \ldots, y_r)$ of $n + r$ variables, we may consider the locus $M^n \subset \mathbb{R}^{n+r}$ defined by the equations

$$F^\alpha(x, y) = c^\alpha, \qquad (c^1, \ldots, c^r) \text{ constants}$$

If the **Jacobian** determinant

$$\left[\frac{\partial(F^1, \ldots, F^r)}{\partial(y^1, \ldots, y^r)}\right](x_0, y_0)$$

at $(x_0, y_0) \in M$ of the locus is not 0, the **implicit function theorem** assures us that locally, near $(x_0, y_0)$, we may solve $F^\alpha(x, y) = c^\alpha$, $\alpha = 1, \ldots, r$, for the $y$'s in terms of the $x$'s

$$y^\alpha = f^\alpha(x^1, \ldots, x^n)$$

We may say that "a portion of $M^n$ near $(x_0, y_0)$ is a submanifold of $\mathbb{R}^{n+r}$." If the Jacobian $\neq 0$ *at all points of the locus*, then the entire $M^n$ is a submanifold.

Recall that the Jacobian condition arises as follows. If $F^\alpha(x, y) = c^\alpha$ can be solved for the $y$'s differentiably in terms of the $x$'s, $y^\beta = y^\beta(x)$, then if, for fixed $i$, we differentiate the identity $F^\alpha(x, y(x)) = c^\alpha$ with respect to $x^i$, we get

$$\frac{\partial F^\alpha}{\partial x^i} + \sum_\beta \left[\frac{\partial F^\alpha}{\partial y^\beta}\right] \frac{\partial y^\beta}{\partial x^i} = 0$$

and

$$\frac{\partial y^\beta}{\partial x^i} = -\sum_\alpha \left(\left[\frac{\partial F}{\partial y}\right]^{-1}\right)^\beta{}_\alpha \left[\frac{\partial F^\alpha}{\partial x^i}\right]$$

provided the subdeterminant $\partial(F^1, \ldots, F^r)/\partial(y^1, \ldots, y^r)$ is not zero. (Here $([\partial F/\partial y]^{-1})^\beta{}_\alpha$ is the $\beta\alpha$ entry of the inverse to the matrix $\partial F/\partial y$; we shall use the convention that for matrix indices, the index to the *left* always is the *row* index, whether it is up or down.) This *suggests* that if the indicated Jacobian is nonzero then we might indeed be able to solve for the $y$'s in terms of the $x$'s, and the implicit function theorem confirms this. The (nontrivial) *proof* of the implicit function theorem can be found in most books on real analysis.

Still more generally, suppose that we have $r$ functions of $n+r$ variables, $F^\alpha(x^1, \ldots, x^{n+r})$. Consider the locus $F^\alpha(x) = c^\alpha$. Suppose that at each point $x_0$ of the locus the Jacobian *matrix*

$$\left(\frac{\partial F^\alpha}{\partial x^i}\right) \qquad \alpha = 1, \ldots, r \qquad i = 1, \ldots, n + r$$

has rank $r$. Then the equations $F^\alpha = c^\alpha$ define an $n$-dimensional submanifold of $\mathbb{R}^{n+r}$, since we may locally solve for $r$ of the coordinates in terms of the remaining $n$.

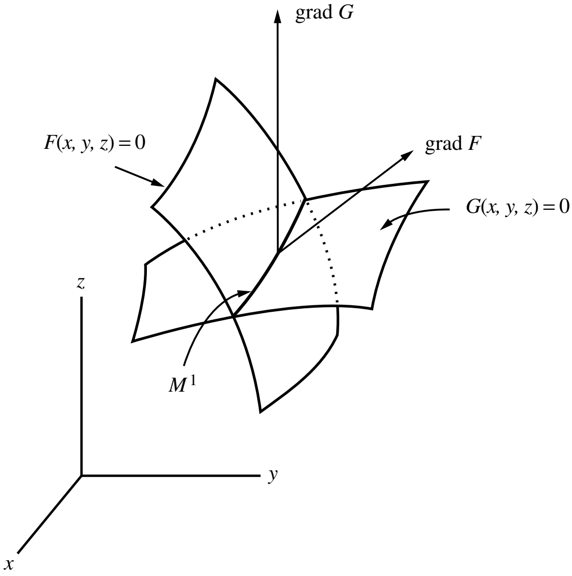


**Figure 1.2**

In Figure 1.2, two surfaces $F = 0$ and $G = 0$ in $\mathbb{R}^3$ intersect to yield a curve $M$.

The simplest case is *one* function $F$ of $N$ variables $(x^1, \dots, x^N)$. If *at each point of the locus* $F = c$ there is always *at least one partial derivative that does not vanish*, then the Jacobian (row) matrix $[\partial F/\partial x^1, \partial F/\partial x^2, \dots, \partial F/\partial x^N]$ has rank 1 and we may conclude that *this locus is indeed an* $(N-1)$*-dimensional submanifold of* $\mathbb{R}^N$. This criterion is easily verified, for example, in the case of the 2-sphere $F(x, y, z) = x^2 + y^2 + z^2 = 1$ of Example (ii). The column version of this row matrix is called in calculus the gradient vector of $F$. In $\mathbb{R}^3$ this vector

$$\begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{bmatrix}$$

is orthogonal to the locus $F = 0$, and we may conclude, for example, that if this gradient vector has a nontrivial component in the $z$ direction at a point of $F = 0$, then locally we can solve for $z = z(x, y)$.

A submanifold of dimension $(N-1)$ in $\mathbb{R}^N$, that is, of "**codimension**" 1, is called a **hypersurface**.

**(iii)** The $x$ axis of the $xy$ plane $\mathbb{R}^2$ can be described (perversely) as the locus of the quadratic $F(x, y) := y^2 = 0$. Both partial derivatives vanish on the locus, the $x$ axis, and our criteria would not allow us to say that the $x$ axis is a 1-dimensional submanifold of $\mathbb{R}^2$. Of course the $x$ axis *is* a submanifold; we should have used the usual description $G(x, y) := y = 0$. Our Jacobian criteria are *sufficient* conditions, not necessary ones.

**(iv)** The locus $F(x, y) := xy = 0$ in $\mathbb{R}^2$, consisting of the union of the $x$ and $y$ axes, is not a 1-dimensional submanifold of $\mathbb{R}^2$. It seems "clear" (and can be proved) that in a neighborhood of the intersection of the two lines we are not going to be able to describe the locus in the form of $y = f(x)$ or $x = g(y)$, where $f$, $g$, are differentiable functions. The best we can say is that this locus *with the origin removed* is a 1-dimensional submanifold.

### 1.1b. The Geometry of Jacobian Matrices: The "Differential"

The **tangent space** to $\mathbb{R}^n$ at the point $x$, written here as $\mathbb{R}^n_x$, is by definition the vector space of all vectors in $\mathbb{R}^n$ based at $x$ (i.e., it is a copy of $\mathbb{R}^n$ with origin shifted to $x$).

Let $x^1, \ldots, x^n$ and $y^1, \ldots, y^r$ be coordinates for $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively. Let $F : \mathbb{R}^n \to \mathbb{R}^r$ be a **smooth** map. ("Smooth" ordinarily means infinitely differentiable. For our purposes, however, it will mean differentiable at least as many times as is necessary in the present context. For example, if $F$ is once continuously differentiable, we may use the chain rule in the argument to follow.) In coordinates, $F$ is described by giving $r$ functions of $n$ variables

$$y^\alpha = F^\alpha(x) \qquad \alpha = 1, \ldots, r$$

or simply $y = F(x)$. We will frequently use the more dangerous notation $y = y(x)$.

Let $y_0 = F(x_0)$; the Jacobian *matrix* $(\partial y^\alpha / \partial x^i)(x_0)$ has the following significance.

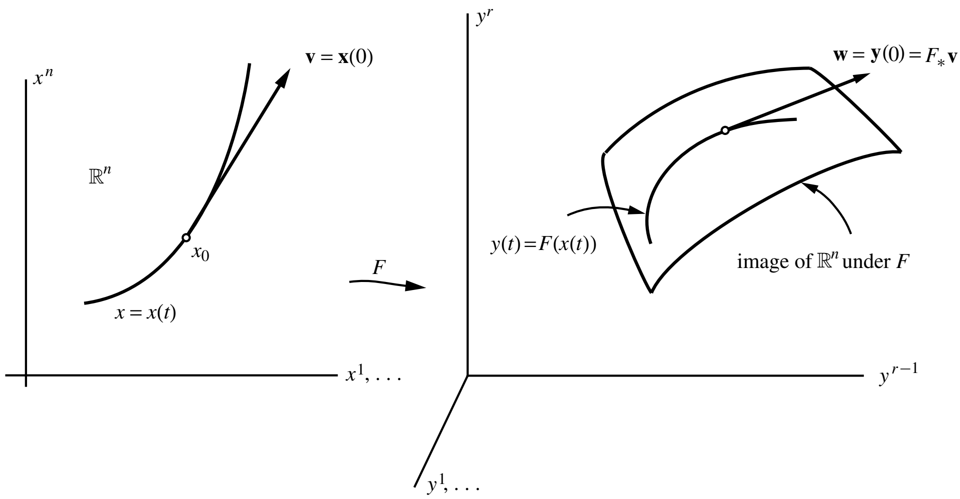


**Figure 1.3**

Let $\mathbf{v}$ be a tangent vector to $\mathbb{R}^n$ at $x_0$. Take *any* smooth curve $x(t)$ such that $x(0) = x_0$ and $\dot{x}(0) := (dx/dt)(0) = \mathbf{v}$, for example, the straight line $x(t) = x_0 + t\mathbf{v}$. The image of this curve

$$y(t) = F(x(t))$$

has a tangent vector $\mathbf{w}$ at $y_0$ given by the chain rule

$$w^\alpha = \dot{y}^\alpha(0) = \sum_{i=1}^{n} \left(\frac{\partial y^\alpha}{\partial x^i}\right)(x_0)\dot{x}^i(0) = \sum_{i=1}^{n} \left(\frac{\partial y^\alpha}{\partial x^i}\right)(x_0)v^i$$

The assignment $\mathbf{v} \mapsto \mathbf{w}$ is, from this expression, independent of the curve $x(t)$ chosen, and defines a *linear transformation*, the **differential** of $F$ at $x_0$

$$F_* : \mathbb{R}^n_{x_0} \to \mathbb{R}^r_{y_0} \qquad F_*(\mathbf{v}) = \mathbf{w} \tag{1.1}$$

whose matrix is simply the Jacobian matrix $(\partial y^\alpha/\partial x^i)(x_0)$. *This interpretation of the Jacobian matrix, as a linear transformation sending tangents to curves into tangents to the image curves under $F$, can sometimes be used to replace the direct computation of matrices*. This philosophy will be illustrated in Section 1.1d.

### 1.1c. The Main Theorem on Submanifolds of $\mathbb{R}^N$

The main theorem is a geometric interpretation of what we have discussed. Note that the statement "$F$ has rank $r$ at $x_0$," that is, $[\partial y^\alpha/\partial x^i](x_0)$ has rank $r$, is geometrically the statement that the differential

$$F_* : \mathbb{R}^n_{x_0} \to \mathbb{R}^r_{y_0=F(x_0)}$$

is **onto** or "surjective"; that is, given any vector $\mathbf{w}$ at $y_0$ there is at least one vector $\mathbf{v}$ at $x_0$ such that $F_*(\mathbf{v}) = \mathbf{w}$. We then have

**Theorem (1.2):** *Let $F : \mathbb{R}^{r+n} \to \mathbb{R}^r$ and suppose that the locus*

$$F^{-1}(y_0) := \{x \in \mathbb{R}^{r+n} \mid F(x) = y_0\}$$

*is not empty. Suppose further that for all $x_0 \in F^{-1}(y_0)$*

$$F_* : \mathbb{R}^{n+r}_{x_0} \to \mathbb{R}^r_{y_0}$$

*is onto. Then $F^{-1}(y_0)$ is an $n$-dimensional submanifold of $\mathbb{R}^{n+r}$.*

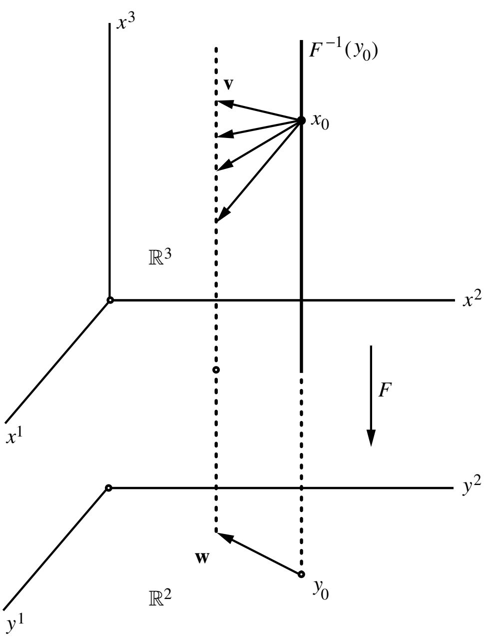


**Figure 1.4**